Nonequilibrium potentials for local codimension-2 bifurcations of dissipative flows

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Nonequilibrium potentials are constructed which serve as macroscopic generalized thermodynamic potentials in dissipative systems far from thermodynamic equilibrium undergoing a local bifurcation of codimension 2 of a fixed point. The cases of two vanishing linear stability coefficients, of one vanishing and one purely imaginary pair, and of two purely imaginary pairs of linear stability coefficients are treated. As a result we establish the existence and form of a nonequilibrium potential for systems sufficiently close to codimension-1 or codimension-2 bifurcations for all cases where locally stable attractors exist in the phase diagram in parameter space. The attractors of the system, their basins of attraction in configuration space, and their bifurcations are determined by extremum properties of the nonequilibrium potentials.

I. INTRODUCTION

One of the goals of nonequilibrium statistical physics is to generalize the successful formalism of equilibrium thermodynamics, based on the use of thermodynamic potentials, in order to make it applicable to systems outside and even far from thermodynamic equilibrium. Some results on this problem have been obtained by us in a series of preceding papers.[1-4] The problem was there considered for a dynamical system described by the autonomous set of differential equations

$$\dot{q}^{\nu}=K^{\nu}(q) \quad (\nu=1,2,\dots,n) \tag{1.1}$$

for which a non-negative matrix of transport coefficients $Q^{\nu\mu}(q)$ is given on physical grounds, e.g., as the correlation matrix of a set of n noise sources with Gaussian statistics and white spectrum appearing additively in (1.1),

$$\dot{q}^{\nu}=K^{\nu}(q)+\sqrt{\eta}g_i^{\nu}(q)\xi_i \;, \tag{1.2}$$

$$\begin{aligned} g_i^{\nu}(q)g_i^{\mu}(q)&=Q^{\nu\mu}(q) \;, \\ \langle \xi_i(t)\xi_j(0)\rangle &=\delta_{ij}\delta(t) \;. \end{aligned} \tag{1.3}$$

For the sake of concreteness (but, in fact, without relevance to the following) let us assume that (1.2) is written in Ito calculus. Then, if a nonequilibrium potential $\phi(q)$ for this system exists, it must satisfy the Hamilton-Jacobi equation

$$\tfrac{1}{2}Q^{\nu\mu}(q)\frac{\partial\phi}{\partial q^{\nu}}\frac{\partial\phi}{\partial q^{\mu}}+K^{\nu}(q)\frac{\partial\phi}{\partial q^{\nu}}=0 \tag{1.4}$$

with the condition that ϕ is minimal in attractors, maximal in repellors, and stationary in saddles.[1-3]

As a consequence of (1.4) Eq. (1.1) can be put into the thermodynamical form

$$\dot{q}^{\nu}=-\tfrac{1}{2}Q^{\nu\mu}\frac{\partial\phi}{\partial q^{\mu}}+r^{\nu} \tag{1.5}$$

with

$$r^{\nu}\frac{\partial\phi}{\partial q^{\nu}}=0 \;, \tag{1.6}$$

i.e., K^{ν} is split into a force driving the system towards the minima of ϕ and a force which conserves ϕ.

As in equilibrium thermodynamics, ϕ contains the essential information on the "statics" of the deterministic system (1.1), i.e., information on the behavior for $t\to\infty$, in the form of the extremum conditions mentioned, and it contains the global statement about the deterministic dynamics that ϕ can only decrease or remain constant. Again, as in equilibrium thermodynamics, ϕ does not fix K^{ν} uniquely, even for a given matrix $Q^{\nu\mu}$, since r^{ν} in Eq. (1.5) is not yet uniquely fixed by Eq. (1.6).

The formal analogy of (1.4) to a Hamiltonian dynamical system in the Hamilton-Jacobi formalism was used in Refs. 1 and 2 to show that, in general, Eq. (1.4) corresponds to a nonintegrable Hamiltonian system, and, as a result, even though in the vicinity of a given point there exist infinitely many differentiable local solutions of Eq. (1.4), a global solution of Eq. (1.4) which is simultaneously single valued, smooth, and everywhere differentiable does not, in general, exist. Methods for discovering integrable special cases of (1.4) where a global, single-valued, everywhere differentiable solution exists were discussed in Ref. 5. Furthermore, an extremum principle based on the weak-noise limit $\eta\to 0$ (but $\eta>0$) of the stochastic process (1.2) was used in Ref. 3 which singles out a unique single-valued, continuous but not everywhere differentiable global solution of Eq. (1.4) even in the general case where Eq. (1.4) corresponds to a nonintegrable Hamiltonian system. The extremum principle is based on the relation

$$\phi(q)=-\lim_{\eta\to 0}[\eta\ln W(q,\eta)] \;, \tag{1.7}$$

where $W(q,\eta)$ is the stationary probability density of the stochastic process (1.2) for $\eta > 0$. In all this work $W(q,\eta)$ is assumed to be unique. In order to obtain the extremum principle $W(q,\eta)$ is represented as a functional integral by well-known methods, which is evaluated for $\eta \to 0$ in saddle-point approximation. Closely related mathematical work is described in Ref. 6.

The extremum principle shows that the potential ϕ is very closely associated with the attractors of the system. Explicitly, it reads

$$\phi(q) = \min_{(i)} \{\phi_i(q)\} \tag{1.8}$$

with

$$\phi_i(q) = \min \int_{q(-\infty)\in \mathscr{A}_i}^{q(0)=q} d\tau\, L_0[q(\tau),\dot{q}(\tau)] + C(\mathscr{A}_i) \tag{1.9}$$

and

$$L_0 = \tfrac{1}{2} Q_{\nu\mu}(\dot{q}^{\,\nu} - K^\nu)(\dot{q}^{\,\mu} - K^\mu) \quad [Q_{\nu\mu} = (Q^{-1})^{\nu\mu}] \, . \tag{1.10}$$

Here the attractors of the system are denoted by $\mathscr{A}_i$ $(i = 1,2,\ldots,m)$. The function ϕ_i is the potential evaluated by (1.7) for the single attractor $\mathscr{A}_i$ by taking the minimum in (1.9) over all trajectories starting in $\mathscr{A}_i$ at time $t \to -\infty$ and ending in q at $t=0$. The constants $C(\mathscr{A}_i)$ are evaluated by a method given in Ref. 4. The unique, continuous, single-valued but not everywhere differentiable nonequilibrium potential ϕ is then given by (1.8) by taking the minimum over all coexisting attractors. A detailed example for the application of the extremum principle to the case of several coexisting attractors was discussed in Ref. 4.

The existence of a simple solution of Eq. (1.4) for dynamical systems (1.1) in a sufficiently close neighborhood of a codimension-1 bifurcation even far from thermodynamic equilibrium has been known for some time. The first use of such a nonequilibrium potential for the case of a Hopf bifurcation, to the best of our knowledge, has been made by Haken[7] in the development of his nonlinear theory of a single-mode laser near threshold. The statistical significance of Haken's laser potential in the sense of Eq. (1.7) became clear from Risken's steady-state probability density of a laser amplitude near threshold.[8] The interpretation of the laser potential as a generalized thermodynamic potential was given later[9] and permitted the familiar reinterpretation of the laser threshold as a second-order phase transition far from thermodynamic equilibrium.[9,10] The same kind of description was later applied to bifurcations in hydrodynamics. A nonequilibrium potential for the Bénard instability in one spatial dimension was first given in Ref. 11 and was later generalized to include, e.g., non-Boussinesq effects,[12] rotational symmetry,[13] the effects of walls and defects,[14,15] etc. It is now quite generally appreciated that the amplitude equation proposed by Landau[16] and Stuart[17] for the description of a continuous instability can be formulated in the form (1.5), (1.6) with a potential ϕ of the same form appearing in Landau's theory of second-order phase transitions.

The existence of a potential ϕ for systems sufficiently close to a bifurcation of codimension 1 is a simple consequence of the center-manifold theorem (cf., e.g., Refs. 18 and 19) applied to such systems. The center-manifold theorem ensures that sufficiently close to the bifurcation point, and on a sufficiently long time scale, the dynamical system can be reduced to one with only a single dynamical variable. In this case, Eq. (1.4) is always integrable and allows one to find ϕ in a straightforward manner.

Recently, an increasing number of mathematical[18–20] and physical[21–28] papers has been devoted to problems involving bifurcations of codimension 2 of a fixed point. An extensive review of such problems from the point of view of applied mathematics has been given by Guckenheimer and Holmes.[19] Recent experimental work was presented in Ref. 27.

A codimension-2 bifurcation appears when two codimension-1 bifurcations coalesce and occur simultaneously at the same point in parameter space. As a result of the coalescence, the two bifurcating degrees of freedom interact strongly near the bifurcation point of codimension 2, which leads to the appearance of further codimension-1 bifurcations in the neighborhood of the codimension-2 point. Mathematically, codimension-2 bifurcations may be of three different main types, depending on the nature of the coalescing codimension-1 bifurcations.

