The nucleus as a source in Kerr-Newman geometry

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The nucleus is treated as a source in Kerr-Newman geometry, under the assumption that the angular momentum of the source is equal to the intrinsic spin angular momentum of the nucleus. The *spin radius a* of the nucleus, which figures in the Kerr-Newman metric, is of the order of the Compton wavelength of the nucleus. The radial functions in Chandrasekhar's separated Dirac equation in Kerr geometry, are transformed so as to yield a pair of simultaneous first-order differential equations with real coefficients, as in flat space. The magnetic quantum number m appears explicitly in the differential equations, thus lifting the hyperfine splitting degeneracy of Dirac's equation in flat space.

I. INTRODUCTION

The effect of space-time curvature on atomic spectra is generally considered to be negligibly small, because of the smallness of the gravitational *mass radius* (the Schwarzschild radius) r_m, given by

$$r_m = \frac{2Gm_N}{c^2} , \tag{1}$$

where m_N denotes the mass of the nucleus. In the case of the proton, for example, $r_m = 2.48\times10^{-51}$ cm. Were it for the mass alone, space would be essentially flat outside a radius of order 10^{-50} cm from the center of the proton.

The curvature due to the charge eZ of the nucleus, gives a much larger *charge radius* r_e, with

$$r_e = \frac{\sqrt{G}eZ}{c^2} , \tag{2}$$

but which is still very small in comparison with nuclear dimensions. In the case of the proton, r_e is equal to 1.38×10^{-34} cm.

The situation is radically different when we come to consider the effect of the space-time curvature stemming from the nuclear spin I_N. The *spin radius,* denoted by a in the following, is of the order of the Compton wavelength of the nucleus.

The solution of Einstein's field equations for an uncharged mass point possessing an angular momentum was first derived by Kerr.[1] It was extended to the case of a rotating charged mass point by Newman *et al.*[2] In the Kerr-Newman metric there appears a constant a which has the dimension of a length. a is equal to $(1/c)$ times the angular momentum per unit mass of the source.

We make the assumption that the intrinsic spin angular momentum of the nucleus, of magnitude $I_N\hbar$, is to be identified with the angular momentum of a Kerr-Newman source. According to this assumption,

$$a = \frac{I_N\hbar}{cm_N} . \tag{3}$$

For the proton, $a = 1.05\times10^{-14}$ cm. Indeed, a is equal to $\frac{1}{2}$ the Compton wavelength of the proton, the factor $\frac{1}{2}$ stemming from the value of $\frac{1}{2}$ of the spin of the proton.

The above assumption is in line with the fact that a Kerr-Newman source has, asymptotically, a magnetic dipole moment μ_D given by

$$\mu_D = eZa . \tag{4}$$

It follows from Eqs. (3) and (4) that the gyromagnetic ratio is equal to 2, as in Dirac's theory of the electron.

II. SEPARATION OF DIRAC'S EQUATION IN KERR-NEWMAN GEOMETRY

We recall that in flat space the wave functions of the Dirac solution depend on two radial functions, $f(r)$ and $g(r)$, which, on writing

$$F(r) = rf(r), \quad G(r) = rg(r) , \tag{5}$$

obey the differential equations[3]

$$\frac{dF}{dr} - \frac{k_0}{r}F + \frac{1}{\hbar c}\left[E + \frac{Ze^2}{r} - E_0\right]G = 0 , \tag{6}$$

$$\frac{dG}{dr} + \frac{k_0}{r}G - \frac{1}{\hbar c}\left[E + \frac{Ze^2}{r} + E_0\right]F = 0 . \tag{7}$$

Here, $E_0 = m_e c^2$, and k_0 is given by

$$k_0 = -(j+\tfrac{1}{2}), \quad j = l + \tfrac{1}{2} , \tag{8}$$

$$k_0 = +(j+\tfrac{1}{2}), \quad j = l - \tfrac{1}{2} . \tag{9}$$

In spaces whose line element is given by

$$ds^2 = e^{\nu}c^2dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + \sin\theta^2 d\varphi^2) , \tag{10}$$

where λ and ν are functions of r only, Brill and Wheeler[4] derived Dirac's equation in the form

$$e^{-(1/2)\lambda}\frac{dF}{dr} - \frac{k}{r}F + \left[e^{-(1/2)\nu}\frac{1}{\hbar c}\left[E + \frac{Ze^2}{r}\right] - \mu_e\right]G = 0 , \tag{11}$$

$$e^{-(1/2)\lambda}\frac{dG}{dr}+\frac{k}{r}G+\left[-e^{-(1/2)\nu}\frac{1}{\hbar c}\left[E+\frac{Ze^2}{r}\right]-\mu_e\right]F=0 . \tag{12}$$

μ_e is the reciprocal of the Compton wavelength of the electron

$$\mu_e=\frac{m_e c}{\hbar} . \tag{13}$$

Note that the differential operators in Eqs. (6) and (7) and in (14) and (15) are real. Equations (11) and (12) apply to the Schwarzschild metric (uncharged mass point) and to the Reissner-Nordström metric (charged mass point), with $e^{\nu}=e^{-\lambda}=(\Delta/r^2)$, where the respective functions $\Delta(r)$ are given in Eqs. (29) and (30) below.

The metric tensor g_{ik} of the Kerr-Newman space is not diagonal, as in Eq. (10), but contains also off-diagonal elements. Dirac's equation in Kerr geometry was separated by Chandrasekhar.[5] Chandrasekhar denotes the two radial functions by $R_{-1/2}(r)$ and $R_{+1/2}(r)$, which he shows obey the simultaneous differential equations

$$\left[\frac{d}{dr}+\frac{iK}{\Delta}\right]R_{-1/2}=(\lambda+i\mu r)R_{+1/2} , \tag{14}$$

$$\Delta\left[\frac{d}{dr}-\frac{iK}{\Delta}+\frac{(r-M)}{\Delta}\right]R_{+1/2}=2(\lambda-i\mu r)R_{-1/2} , \tag{15}$$

where K and Δ are real functions of r to be given below, and $M=(Gm_N/c^2)$. Chandrasekhar eliminates $R_{+1/2}$ from Eqs. (14) and (15) and arrives at a second-order ordinary differential equation for $R_{-1/2}$, with complex coefficients.

We wish to retain the simultaneous first-order form of Eqs. (14) and (15), but to put them in a form which will manifestly resemble the pair of equations (6) and (7), and the pair (11) and (12). A clue to the transformation required is to be found in Chandrasekhar's observation that "$\sqrt{\Delta}R_{+1/2}$ and $\sqrt{2}R_{-1/2}$ are proportional to complex conjugate functions." Accordingly, we put, formally,

$$\begin{aligned} R_{-1/2}&=[F(r)+iG(r)] ,\\ R_{+1/2}&=\sqrt{(2/\Delta)}[F(r)-iG(r)] , \end{aligned} \tag{16}$$

without, however, assuming that the functions $F(r)$ and $G(r)$ are real, an aspect which will prove to be of importance when we come to solve the equations. Substituting in Eqs. (14) and (15), and using the relation

$$\frac{d\Delta}{dr}=2(r-M) , \tag{17}$$

we obtain

$$\left[\frac{d}{dr}+\frac{iK}{\Delta}\right](F+iG)=\sqrt{(2/\Delta)}(\lambda+i\mu r)(F-iG) , \tag{18}$$

$$\left[\frac{d}{dr}-\frac{iK}{\Delta}\right](F-iG)=\sqrt{(2/\Delta)}(\lambda-i\mu r)(F+iG) . \tag{19}$$

Adding and subtracting Eqs. (18) and (19) gives, respectively,

$$\frac{dF}{dr}=\frac{k}{\sqrt{\Delta}}F+\left[\frac{K}{\Delta}+\frac{\mu_e r}{\sqrt{\Delta}}\right]G , \tag{20}$$

$$\frac{dG}{dr}=-\frac{k}{\sqrt{\Delta}}G+\left[-\frac{K}{\Delta}+\frac{\mu_e r}{\sqrt{\Delta}}\right]F . \tag{21}$$

Here, we have put

$$\sqrt{2}\lambda=k,\ \ \sqrt{2}\mu=\mu_e . \tag{22}$$

The differential operators in Eqs. (20) and (21) are now real.