The first main type appears if the flow (1.1), linearized around a fixed point [where $K^\nu(q)=0$] and diagonalized, has two vanishing simple eigenvalues at the bifurcation point. This case is physically relevant, e.g., for thermohaline convection in a fluid layer[21] in which a horizontal fluid layer is externally stressed by a temperature gradient and a salt-concentration gradient. Another example is provided by thermal convection in a binary fluid subject to a temperature gradient.[24]

The second and third main types of codimension-2 bifurcations occur, respectively, if the flow (1.1), linearized and diagonalized near a fixed point, has a purely imaginary complex-conjugate pair and a simple vanishing eigenvalue or a pair of purely imaginary eigenvalues. An important example of the latter case occurs in laser physics, when two laser modes with different frequencies pass the lasing threshold simultaneously (cf., e.g., Ref. 29). An example of the former case occurs in a simple chemical reaction-diffusion system.[30]

Bifurcations in fluctuating systems have been studied by Knobloch and Wiesenfeld.[31] They considered the derivation of stochastic normal forms near bifurcations of codimension 1 and 2. However, a solution of the resulting Fokker-Planck equation on the center-manifold for codimension-2 bifurcations were not given there, which is among the goals of the present paper.

Nonequilibrium potentials of the type discussed in Eqs. (1.1)–(1.10) have so far not been studied for the case of codimension-2 bifurcations. It is the purpose of the present paper to present such a study. The center-manifold analysis near the bifurcation point of codimension 2 leads, essentially, to a reduction of the original dynamical system (1.1) to one with two degrees of freedom. The dynamics of these two degrees of freedom in a vicinity of the bifurcating fixed point is governed by two amplitude equations, which can be brought into normal forms[19] characteristic of the particular type of bifurcation under study. Assuming a simple form of the transport matrix the Hamilton-Jacobi equation (1.4) associated with the two-dimensional flow on the center manifold can be written down. Its solution is then the nontrivial task.

Quite remarkably the Hamilton-Jacobi equations turn out to have smooth solutions in the vicinity of the bifurcating fixed point in the form of a formal power series in the small parameters which measure the distance from the codimension-2 bifurcation point. If the two-dimensional flow on the center manifold retains a stable attractor (not necessarily a fixed point) in the vicinity of the bifurcating fixed point, the solution of the Hamilton-Jacobi equation coincides with the function associated with that attractor by Eq. (1.9). It can therefore be identified locally [eventually after taking the minimum over several coexisting attractors according to Eq. (1.8)] with the nonequilibrium potential.

As a result we establish the existence and form of a local nonequilibrium potential for systems sufficiently close to codimension-2 bifurcations of a fixed point for all cases where locally stable attractors exist in the neighborhood of the bifurcating fixed point. In those cases where the latter condition is not satisfied, i.e., where an attractor does not exist locally, the nonequilibrium potential, in principle, is not determined by the form of the flow near the bifurcating fixed point as can be seen from Eq. (1.9).

A restriction of our analysis is the fact that spatial derivatives in the amplitude equations are not permitted. Nothing is known about the existence of a nonequilibrium potential if spatial derivatives are permitted in codimension-2 bifurcations, as, e.g., in Ref. 24. For codimension-1 bifurcations, spatial derivatives can be taken into account easily by generalizing ϕ to become a functional of the spatially varying amplitude.[9,11]

The technical part of the present paper is organized as follows. In Sec. II and Appendix A we discuss the nonequilibrium potential for the case of codimension-1 bifurcation of a fixed point. In Sec. III–V and Appendixes A–C the nonequilibrium potentials for the three types of codimension-2 bifurcations are determined.

Except for the case of two coalescing Hopf bifurcations we also determine the leading correction in η to the limit (1.7), i.e., the prefactor $Z(q)$ in $W(q,\eta) \sim Z(q)\exp[-\phi(q)/\eta]$. This is done in Appendix D.

Section VI contains our final conclusions. We show explicitly that results on codimension-2 bifurcations obtained earlier in the literature can be obtained by studying the extrema of the nonequilibrium potentials which we construct. In some cases we are able to correct or extend earlier results. Last but not least we obtain new results on steady-state probability densities for weak noise, which, in future work, may be used to calculate mean exit times for the attractors appearing in codimension-2 bifurcations.

II. LOCAL CODIMENSION-ONE BIFURCATIONS OF FIXED POINTS

In this section we describe briefly how codimension-1 bifurcations of an equilibrium state can be analyzed with the help of a non-equilibrium potential. An equilibrium state $q=q_0$ of (1.1) satisfies $K^\nu(q_0)=0$. A local codimension-1 bifurcation of this equilibrium point occurs if the matrix of coefficients of the flow (1.1) linearized near $q=q_0$ has a single vanishing eigenvalue or a pair of complex-conjugate purely imaginary eigenvalues. In the latter case a Hopf bifurcation occurs. In the former case one may have a saddle-node bifurcation (collision of a stable and an unstable equilibrium state under mutual annihilation), a transcritical bifurcation (collision of a stable and an unstable equilibrium state under exchange of stability), or a pitchfork bifurcation [bifurcation of a pair of two stable (supercritical) or unstable (subcritical) equilibria from a stable equilibrium state which thereby becomes unstable].

In the case of a single vanishing eigenvalue the center-manifold theorem permits a reduction of the dimensionality of the state space to one; similarly, this occurs in the case of a Hopf bifurcation, where the remaining dimensionality is two. Furthermore, close to the bifurcation point, and in a small neighborhood of the bifurcating equilibrium state, the theory of normal forms permits a reduction of Eq. (1.1) to the forms (cf. Ref. 19)

$$\dot{x}=\mu-x^2 \quad \text{(saddle-node bifurcation)}\,, \tag{2.1}$$

$$\dot{x}=\mu x-x^2 \quad \text{(transcritical bifurcation)}\,, \tag{2.2}$$

$$\dot{x}=\mu x-x^3 \quad \text{(pitchfork bifurcation)}\,, \tag{2.3}$$

$$\dot{x}=-y+x[\mu-(x^2+y^2)]\,, \tag{2.4a}$$

$$\dot{y}=x+y[\mu-(x^2+y^2)] \quad \text{(Hopf bifurcation)}\,. \tag{2.4b}$$

Here $\mu=0$ is the bifurcation point and the size of μ is a measure of the distance from the bifurcation point in (one-dimensional) parameter space.

In order to define nonequilibrium potentials for the systems (2.1)–(2.4) a transport matrix must still be given. In the cases (2.1)–(2.3) the matrix consists of a single coefficient $Q(x,\mu)$. In view of the local character of the flows (2.1)–(2.3) near $x=0$, $\mu=0$ it is natural to give also a local representation of $Q(x,\mu)$ and to replace it by its value at $x=0=\mu$, $Q_0=Q(0,0)$, assuming that this value is nonvanishing. Equation (1.4) is then easily solved with the result

$$\phi=\frac{1}{Q_0}(-2\mu x+\tfrac{2}{3}x^3) \quad \text{(saddle-node)}\,, \tag{2.5}$$

$$\phi=\frac{1}{Q_0}(-\mu x^2+\tfrac{2}{3}x^3) \quad \text{(transcritical)}\,, \tag{2.6}$$

$$\phi=\frac{1}{Q_0}(-\mu x^2+\tfrac{1}{2}x^4) \quad \text{(pitchfork)}\,. \tag{2.7}$$

The bifurcations of the flows (2.1)–(2.3) are now associated with the extremum properties of these local potentials in a straightforward way.

In the case of the Hopf bifurcation the existence of a nonequilibrium potential is slightly less trivial, since the flow (2.4) is still two dimensional. The transport matrix is now a symmetric (2×2) matrix. Again we may use a local approximation and replace the coefficients by their values at $x=y=0=\mu$. We may also use the rotational invariance of (2.4) in the (x,y) plane and rotate the coordinates x,y in order to diagonalize the transport matrix

$$Q^{\nu\mu}=\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} . \tag{2.8}$$

In this general case a nonequilibrium potential of Eq. (1.4) is constructed in Appendix A along the lines of Refs. 1–4. It is shown there that sufficiently close to the bifurcation point the difference between Q_1 and Q_2 may be neglected. Then the transport matrix (2.8) preserves the rotational symmetry of (2.4)

$$Q_1=Q_2=Q_0 , \tag{2.9}$$

and a very simple nonequilibrium potential

$$\phi=\frac{1}{Q_0}[-\mu(x^2+y^2)+\tfrac{1}{2}(x^2+y^2)^2] \tag{2.10}$$

is obtained as a solution of (1.4). The bifurcation of the flow (2.4) is again reflected by changes in the extrema of (2.10). In particular, the limit cycle

$$x^2+y^2=\mu \tag{2.11}$$

appearing for $\mu>0$ corresponds to a continuously degenerate set of minima of the potential (2.10).

Let us finally mention that (2.10) has been widely used in quantum optics to describe the Hopf bifurcations at the laser threshold[7,8] and at the threshold of optical parametric oscillators.[32] Similarly, Eq. (2.7) and extensions of it have found widespread use for the Bénard instability[33] and the Taylor instability.[34]

III. SIMULTANEOUS VANISHING OF TWO EIGENVALUES

A. General form of the nonequilibrium potential

For codimension-2 bifurcations characterized by a simultaneous vanishing of two eigenvalues, the center-manifold theorem ensures that in the vicinity of the bifurcation point the number of relevant variables can also be reduced to two. Therefore, close to the bifurcation point, such systems behave like weakly damped systems with a two-dimensional phase space.[19] It is useful to consider the nonequilibrium potential of such systems from a general point of view before applying the results. The dynamical systems to be studied satisfy the equations of motion

$$\begin{aligned} \dot{x}&=v , \\ \dot{v}&=-\frac{\partial H}{\partial x}+\mu v g(x) , \end{aligned} \tag{3.1}$$

where H is a Hamiltonian of the form

$$H=\frac{v^2}{2}+V(x) , \tag{3.2}$$

$\mu g(x)$ represents the divergence of the flow in phase space, and we assume a transport matrix to be given in the diagonal form

$$Q^{\nu\mu}=\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} . \tag{3.3}$$

The Hamilton Jacobi equation for ϕ reads

$$\frac{Q_1}{2}\left[\frac{\partial\phi}{\partial x}\right]^2+\frac{Q_2}{2}\left[\frac{\partial\phi}{\partial v}\right]^2+v\frac{\partial\phi}{\partial x} +[-V'(x)+\mu v g(x)]\frac{\partial\phi}{\partial v}=0 . \tag{3.4}$$

Since the strength of the dissipation μ is small, it is useful to introduce

$$E=\frac{v^2}{2}+V(x) \tag{3.5}$$

and to eliminate v by

$$v=v(x,E)=\sqrt{2[E-V(x)]} . \tag{3.6}$$

Defining

$$\begin{aligned} \phi^+(x,E)&=\phi(x,v(x,E)),\quad v>0 \\ \phi^-(x,E)&=\phi(x,-v(x,E)),\quad v<0 \end{aligned} \tag{3.7}$$

we obtain

$$\frac{Q_1}{2}\left[\frac{\partial\phi^\pm}{\partial x}\right]^2+\tfrac{1}{2}[Q_1 V'^2(x)+Q_2 v^2(x,E)]\left[\frac{\partial\phi^\pm}{\partial E}\right]^2+Q_1 V'(x)\frac{\partial\phi^\pm}{\partial x}\frac{\partial\phi^\pm}{\partial E}+\mu v^2(x,E)g(x)\frac{\partial\phi^\pm}{\partial E}\pm v(x,E)\frac{\partial\phi^\pm}{\partial x}=0 . \tag{3.8}$$

As in a conservative system the potential ϕ should be constant, we look for a solution in the form

$$\phi^\pm=\sum_{n=1}^{\infty}\mu^n\phi_n^\pm(x,E) \tag{3.9}$$

and obtain in lowest order in μ,

$$\frac{\partial\phi_1^\pm}{\partial x}=0 , \tag{3.10}$$

which is solved by

$$\phi_1^\pm(x,E)=F_1^\pm(E) , \tag{3.11}$$

where $F_1^\pm(E)$ is a still arbitrary pair of functions of E.

For $v=0$ the potential $\phi(x,v)$ must be continuous for all x. Hence

$$\phi^+(x,V(x))=\phi^-(x,V(x)) \tag{3.12}$$

must hold for all x, from which we conclude that

$$F_1^+(E)=F_1^-(E)\equiv F_1(E) . \tag{3.13}$$

In the next order in μ we obtain the equation

$$\frac{1}{2}[Q_1 V'^2(x)+Q_2 v^2(x,E)]F_1'^2(E) + v^2(x,E)g(x)F_1'(E) \pm v(x,E)\frac{\partial \phi_2^{\pm}}{\partial x}=0 . \quad (3.14)$$

In order to find an equation for $F_1(E)$ we integrate (3.14) over x for fixed E, assuming that

$$E \geq V(x) \quad (3.15)$$

in a finite interval $x_1(E) \leq x \leq x_2(E)$ where $v(x_{1,2},E)=0$ (Fig. 1). If such a finite interval does not exist, the system (3.1) for $\mu=0$ does not have closed trajectories. Such a system is globally unstable and a nonequilibrium potential is not determined by the local flow. For stable systems the integral is taken from x_1 to x_2 for $v>0$ and from x_2 back to x_1 for $v<0$. Then we have

$$\int_{x_1}^{x_2} dx \frac{\partial \phi^+}{\partial x} + \int_{x_2}^{x_1} dx \frac{\partial \phi^-}{\partial x}=0 \quad (3.16)$$

since the potential must be single valued. Using Eq. (3.14) we find

$$\int_{x_1}^{x_2} dx \left\{ v(x,E)g(x)F_1'(E) + \frac{1}{2v(x,E)}[Q_1 V'^2(x)+Q_2 v^2(x,E)]F_1'^2(E) \right\} =0 . \quad (3.17)$$

Assuming $F_1'(E) \neq 0$ this leads us to

$$F_1'(E)=-\frac{2\bar{v}_g(E)}{Q_2\bar{v}(E)+Q_1\bar{w}(E)} \quad (3.18)$$

with

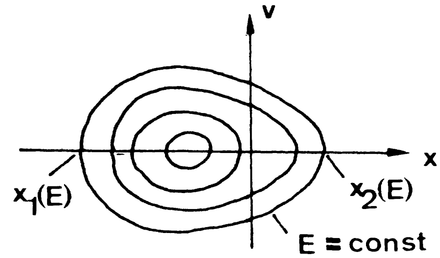

FIG. 1. Phase portrait of the system (3.1) at $\mu=0$. Integral (3.16) should be taken along an $E=$const contour of the (x,v) space.

$$\bar{v}(E)=\frac{1}{x_2-x_1}\int_{x_1}^{x_2} v(x,E)dx , \quad (3.19)$$

$$\bar{w}(E)=\frac{1}{x_2-x_1}\int_{x_1}^{x_2} V''(x)v(x,E)dx \geq 0 , \quad (3.20)$$

$$\bar{v}_g(E)=\frac{1}{x_2-x_1}\int_{x_1}^{x_2} g(x)v(x,E)dx . \quad (3.21)$$

In order to obtain (3.20) we have performed an integration by parts. Hence

$$F_1(E)=-2\int_{E_0}^{E} \frac{\bar{v}_g(\tilde{E})}{Q_2\bar{v}(\tilde{E})+Q_1\bar{w}(\tilde{E})}d\tilde{E}+F_1(E_0) . \quad (3.22)$$

Here E_0 is the smallest possible value of E, where $x_1(E_0)=x_2(E_0)$.

The equation for $\phi_2^{\pm}$ can now be solved and we find

$$\phi_2^{\pm}(x,E)=\pm\frac{2\bar{v}_g(E)}{Q_2\bar{v}(E)+Q_1\bar{w}(E)}\left[\int_{x_1}^{x} g(\tilde{x})v(\tilde{x},E)d\tilde{x}-\frac{\bar{v}_g(E)}{Q_2\bar{v}(E)+Q_1\bar{w}(E)}\int_{x_1}^{x}\frac{Q_1 V'^2(\tilde{x})+Q_2 v^2(\tilde{x},E)}{v(\tilde{x},E)}d\tilde{x}\right]+F_2^{\pm}(E) . \quad (3.23)$$

The continuity of $\phi(x,v)$ at $v=0$ implies again

$$F_2^{\pm}(E)=F_2(E) . \quad (3.24)$$

In order to fix the arbitrary function $F_2(E)$ the solvability conditions of the equation satisfied by $\phi_2^{\pm}$ must be studied, following the steps given after Eq. (3.14). It turns out that the integral terms of (3.23) drop out from the solvability condition due to their change of sign for $v \rightarrow -v$. The condition obtained for F_2 then simply is

$$F_2'(E)=0 . \quad (3.25)$$

A repetition of these steps carries the expansion to arbitrary order. We recall now that we are here actually interested only in the system in the vicinity of the bifurcation point, i.e., for μ very small. In fact, the actual equations of the form (3.1) which we shall use in Secs. III B and III C are given by normal forms which are only valid to the lowest nontrivial order of μ. Hence, consistency requires that we determine ϕ in the same limit, i.e., again in the lowest nontrivial order of μ. To this order our result is simply

$$\phi(x,v)=\mu\phi_1(E) = -2\mu\int_{E_0}^{E}\frac{\bar{v}_g(\tilde{E})}{Q_2\bar{v}(\tilde{E})+Q_1\bar{w}(\tilde{E})}d\tilde{E}+\mu\phi_1(E_0) \quad (3.26)$$

with Eqs. (3.5) and (3.19)–(3.21). We expect that the extremum properties of this potential completely determine the bifurcations of the system for sufficiently small μ in all cases where attractors exist in the local region of state space which is studied. The equipotential lines of (3.26) are given by the contours of constant energy $E=$const. [Deformations from constant-energy contours proportional to μ appear by the use of the correction (3.23)]. Extrema of the potential satisfy either

$$\phi'(E)=0 \tag{3.27}$$

or

$$V'(x)=0, \quad v=0 . \tag{3.28}$$

The condition (3.28) determines fixed points at the extrema of the "mechanical" potential $V(x)$ appearing in the Hamiltonian (3.2). The condition (3.27) determines limit cycles. It is equivalent to the condition

$$\overline{v}_g(E)=0 , \tag{3.29}$$

which expresses the physical condition that the integral over local energy losses and gains along a limit cycle must vanish.