III. REDUCTION OF THE EQUATIONS FOR THE CASE OF A NUCLEAR SOURCE

Equations (14) and (15) apply to an uncharged rotating source in Kerr geometry. The extension to the case of a rotating charged source was made by Page,[6] following the method of separation of Chandrasekhar. We shall refer to this work as (A). In (A), relativistic units are used, in which $G=c=\hbar=1$. The relation between the relativistic quantities L, T, M, and Q (charge), and l, t, m, and eZ, expressed in cgs units, is

$$L=l,\ \ T=ct,\ \ M=\frac{Gm_N}{c^2},\ \ Q=\frac{\sqrt{G}eZ}{c^2} . \tag{23}$$

Page used a factor $\exp(im\varphi+i\sigma T)$, compared to the standard factor of $\exp[im\varphi-(iEt/\hbar)]$. Hence

$$\sigma=-\frac{E}{\hbar c}=-\frac{m_e c\epsilon}{\hbar},\ \ \epsilon=\frac{E}{m_e c^2} . \tag{24}$$

By (3) and (13), the terms $a\mu_e$ and $a\sigma$ appearing in (A) are given by

$$a\mu_e=\frac{I_N m_e}{m_N}\equiv\omega , \tag{25}$$

$$a\sigma=-\omega\epsilon . \tag{26}$$

The basic function $\Delta(r)$, which in the case of a black hole determines the event horizon, namely,

$$\Delta=(r^2-2Mr+a^2+Q^2) , \tag{27}$$

becomes, in view of (23), (1), and (2),

$$\Delta=r^2\left[1-\frac{r_m}{r}+\left[\frac{r_e}{r}\right]^2+\left[\frac{a}{r}\right]^2\right] . \tag{28}$$

For the Schwarzschild metric, and for the Reissner-Nordström metric, we have, respectively,

$$\Delta_S=r^2\left[1-\frac{r_m}{r}\right] , \tag{29}$$

$$\Delta_{\rm RN}=r^2\left[1-\frac{r_m}{r}+\left[\frac{r_e}{r}\right]^2\right] . \tag{30}$$

In contrast to the Schwarzschild metric, $\Delta_{\rm NR}$ is everywhere positive for the nuclear source, and so is Δ in Eq. (28) for the Kerr-Newman metric. The phenomenon of black holes does not appear, therefore, in the case of a nuclear source.

Since for the proton, for example,

$$\frac{r_m}{a}=2.36\times10^{-37},\ \ \frac{r_e}{a}=1.31\times10^{-20}\ , \tag{31}$$

we have, on putting in (28)

$$r=ax\ , \tag{32}$$

$$\Delta=a^2(1+x^2-2.36\times10^{-37}x+1.73\times10^{-40})\ . \tag{33}$$

Except for the immediate vicinity of the origin, i.e., when $x=O(10^{-37}\text{ cm})$, we have, to a high approximation,

$$\Delta\cong a^2(1+x^2)\ . \tag{34}$$

The function $K(r)$, given in (A) as

$$K(r)=(r^2+a^2)\sigma-eZr+am\ , \tag{35}$$

reduces to

$$K\equiv a\chi,\ \ \chi=[-\omega\epsilon(1+x^2)+m-\alpha Zx]\ . \tag{36}$$

Here α denotes the fine-structure constant, m is the magnetic quantum number, $-e$ the charge of the electron, and eZ the charge of the nucleus.

Substituting (34) and (36) in Eqs. (20) and (21), we arrive at the following form of Dirac's equation in Kerr-Newman geometry:

$$(1+x^2)\frac{dF}{dx}=k(1+x^2)^{1/2}F+[-\omega\epsilon(1+x^2)-\alpha Zx+m+\omega x(1+x^2)^{1/2}]G\ , \tag{37}$$

$$(1+x^2)\frac{dG}{dx}=-k(1+x^2)^{1/2}G+[\omega\epsilon(1+x^2)+\alpha Zx-m+\omega x(1+x^2)^{1/2}]F\ . \tag{38}$$

Following Chandrasekhar, Page showed that the wave functions of Dirac's equation in Kerr-Newman geometry can be represented, in Boyer-Lindquist[7] coordinates, in the form

$$\psi\simeq\exp(i\sigma T+im\varphi)[(r\pm ia\cos\theta)^{-1}]R(r)S(\theta)\ , \tag{39}$$

where

$$R(r)=R_{-1/2}(r)=F(r)+iG(r)\ . \tag{40}$$

The differential equation (real) obeyed by the angular function $S(\theta)$ given in (A) is