The stability of the fixed points and limit cycles is determined by the second derivatives of ϕ, which will be studied below for various special cases. In the case of a Brownian motion $Q_1=0$ the potential (3.26) has a more general meaning. Arguments along the lines of Ref. 35 show that the stationary probability distribution of the process (1.2), (3.1)–(3.3) with a *weak damping* $\mu \ll 1$ is then $\exp[-\phi(x,v)/\eta]$ not only for $\eta \to 0$ but at *arbitrary noise intensity.*

B. Case without inversion symmetry

The normal form of this case can be taken as[19]

$$\dot{x}=v , \qquad \dot{v}=\mu_1+\mu_2 v+x^2+bxv . \tag{3.30}$$

Scaling the variables and the bifurcation parameters by[19]

$$\mu_1=\epsilon^4\nu_1, \quad \mu_2=\epsilon^2\nu_2, \quad x=\epsilon^2 x, \quad v=\epsilon^3\overline{v}, \quad t=\overline{t}/\epsilon \tag{3.31}$$

and omitting the bars, henceforth, we obtain

$$\dot{x}=v , \qquad \dot{v}=\nu_1+x^2+\epsilon(\nu_2 v+bxv) . \tag{3.32}$$

The transport matrix we use is of the type of (3.3) in the rescaled variables. The bifurcation point is characterized by $\nu_1=\nu_2=0$. We shall be interested in the two cases $b=\pm 1$. Equation (3.32) is of the form (3.1) with

$$V(x)=-\nu_1 x-\tfrac{1}{3}x^3 , \tag{3.33}$$

$$g(x)=\nu_2+bx , \tag{3.34}$$

i.e., the results of the preceding section are immediately applicable with μ of Sec. III A being identified as ϵ in (3.32). From Eq. (3.33) it follows that for $\nu_1>0$, $V(x)$ is monotonously decreasing, i.e., the system is unstable and leaves the neighborhood of $x=v=0$. We therefore exclude this case from further consideration.

For $\nu_1<0$ the potential $V(x)$ has a minimum at

$$x_{\min}=-\sqrt{|\nu_1|}=-x_0 , \qquad V_{\min}=-\tfrac{2}{3}|\nu_1|^{3/2}=-V_0 , \tag{3.35}$$

and a maximum V_0 at x_0. For

$$-V_0 \le E \le V_0 \tag{3.36}$$

there exists a finite interval $x_1(E) \le x \le x_2(E)$ with $E \ge V(x)$, where $x_{1,2}$ are the two smallest roots of

$$x^3-3|\nu_1|x+E=0 . \tag{3.37}$$

In this interval the expression on the right-hand side of Eq. (3.26) is defined. Outside this interval (i.e., for $|E|>V_0$) the system is unstable, and we exclude this region from further consideration. Let us first look for limit cycles, which must satisfy $\phi'(E_c)=0$, from which

$$\overline{v}_g(E_c)=0 \tag{3.38}$$

follows. Using (3.21) with (3.6) and (3.34) we find the condition

$$\nu_2=-b\frac{\overline{v}_1(E_c)}{\overline{v}(E_c)} \tag{3.39}$$

where we introduced the abbreviation

$$\overline{v}_n(E)=\frac{1}{x_2-x_1}\int_{x_1}^{x_2} dx\, x^n v(x,E) . \tag{3.40}$$

For $E_c=-V_0$ the interval $x_1 \le x \le x_2$ shrinks to a point $x=x_1=x_2=-x_0$ and the limit cycle disappears. This happens for

$$\nu_2=-b\frac{\overline{v}_1(-V_0)}{\overline{v}(-V_0)}=b\sqrt{|\nu_1|} . \tag{3.41}$$

On the other end of the allowed interval for E_c, at $E_c=V_0$, the limit cycle is as large as possible and ceases to exist upon further increase of E_c. This is realized for

$$\nu_2=-b\frac{\overline{v}_1(V_0)}{\overline{v}(V_0)}=\tfrac{5}{7}b\sqrt{|\nu_1|} . \tag{3.42}$$

The limit cycle therefore exists in the parameter interval

$$\nu_1<0, \quad \tfrac{5}{7}\sqrt{|\nu_1|}<\frac{\nu_2}{b}<\sqrt{|\nu_1|} . \tag{3.43}$$

The limit cycle is stable if

$$\phi''(E_c)>0 . \tag{3.44}$$

This condition is equivalent to $\overline{v}_g'(E_c)<0$ or, via Eqs. (3.21), (3.26), (3.34), and (3.41), to

$$b\left[\frac{\overline{v}_1(E_c)}{\overline{v}(E_c)}\right]'<0 . \tag{3.45}$$

It is easy to see geometrically, and it is also possible to show analytically, that $[\overline{v}_1(E)/\overline{v}(E)]'>0$ for $-V_0<E<V_0$. Therefore, for $b>0$ the stability condition (3.45) is not satisfied, i.e., the limit cycle is not stable. For $b<0$, on the other hand, the condition (3.45) is satisfied, and the limit cycle is stable in the interval $-|b|\sqrt{|\nu_1|}<\nu_2<-5|b|\sqrt{|\nu_1|}/7$.

Fixed points of the system exist, according to Eq. (3.28), at $x=\pm x_0$, $v=0$ where $E=\pm V_0$. The matrix of second derivatives of ϕ with respect to x,v at the fixed points is given by

$$\begin{bmatrix} \phi'(\pm V_0)V''(\pm x_0) & 0 \\ 0 & \phi'(\pm V_0) \end{bmatrix} . \tag{3.46}$$

As a result, the fixed point $x = x_0$, $v = 0$, $E = V_0$ is a saddle because of $V''(x_0) < 0$ regardless of the sign of $\phi'(V_0)$. The other fixed point $x = -x_0$, $v = 0$, $E = -V_0$ has $V''(-x_0) > 0$ and is therefore stable if and only if $\phi'(-V_0) > 0$. This latter condition is equivalent to

$$\bar{v}_g(-V_0) < 0 \tag{3.47}$$

which is satisfied if

$$\nu_2 < b\sqrt{|\nu_1|} \; . \tag{3.48}$$

This completes the discussion of the extrema of the nonequilibrium potential and determines the local bifurcations in a neighborhood of the codimension-2 point $\nu_1 = \nu_2 = 0$.

We summarize the results as follows. Only $\nu_1 < 0$ is of interest to us since otherwise no attractors exist close to the origin in state space. For the case $b = 1$ there does not exist a stable attractor even for $\nu_1 < 0$, as long as $\nu_2 > \sqrt{|\nu_1|}$. Lowering ν_2 for fixed $\nu_1 < 0$, a stable fixed point $x = -\sqrt{|\nu_1|}$, $v = 0$ and an unstable limit cycle $E = E_c$ appear at $\nu_2 = \sqrt{|\nu_1|}$. There is now a nonequilibrium potential in state space in the neighborhood of the attractor defined for $-2|\nu_1|^{3/2}/3 \le E \le E_c \le 2|\nu_1|^{3/2}/3$ which attains a maximum at the limit cycle $E = E_c$. The appearance of the unstable limit cycle together with the stable fixed point indicates that there is a subcritical Hopf bifurcation for $\nu_2 \to \sqrt{|\nu_1|} - 0$. The shape of the potential for $E > E_c$ is not determined by the attractor at $E = -V_0$. Rather, the decrease for $E > E_c$ of the function defined by Eq. (3.26) for $E_c < E < V_0$ indicates that there must exist another attractor of the system far from the origin in state space if the system is to be globally stable. This additional attractor then determines the form of the potential in the domain $E > E_c$ via Eq. (1.9).

Decreasing ν_2 further to $\nu_2 = \frac{5}{7}\sqrt{|\nu_1|}$ the value E_c increases and reaches the boundary at $E_c = V_0$ where the unstable limit cycle forms a homoclinic orbit of the saddle at $x = x_0$, $E = V_0$. For $\nu_2 < \frac{5}{7}\sqrt{|\nu_1|}$ the maximum of the potential at a limit cycle has disappeared and the attractor at $E = -V_0$ now determines the potential in the whole domain $-V_0 < E < V_0$.

For the case $b = -1$ there is no attractor even in the case $\nu_1 < 0$, as long as $\nu_2 > -\frac{5}{7}\sqrt{|\nu_1|}$. Decreasing ν_2 for fixed $\nu_1 < 0$, an attractor appears first for $\nu_2 = -\frac{5}{7}\sqrt{|\nu_1|}$ as a homoclinic orbit $E = V_0$ of a saddle at $E = V_0$. For $-5\sqrt{|\nu_1|}/7 > \nu_2 > -\sqrt{|\nu_1|}$ a stable limit cycle exists for $E = E_c$, and a nonequilibrium potential is defined for $-V_0 < E < V_0$ which is minimal at $E = E_c$, and, as a function of E, has boundary maxima at $E = \pm V_0$. At $\nu_2 = -\sqrt{|\nu_1|}$ the limit cycle at $E = E_c$ and the boundary maximum at $E = -V_0$ collide and disappear together by an inverted supercritical Hopf bifurcation. For $\nu_2 < -\sqrt{|\nu_1|}$ the potential has a boundary minimum at $E = -V_0$ where the system has a stable fixed point and a boundary maximum at $E = V_0$ where the system has a saddle. The discussion of the extrema of the nonequilibrium potential has thus reproduced all those features of the bifurcation (cf. Ref. 19), which are associated with attractors and are, therefore, in principle, observable in a physical system.

C. Case with cubic symmetry

In the presence of the point symmetry

$$(x,v) \to (-x,-v) \tag{3.49}$$

the normal form (3.30) cannot be applied and is replaced by[19]

$$\begin{aligned} \dot{x} &= v \; , \\ \dot{v} &= \mu_1 x + \mu_2 v - a x^3 + b x^2 v \; . \end{aligned} \tag{3.50}$$

Scaling the bifurcation parameters and variables by[19]

$$\mu_1 = \epsilon^2 \nu_1, \quad \mu_2 = \epsilon^2 \nu_2, \quad x = \epsilon \bar{x}, \quad v = \epsilon^2 \bar{v}, \quad t = \bar{t}/\epsilon \tag{3.51}$$

and omitting the bars, henceforth we obtain

$$\begin{aligned} \dot{x} &= v \; , \\ \dot{v} &= \nu_1 x + \epsilon \nu_2 v - a x^3 + \epsilon b x^2 v \; . \end{aligned} \tag{3.52}$$

This is again of the form (3.1) with

$$\begin{aligned} V(x) &= -\tfrac{1}{2}\nu_1 x^2 + \frac{a}{4} x^4 \; , \\ g(x) &= \nu_2 + b x^2 \; , \end{aligned} \tag{3.53}$$

and the results of Sec. III apply. We shall be interested in the cases $a = \pm 1$ and $b = \pm 1$.

1. The case $a = 1$

Let us first discuss the case $a = 1$. Then, for $\nu_1 < 0$ the potential (3.26) is defined for $0 \le E < \infty$ in the interval $x_1(E) \le x \le x_2(E)$. For $\nu_1 > 0$, the potential (3.26) is defined for $-V_0 \le E < \infty$ in the intervals

$$\begin{aligned} &x_1(E) \le x \le x_2(E), \quad x_3(E) \le x \le x_4(E) \quad (-V_0 \le E \le 0) \\ &x_1(E) \le x \le x_2(E) \quad (0 < E < \infty) \; . \end{aligned} \tag{3.54}$$

Here, the x_i are the real roots of

$$E + \tfrac{1}{2}\nu_1 x^2 - \tfrac{1}{4} x^4 = 0 \; , \tag{3.55}$$

ordered according to their size, and

$$V_0 = \tfrac{1}{4}\nu_1^2 \; . \tag{3.56}$$

Limit cycles satisfying $\phi'(E_c) = 0$ now exist if

$$\frac{\nu_2}{b} = -\frac{\bar{v}_2(E_c)}{\bar{v}(E_c)} \; . \tag{3.57}$$

As $\bar{v}_2(E)/\bar{v}(E) > 0$, solutions exist only if

$$\frac{\nu_2}{b} < 0 \; . \tag{3.58}$$

Stability of limit cycles requires $\phi''(E_c) > 0$, which is equivalent to

$$b\left[\frac{\bar{v}_2(E_c)}{\bar{v}(E_c)}\right]' < 0 \; . \tag{3.59}$$

If $\nu_1 < 0$ it is obvious, geometrically, that $\bar{v}_2(E)/\bar{v}(E)$ monotonously increases with E starting from

$\bar{v}_2(0)/\bar{v}(0)=0$. Hence, Eq. (3.57) has a single solution in this case, which is stable only if $b=-1$; for $b=+1$ it is unstable. The limit cycle for $\nu_1<0$ collapses to a single point if $E_c=0$, which happens for $\nu_2=0$. Hence $\nu_2=0$, $\nu_1<0$ is a line of Hopf bifurcations which are subcritical for $b=1$ and supercritical for $b=-1$.

If $\nu_1>0$, on the other hand, $\bar{v}_2(E)/\bar{v}(E)$ first decreases from its value $\bar{v}_2(-V_0)/\bar{v}(-V_0)=\nu_1$ for increasing $E>-V_0$ and increases again for sufficiently large positive E.[18,19] Hence, there is a minimum

$$\left.\frac{\bar{v}_2(E)}{\bar{v}(E)}\right|_{\min}=C>0 \tag{3.60}$$

such that for

$$-\frac{\nu_2}{b}<C \tag{3.61}$$

no solution of (3.57) exists; for

$$\nu_1>-\frac{\nu_2}{b}>C \tag{3.62}$$

there are two solutions at $E=E_{c1}$, $E=E_{c2}$ $(E_{c1}<E_{c2})$ and for

$$-\frac{\nu_2}{b}>\nu_1 \tag{3.63}$$

there is one solution, at $E=E_{c2}$. For $b=+1$ the solution $E=E_{c1}$ is stable and $E=E_{c2}$ is unstable. For $b=-1$ the situation is reversed. The constant C in Eq. (3.60) can only be evaluated by actually doing the necessary elliptic integrals,[19]

$$C=\bar{C}\nu_1 \ . \tag{3.64}$$

The limit cycle at $E=E_{c1}$ splits into two symmetrical cycles if $E_{c1}\le 0$, which happens first for

$$\frac{\nu_2}{b}=-\frac{\bar{v}_2(0)}{\bar{v}(0)}=-\tfrac{4}{5}\nu_1 \ . \tag{3.65}$$

The two symmetrical cycles collapse to the two points $x=\pm\sqrt{|\nu_1|}$, $v=0$ when $E_{c1}=-V_0$, which is realized for

$$\frac{\nu_2}{b}=-\frac{\bar{v}_2(-V_0)}{\bar{v}(-V_0)}=-\nu_1 \ . \tag{3.66}$$

It remains to investigate the fixed points of the potential. For $\nu_1<0$ there is only one fixed point at $E=0$, $x=v=0$, whose stability requires $\phi'(0)>0$. This is satisfied for $\nu_2<0$, for $\nu_2>0$ the fixed point is unstable. For $\nu_1>0$ there are two equivalent fixed points at $E=-V_0$, $x=\pm\sqrt{\nu_1}$, $v=0$. The stability condition $\phi'(-V_0)>0$ leads to the condition $\nu_2<-b\nu_1$. Another fixed point satisfying (3.28) is located at $E=0$, $x=v=0$. Due to $V''(0)<0$ it is identified as a saddle as in Eq. (3.46). An explicit expression of the potential is given in Appendix B for $Q_1=0$.

The bifurcations in the vicinity of $\nu_1=\nu_2=0$ for the case $a=1$ can now be summarized as follows. For $b=-1$ there is a supercritical Hopf bifurcation along the line $\nu_2=0$, $\nu_1<0$ which takes the boundary minimum of the potential at $E=0$ existing for $\nu_2<0$ into a boundary maximum for $\nu_2>0$ with the simultaneous appearance of a new minimum at $E=E_c>0$ describing a stable limit cycle. Crossing the line $\nu_1=0$ to the half-plane $\nu_1>0$ the fixed point at $E=0$ splits into a symmetrical pair at $E=-V_0$ via a pitchfork bifurcation. The pair is stable for $\nu_2<\nu_1$ and unstable for $\nu_2>\nu_1$. Decreasing ν_2 from positive values for fixed ν_1, there exists first a boundary maximum of ϕ at $E=-V_0$ describing a pair of unstable fixed points and a minimum at $E=E_{c2}>0$ describing a simple stable symmetrical limit cycle. When ν_2 crosses below $\nu_2=\nu_1$ there occurs a reversed subcritical Hopf bifurcation which turns the unstable fixed points into stable ones and creates a symmetrical pair of unstable limit cycles within the symmetrical stable limit cycle surrounding them. When ν_2 crosses below $\nu_2=\frac{4}{5}\nu_1$ the pair of unstable limit cycles turns into a single unstable limit cycle forming a homoclinic orbit of the saddle at $x=0$, $v=0$ for $\nu_2=\frac{4}{5}\nu_1$. The two symmetrical limit cycles, the inner one unstable, the outer one stable, coalesce and annihilate each other at $\nu_2=\bar{c}\nu_1$ and for $\nu_2<\bar{c}\nu_1$ only the pair of stable fixed points remains.