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{dS}{d\theta}\right]+\frac{\omega\sin\theta}{(k+\omega\cos\theta)}\frac{dS}{d\theta}+\left[\left[\frac{1}{2}+\omega\epsilon\cos\theta\right]^2-\left[\frac{m-(1/2)\cos\theta}{\sin\theta}\right]^2-\frac{3}{4}+2\omega\epsilon m-\omega^2\epsilon^2+\frac{\omega}{(k+\omega\cos\theta)}\left[\frac{1}{2}\cos\theta+\omega\epsilon\sin^2\theta-m\right]-\omega^2\cos^2\theta+k^2\right]S=0\ . \tag{41}$$

The second-order differential equation (complex) for $R(r)$ given in (A), and which is equivalent to the pair (37) and (38), is

$$(1+x^2)^{1/2}\frac{d}{dx}\left[(1+x^2)^{1/2}\frac{dR}{dx}\right]-\frac{i\omega(1+x^2)}{(k+i\omega x)}\frac{dR}{dx}+\left[\frac{\chi^2-ix\chi}{(1+x^2)}-2i\omega\epsilon x-i\alpha Z+\frac{\omega\chi}{(k+i\omega x)}-\omega^2x^2-k^2\right]R=0\ . \tag{42}$$

Equations (41) and (42) are a pair of simultaneous equations for the determination of the coupled eigenvalues k and ϵ, a phenomenon encountered previously by Teukolski[8] in a related astrophysical problem. k is no longer an integer, like k_0 given in Eqs. (8) and (9), but is a function of ϵ.

Another version of Dirac's equation in Kerr-Newman geometry, which facilitates comparison with Eq. (42), is as follows. Write

$$R=F+iG,\ \ T=F-iG\ , \tag{43}$$

whereby Eqs. (37) and (38) become

$$(1+x^2)\frac{dR}{dx}=i[\omega\epsilon(1+x^2)+\alpha Zx-m]R+(k\sqrt{1+x^2}+i\omega x\sqrt{1+x^2})T\ , \tag{44}$$

$$(1+x^2)\frac{dT}{dx}=-i[\omega\epsilon(1+x^2)+\alpha Zx-m]T+(k\sqrt{1+x^2}-i\omega x\sqrt{1+x^2})R\ . \tag{45}$$

Elimination of T from these equations leads to Eq. (42).

A significant feature of Dirac's equation in Kerr-Newman geometry, as given in Eqs. (37), (38), and (41), or in Eqs. (41) and (42), is that the magnetic quantum number m is built in these equations, in contrast to the degeneracy of the hyperfine-splitting levels in Dirac's equation in flat space. The hyperfine splitting need not, therefore, be treated as an external perturbation.[9]

Finally, we wish to point out a curious result. If we write in Eq. (42)

$$x=-i\cos\tau,\ \ R=\sqrt{\sin\tau}\,Q(\tau)\ , \tag{46}$$

so that the segment of the imaginary axis of x extending from $x=-i$ to $x=+i$ corresponds to the range $0<\tau<\pi$, we obtain an equation for Q which resembles the equation (41) for $S(\theta)$, except for the terms $i\alpha Z$:

$$\frac{1}{\sin\tau}\frac{d}{d\tau}\left[\sin\tau\frac{dQ}{d\tau}\right]+\frac{\omega\sin\tau}{(k+\omega\cos\tau)}\frac{dQ}{d\tau}$$
$$+\left[\left[\frac{1}{2}+\omega\epsilon\cos\tau\right]^2-\left[\frac{m+i\alpha Z\cos\tau-(1/2)\cos\tau}{\sin\tau}\right]^2-\frac{3}{4}+2\omega\epsilon(m+i\alpha Z\cos\tau)-\omega^2\epsilon^2\right.$$
$$\left.+\frac{\omega}{k+\omega\cos\tau}\left[\frac{1}{2}\cos\tau+\omega\epsilon\sin^2\tau-m-i\alpha Z\cos\tau\right]-\omega^2\cos^2\tau+k^2+i\alpha Z\right]Q=0\ . \quad (47)$$

For values of $r \gg a$ ($x \gg 1$), Eqs. (37) and (38) reduce to the flat-space equations (6) and (7). Not only is the asymptotic behavior of the two solutions similar, but also throughout the region where the amplitude of Dirac's wave function is appreciable, the deviation is small. However, a close inspection of the behavior of the solution of Eqs. (37) and (38) near the origin shows that we are faced here with a novel type of eigenvalue problem in quantum mechanics. A solution of Dirac's equation in Kerr-Newman geometry will be given in a forthcoming paper.[10]

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