In the case $b=1$ the negative ν_1 half-plane is still separated by a line of Hopf bifurcations at $\nu_2=0$ but this time the Hopf bifurcation is subcritical and turns a boundary minimum of ϕ at $E=0$ and a maximum at $E=E_c$ existing for $\nu_2<0$ into a boundary maximum for $\nu_2>0$. For $\nu_1<0$ a stable attractor therefore exists only in the domain $\nu_2<0$ and the nonequilibrium potential is only determined by this attractor for $0\le E\le E_c$. For $\nu_1<0$, $\nu_2<0$, $E>E_c$ and $\nu_1<0$, $\nu_2>0$ the system is unstable and the nonequilibrium potential cannot be determined by local considerations. We therefore exclude these regions from further consideration. Crossing the line $\nu_1=0$, $\nu_2<0$ into the positive ν_1 half-plane the stable fixed point at $E=0$, $x=0=v$ undergoes a pitchfork bifurcation and turns into a symmetrical pair of stable fixed points at $E=-V_0$, $x=\pm\sqrt{\nu_1}$, $v=0$, which is surrounded by the unstable (symmetrical) limit cycle $E=E_{c2}>0$. Increasing now ν_2 from negative values at fixed $\nu_1>0$ the pair of stable fixed points undergoes a supercritical Hopf bifurcation at $\nu_2=-\nu_1$ and turns unstable by ejecting a pair of stable limit cycles $E=E_{c1}$ which is surrounded by the unstable limit cycle $E=E_{c2}$ $(E_{c2}>E_{c1})$. At $\nu_2=-\frac{4}{5}\nu_1$ the two stable limit cycles form a homoclinic double loop of the saddle at $x=0=v$ and merge into a single symmetrical stable limit cycle for $\nu_2>-\frac{4}{5}\nu_1$. For $\nu_2=-\bar{c}\nu_1$, finally, the two limit cycles coalesce and annihilate each other. Thus for $\nu_2>-\bar{c}\nu_1$ there exists no stable attractor and the nonequilibrium potential is not determined by the local flow. This region is therefore again excluded from consideration.

2. *The case $a=-1$*

Finally, we analyze the case $a=-1$. It is clear from Eq. (3.52) that the half-plane $\nu_1>0$ is a region without attractors where the local flow does not determine the nonequilibrium potential. For $\nu_1<0$ the solution (3.26) is defined for $0\le E\le V_0=|\nu_1|^2/4$.

A limit cycle exists for

$$\nu_2 = -b\frac{\overline{v}_2(E_c)}{\overline{v}(E_c)} \tag{3.67}$$

and is stable if

$$b\left[\frac{\overline{v}_2(E_c)}{\overline{v}(E_c)}\right]' < 0 . \tag{3.68}$$

Since $\overline{v}_2(E)/\overline{v}(E)$ is positive and monotonously increasing for $0<E<V_0$ there is a single limit cycle in the region $\nu_2/b<0$ and it is stable for $b=-1$ and unstable for $b=1$. The limit cycle coalesces to the point $E=0$, $x=v=0$ for $E_c=0$ which happens for $\nu_2=0$. The limit cycle can also disappear for $E_c=V_0$ by forming there a heteroclinic connection of two saddle points $E=V_0$, $x=\pm\sqrt{|\nu_1|}$, $v=0$. This happens for $\nu_2/b=-|\nu_1|/5$.

A fixed point exists at $E=0$ where ϕ has a boundary extremum. It is a minimum for $\nu_2<0$. The boundary extremum at $E=V_0$ is always a saddle at $x=\pm\sqrt{|\nu_1|}$, $v=0$. Thus, for $b=-1$, $\nu_1<0$ and $\nu_2<0$ there is a stable fixed point at $x=0=v$ and the potential is defined for $0\le E\le V_0$. Crossing the line $\nu_2=0$, $\nu_1<0$ there occurs a supercritical Hopf bifurcation which turns the minimum of ϕ at $E=0$ into a maximum by ejecting a new minimum at $E=E_c$ describing a stable limit cycle. At $\nu_2=|\nu_1|/5$ the limit cycle forms a heteroclinic orbit of the points $x=\pm\sqrt{|\nu_1|}$, $v=0$ and disappears, leaving the local flow without any attractor. Approaching the line $\nu_1=0$ for $\nu_2<0$ the domain $0\le E\le V_0$ where the potential can be defined shrinks to zero and disappears. For $b=1$ and $\nu_1<0$, there still is a stable fixed point at $x=v=0$ for $\nu_2<0$, but in addition there is an unstable limit cycle at $E=E_c$ which forms the border of the domain in which ϕ can be defined. A subcritical Hopf bifurcation at $\nu_2=0$, $\nu_1<0$ turns the stable fixed point into an unstable one and leaves the system without any attractors. The region $\nu_1<0$, $\nu_2>0$ is therefore excluded from consideration.

IV. SIMULTANEOUS VANISHING OF A REAL EIGENVALUE AND THE REAL PART OF A COMPLEX-CONJUGATE PAIR

The local normal form of this second type of codimension-2 bifurcation reads

$$\begin{aligned}\dot r &= \mu_1 r + arz + cr^3 + drz^2 ,\\ \dot z &= \mu_2 + br^2 - z^2 + er^2 z + fz^3 ,\\ \dot\theta &= \omega .\end{aligned} \tag{4.1}$$

Henceforth, we concentrate on the case that the coefficients a and b have opposite sign, $ab<0$. A special feature of the case $ab>0$ is considered in Appendix C. Rescaling the variables and parameters by

$$\begin{aligned}&r=\epsilon\overline{r},\quad z=\epsilon\overline{z},\quad \mu_1=\epsilon^2\nu_1,\quad \mu_2=\epsilon^2\nu_2 ,\\ &t=\overline{t}/\epsilon,\quad \omega=\epsilon\overline{\omega}\end{aligned} \tag{4.2}$$

and omitting the bars henceforth we obtain $\dot\theta=\omega$ and

$$\begin{aligned}\dot r &= arz + \epsilon(\nu_1 r + cr^3 + drz^2)\\ &= arz + \epsilon g_r(r,z) ,\\ \dot z &= \nu_2 + br^2 - z^2 + \epsilon z(er^2 + fz^2)\\ &= \nu_2 + br^2 - z^2 + \epsilon z g_z(r,z) .\end{aligned} \tag{4.3}$$

The transport matrix we choose to be diagonal and isotropic in the (r,θ) plane

$$Q^{\nu\mu} = \begin{bmatrix} Q_1 & 0 & 0\\ 0 & Q_2 & 0\\ 0 & 0 & \dfrac{Q_1}{r^2}\end{bmatrix} . \tag{4.4}$$

The Hamilton-Jacobi equation for ϕ reads

$$\frac{Q_1}{2}\left[\frac{\partial\phi}{\partial r}\right]^2 + \frac{Q_1}{2r^2}\left[\frac{\partial\phi}{\partial\theta}\right]^2 + \frac{Q_2}{2}\left[\frac{\partial\phi}{\partial z}\right]^2 + (arz+\epsilon g_r)\frac{\partial\phi}{\partial r} + (\nu_2 + br^2 - z^2 + \epsilon z g_z)\frac{\partial\phi}{\partial z} + \omega\left[\frac{\partial\phi}{\partial\theta}\right] = 0 . \tag{4.5}$$

Due to the rotational symmetry of the flow (4.3) θ is a cyclic variable, and ϕ is independent of θ since otherwise ϕ could not be periodic in θ. This conclusion depends only on rotational symmetry and holds even if ω in (4.1) and Q^{33} in (4.4) is an arbitrary function of r and z. (We note in passing that symmetry-breaking perturbations of the normal form may lead to chaotic behavior and hence introduce great complications which have not yet been dealt with in the present formalism. We exclude here the presence of such symmetry-breaking perturbations.)

The Hamilton-Jacobi equation (4.5) can be solved perturbatively in ϵ,

$$\phi = \sum_{n=1}^{\infty} \epsilon^n \phi_n . \tag{4.6}$$

For ϕ_1 we obtain the general solution

$$\phi_1 = F_1(u) \tag{4.7}$$

with

$$u = u(r,z) = r^{2/a}\left[z^2 - \nu_2 - \frac{br^2}{1+a}\right] . \tag{4.8}$$

We introduce u as a new variable instead of z and define

$$z = z(r,u) = \left[ur^{-2/a} + \nu_2 + \frac{br^2}{1+a}\right]^{1/2} . \tag{4.9}$$

Henceforth, we distinguish the potential in the upper and lower z half-plane by the notation $\phi^{\pm}(r,u)=\phi(r,\pm z(r,u))$.

The continuity of ϕ for $z=0$ is then expressed by

$$\phi^{+}(r,u(r,0))=\phi^{-}(r,u(r,0)) , \qquad (4.10)$$

which is why we may keep the simpler notation in Eq. (4.7). For $\phi_2^{\pm}$ we obtain the equation

$$\mp\frac{arz}{2}\frac{\partial\phi_2^{\pm}}{\partial r} = \left[\frac{Q_1}{Q^2}r^{4/a-2}(z^2-\nu_2-br^2)^2+Q_2r^{4/a}z^2\right]F_1'(u)^2 + \left[g_r(r,z)\frac{1}{a}r^{2/a-1}(z^2-\nu_2-br^2) + g_z(r,z)r^{2/a}z^2\right]F_1'(u) , \qquad (4.11)$$

where we have used that g_r and g_z are even in z and $z=z(u,r)$ is always implied. In the following we restrict our attention to the parameter domains

$$b>0,\ \ a<0,\ \ \nu_2\le 0 \qquad (4.12)$$

and

$$b<0,\ \ a>0,\ \ \nu_2>0 , \qquad (4.13)$$

because in these cases there exists a fixed point at $r_0=\sqrt{-\nu_2/b}$, $z=0$ and closed contours $u(r,z)=\text{const}$ surrounding it in the (r,z) plane, and there is a possibility for Hopf bifurcations to occur. [In order to save space we shall not enter a discussion of the bifurcations of the fixed points at $r=0$, $z_0=\pm\sqrt{\nu_2}$ which always occur for $\nu_2>0$; in fact, the scaling (5.2) we have adopted would not be appropriate for such a discussion, rather $\mu_1=\epsilon\nu_1$, $\mu_2=\epsilon^2\nu_2$, would be necessary.] Our restriction of parameter space ensures that we can integrate Eq. (4.11) over r for fixed u in a finite interval $r_1\le r\le r_2$ where $u>-r_0^{2/a}[\nu_2+br_0^2/(1+a)]$ and r_1,r_2 with $0<r_1(u)<\sqrt{-\nu_2/b}<r_2(u)<\infty$ are two positive roots of $z(u,r)=0$. Taking the integral in the (r,z) plane from r_1 to r_2 with $z>0$ and back from r_2 to r_1 with $z<0$ and using

$$\oint \left.\frac{\partial\phi_2}{\partial r}\right|_u dr=0 , \qquad (4.14)$$

we obtain a solvability condition which must be satisfied by $F_1'(u)$,

$$F_1'(u)=-\frac{g(u)}{Q(u)} \qquad (4.15)$$

with

$$g(u)=\frac{1}{a}\int_{r_1}^{r_2}dr[r^{2/a-2}(z^2-\nu_2-br^2)g_r(r,z)z^{-1} + ar^{2/a-1}zg_z(r,z)] ,$$
$$Q(u)=\frac{1}{a^2}\int_{r_1}^{r_2}dr[Q_1(z^2-\nu_2-br^2)^2 + Q_2a^2r^2z^2]r^{4/a-3}z^{-1}\ge 0 , \qquad (4.16)$$

where again $z=z(r,u)$ is implied in the integrands. The expression for $g(u)$ can be simplified using the identity

$$z^2-\nu_2-br^2=-azr\frac{\partial z(r,u)}{\partial r} , \qquad (4.17)$$

and the fact that z vanishes at the boundaries of the integrals. Using the explicit form of g_r,g_z we obtain by partial integration

$$g(u)=\int_{r_1}^{r_2}dr\, r^{2/a-1}z\left[\frac{2\nu_1}{a}+Ar^2+\tfrac{1}{3}Bz^2\right] \qquad (4.18)$$

with

$$A=\frac{2(1+a)}{a}c+e ,\qquad B=\frac{2d}{a}+3f , \qquad (4.19)$$

or, applying the identity (4.17) again in order to eliminate the r^2 term in the integrand,

$$g(u)=\int_{r_1}^{r_2}dr\, r^{2/a-1}\left[\left[\frac{2\nu_1}{a}-\frac{\nu_2}{b}A\right]z + \left[\frac{A}{b}+B\right]\frac{z^3}{3}\right] . \qquad (4.20)$$

Thus, restricting ourselves to the leading order in ϵ our result is

$$\phi(r,z)=-\epsilon\int_{u_0}^{u(r,z)}\frac{g(\tilde u)}{Q(\tilde u)}d\tilde u . \qquad (4.21)$$

Let us now discuss the extrema of this solution. Limit cycles in the (r,z) plane [corresponding to two-tori in the (r,θ,z) space $u(r,\theta,z)=u_c$] satisfy

$$\frac{\partial\phi(u_c)}{\partial u_c}=0 , \qquad (4.22)$$

which is equivalent to

$$g(u_c)=0 \qquad (4.23)$$

if $Q(u_c)\neq 0$. The relation fixing ν_1 as a function of u_c is obtained from Eq. (4.23) with Eq. (4.20) as

$$\nu_1=\nu_1(u_c)\equiv\frac{a}{2b}[A\nu_2-\tfrac{1}{3}(A+bB)\langle z^2(u_c)\rangle] \qquad (4.24)$$

with

$$\langle z^2(u_c)\rangle=\frac{\int_{r_1}^{r_2}dr\, r^{2/a-1}z^3}{\int_{r_1}^{r_2}dr\, r^{2/a-1}z} . \qquad (4.25)$$

Obviously, $\langle z^2\rangle$ is positive, but it need not be montonously increasing with u_c. Therefore, Eq. (4.24) may have more than one solution in a range of ν_1 values. Such additional solutions appear or disappear at bifurcation points which are determined by

$$\frac{d}{du_c}\langle z^2(u_c)\rangle=0 . \qquad (4.26)$$

Whether such points exist or not depends *only* on the value of a,b,ν_2 and not on the constants A,B defined by (4.19).

The only admissible solutions of (4.23) lie in the range $0 \le z^2(r,u) < \infty$. Therefore, according to Eq. (4.8), u_c must satisfy $u_c \ge u_{\min}$ with

$$u_{\min} = -\frac{a\nu_2}{1+a}\left|\frac{\nu_2}{b}\right|^{1/a} . \tag{4.27}$$

In the case of (4.12) for $-1<a<0$ and of (4.13) u_c must lie in the interval

$$u_{\min} \le u_c \le 0 , \tag{4.28}$$

while in the case of (4.12) for $a<-1$

$$0 < u_{\min} \le u_c < \infty . \tag{4.29}$$

A limit cycle collapses to the fixed point

$$r_0^2 = -\frac{\nu_2}{b}, \quad z=0 \tag{4.30}$$

when u_c reaches its lower limit. Then $\langle z^2\rangle$ defined by (4.25) approaches zero and is monotonously increasing with u_c for u_c slightly above $u_{\min}$. Thus, for u_c slightly above $u_{\min}$ there exists only a single limit cycle. We conclude that the system undergoes a Hopf bifurcation for $u_c = u_{\min}$ which happens for

$$\nu_1 = \frac{a}{2b}A\nu_2 \tag{4.31}$$

as follows from (4.22) for $u_c \to u_{\min}$. The size of the limit cycle grows without limit in the case of (4.12) and can then no longer be reliably described by the local expansion (4.1). In the case of (4.13) the limit cycle reaches its largest size at $u_c = 0$ and forms a heteroclinic connection of the fixed points $r=0$, $z=\pm\sqrt{\nu_2}$ before disappearing as soon as $u_c>0$. The critical value $u_c=0$ is reached for

$$\nu_1 = \frac{\nu_2 a}{(2+3a)b}\left[(1+a)A - \frac{ab}{2}B\right] . \tag{4.32}$$

A limit cycle is stable if

$$\phi^{(n)}(u_c) > 0 , \tag{4.33}$$

where n is the order of the first nonvanishing derivative of ϕ. The condition (4.33) is equivalent to

$$g^{(n)}(u_c) < 0 \tag{4.34}$$

and also to

$$\frac{1}{a}\nu_1^{(n)}(u_c) > 0 . \tag{4.35}$$

In general, this condition is tedious to evaluate and we therefore consider only the two limiting cases $u_c \to u_{\min}$ and, for the case of (4.13), $u_c \to 0$. For $u_c \to u_{\min}$ we see from (4.24) that $\nu_1(u_c)/a$ is increasing with $\langle z^2\rangle$ and u_c for $u_c \gtrsim u_{\min}$ if

$$\frac{A+bB}{b} < 0 , \tag{4.36}$$

i.e., in this case the Hopf bifurcation is supercritical and the limit cycle is stable. (This result is in disagreement with a corresponding result in Ref. 19, due to an algebraic error in the evaluation of the stability of the limit cycle.) If $(A+bB)/b>0$ the Hopf bifurcation is subcritical, and the limit cycle is unstable. For $u_c \to 0$ with $b<0, a>0, \nu_2>0$ the derivative of $\nu_1(u_c)/a$ diverges to $+\infty$ if $(A+bB)/b<0$ and to $-\infty$ if $(A+bB)/b>0$. Thus Eq. (4.36) is also the stability condition for the limit cycle near $u_c=0$, which continuously evolves from the Hopf bifurcation. As $\nu_1'(u_c)$ has the same sign for $u_c \to u_{\min}$ and $u_c \to 0$, the number of minima and maxima of $\nu_1(u_c)$ inside this interval must be equal. Therefore, the number of solutions of Eq. (4.24) can only be one stable limit cycle, or, in general, an even number of stable and an odd number of unstable limit cycles.

Finally, we analyze the fixed points of the system described by the boundary extrema of $\phi(u)$ at $u=u_{\min}$ and at $u=0$, for the case of Eq. (4.13). For $u=u_{\min}$ we can satisfy

$$\frac{\partial\phi}{\partial r}=0, \quad \frac{\partial\phi}{\partial z}=0 \tag{4.37}$$

by putting

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial z} = 0 . \tag{4.38}$$

This can be satisfied for $\nu_2/b>0$ and yields (4.30). The trace and determinant of the Hessian of u with respect to r and z taken at this point are

$$\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2(a-2b)(-\nu_2)}{ab} ,$$
$$\begin{vmatrix} \dfrac{\partial^2 u}{\partial r^2} & \dfrac{\partial^2 u}{\partial r\,\partial z} \\ \dfrac{\partial^2 u}{\partial r\,\partial z} & \dfrac{\partial^2 u}{\partial z^2} \end{vmatrix} = \frac{8\nu_2}{a}\left[-\frac{\nu_2}{b}\right]^{2/a-1} , \tag{4.39}$$

and hence both eigenvalues are positive if

$$\frac{\nu_2}{a} > 0 . \tag{4.40}$$

For $\nu_2/a<0$ the fixed point (4.30) is therefore unstable. It is stable if (4.40) is satisfied and

$$\frac{\partial\phi}{\partial u} > 0 \quad (u=u_{\min}) . \tag{4.41}$$

Equation (4.41) is equivalent to

$$\frac{g(u_{\min})}{Q(u_{\min})} < 0 . \tag{4.42}$$

As $g(u)$ increases with ν_1/a and $g(u_{\min})/Q(u_{\min})=0$ for ν_1/a given by Eq. (4.31), we find as the domain of stability of (4.30),

$$\frac{\nu_1}{a} < \frac{\nu_2}{2b}A, \quad \frac{\nu_2}{a} > 0 . \tag{4.43}$$

For the case of Eq. (4.13) another boundary of u exists at $u=0$. [For the case of Eq. (4.12) with $-1<a<0$ the boundary at $u=0$ also exists but is only reached for

$\nu_1 \rightarrow \pm\infty$, depending on the sign of A, and hence is irrelevant.] Putting $\partial u/\partial r = \partial u/\partial z = 0$ in this case we find the fixed points

$$r=0, \quad z=\pm\sqrt{\nu_2} \,. \tag{4.44}$$

However, the determinant of the Hessian of u at these points either diverges to $-\infty$ (if $a>2$) or approaches zero from the negative side (if $0<a<2$). In either case, the fixed points (4.45) are unstable. If $0<a<2$, another solution of $\partial u/\partial r = \partial u/\partial z = 0$ is the manifold $r=0$, z arbitrary, which contains the fixed points (4.44) and is again unstable.

Let us now briefly summarize the implications of the extrema of ϕ for the behavior of the system in the three-dimensional (r,θ,z) space. In the case of Eq. (4.12) $b>0$, $a<0$, $\nu_2<0$ we have to distinguish the two possibilities

(i) $A < -bB$,
(ii) $A > -bB$.

If (i) is realized the system has a stable limit cycle $r=\sqrt{|\nu_2/b|}$, $0<\theta<2\pi$, $z=0$ in the domain $\nu_1 > \frac{1}{2}|\nu_2 a/b|A \equiv \nu_{10}$ and the nonequilibrium potential is everywhere defined by (4.21). The limit cycle shrinks and disappears by a Hopf bifurcation if $\nu_2 \rightarrow 0-$; the domain $\nu_2>0$ is excluded from our considerations. The limit cycle loses its stability by a secondary supercritical Hopf bifurcation when ν_1 is decreased to the critical value $\nu_1=\nu_{10}$ where a stable two torus is born which exists for all $\nu_1<\nu_{10}$ and surrounds the now unstable limit cycle. If $-1<a<0$ new pairs of stable and unstable two tori may appear for $\nu_1<\nu_{10}$ if a has a value such that critical values of u_c satisfying (4.26) exist. If $\nu_2 \rightarrow 0-$ for $\nu_1<0$ the radius of the unstable limit cycle shrinks to zero together with the inner radius of the surrounding two torus until they both disappear for $\nu_2=0$.

If (ii) is realized, the system has a stable limit cycle for $\nu_1>\nu_{10}$ but it coexists with an unstable two-torus surrounding it and limiting the domain in which the nonequilibrium potential is determined by the local flow. For $\nu_1=\nu_{10}$ the unstable two-torus collapses to the limit cycle and destabilizes it by an inverted Hopf bifurcation. The domain $\nu_1<\nu_{10}$ must be excluded from consideration, as no local attractors exist there.

Now we discuss the case of Eq. (4.13), $a>0$, $b<0$, $\nu_2>0$. In this case, the system has two unstable fixed points at $r=0$, $z=\pm\sqrt{\nu_2}$. We must distinguish the two cases

(i) $A > |b|B$,
(ii) $A < |b|B$.

If (i) is realized the system has a stable limit cycle $r=\sqrt{|\nu_2/b|}$, $z=0$, $0\le\theta<2\pi$ for $\nu_1 < -\frac{1}{2}|\nu_2 a/b|A$. The domain where the nonequilibrium potential is defined is limited by the condition $u_{\min} \le u \le 0$. The limit cycle disappears via a Hopf bifurcation for $\nu_2 \rightarrow 0+$ (the domain $\nu_2<0$ is excluded) and it undergoes a supercritical secondary Hopf bifurcation for $\nu_1 = -\frac{1}{2}|\nu_2 a/b|A$. For ν_1 larger than this critical value the limit cycle is unstable and surrounded by a stable two-torus. Further pairs of stable and unstable two tori appear and disappear in this region at points where condition (4.26) is met. If ν_1 is increased further towards the value

$$\nu_1=\nu_{12} \equiv -\frac{1}{2}\left|\frac{\nu_2 a}{b}\right| \frac{2(1+a)A + a|b|B}{2+3a}$$

there remains only one stable two-torus which, at this critical value of ν_1, forms a heteroclinic connection of the two unstable fixed points and disappears. The region of larger ν_1 is excluded as no attractors remain.

If (ii) is realized, the stable limit cycle for $\nu_1 < -\frac{1}{2}|\nu_2 a/b|A$ coexists with a surrounding unstable two-torus, which forms the boundary of the region where the nonequilibrium potential can be locally defined. This region is most extended for $\nu_1=\nu_{12}$. For still smaller values of ν_1 the stable limit cycle remains but the unstable two-torus has disappeared, and the domain where the nonequilibrium potential is defined by the local flow is now determined by the equipotential surface containing the unstable fixed points, $r=0$, $z=\pm\sqrt{\nu_2}$. At $\nu_1 = -\frac{1}{2}|\nu_2 a/b|A$ a subcritical Hopf bifurcation takes place in which the two-torus collapses on the limit cycle and destabilizes it. The domain of larger ν_1 is again excluded as no attractors remain.

Until now we investigated only the nonequilibrium potentials. A systematic asymptotic expansion of the stationary probability density for small noise intensity η yields, as a first correction to $\exp(-\phi/\eta)$ an η-independent prefactor Z. This prefactor is calculated in Appendix D for the systems treated in Secs. III and IV.

V. TWO SIMULTANEOUS HOPF BIFURCATIONS

This last type of codimension-2 bifurcation has the local normal form[19]

$$\begin{aligned}
\dot r_1 &= \mu_1 r_1 - r_1^3 - b r_1 r_2^2 + O(r^5) \,,\\
\dot r_2 &= \mu_2 r_2^2 - c r_2 r_1^3 - d r_2^3 + O(r^5) \,,\\
\dot\theta_1 &= \omega_1 \,,\\
\dot\theta_2 &= \omega_2 \,,
\end{aligned} \tag{5.1}$$

provided there is no resonance of low order between the two frequencies, $m\omega_1+n\omega_2 \ne 0$, $|m|+|n| \le 4$. Otherwise, phase-coupling terms must be taken into account already in the leading terms of the normal form. $O(r^5)$ denotes homogeneous functions of fifth order.[19] In the following we consider the case where $bc<0$. The opposite case where $bc>0$ has a simplifying feature which is treated in Appendix C. Scaling in the form

$$\begin{aligned}
& r_1=\sqrt{\epsilon}\,\bar r_1, \quad r_2=\sqrt{\epsilon}\,\bar r_2, \quad \mu_1=\epsilon\nu_1 \,,\\
& \mu_2=-\epsilon\nu_1\frac{c-1}{b-d}d+\epsilon^2\nu_2 \,,\\
& t=\bar t/\epsilon, \quad \omega_1=\epsilon\bar\omega_1, \quad \omega_2=\epsilon\bar\omega_2 \,,
\end{aligned} \tag{5.2}$$

we obtain, omitting the bars in order to simplify the notation,

$$\dot r_1 = r_1(\nu_1 - r_1^2 - b r_2^2) - \epsilon r_1 g_1(r_1,r_2) ,$$
$$\dot r_2 = r_2 \left[-\nu_1 \frac{c-1}{b-d} d - c r_1^2 - d r_2^2 \right] - \epsilon r_2 g_2(r_1,r_2) , \tag{5.3}$$
$$\dot\theta_1 = \omega_1, \quad \dot\theta_2 = \omega_2$$

with

$$g_1(r_1,r_2) = e r_1^4 + f r_1^2 r_2^2 + g r_2^4 ,$$
$$g_2(r_1,r_2) = -\nu_2 + h r_1^4 + j r_1^2 r_2^2 + k r_2^4 . \tag{5.4}$$

The special form of μ_2 in Eq. (5.2) ensures the existence of a Hopf bifurcation.[19] We choose a transport matrix in the form

$$Q^{\nu\mu} = \begin{bmatrix} Q_1 & 0 & 0 & 0 \\ 0 & Q_2 & 0 & 0 \\ 0 & 0 & \frac{Q_1}{r_1^2} & 0 \\ 0 & 0 & 0 & \frac{Q_2}{r_2^2} \end{bmatrix} . \tag{5.5}$$

The Hamilton-Jacobi equation for ϕ reads

$$\frac{Q_1}{2}\left[\frac{\partial\phi}{\partial r_1}\right]^2 + \frac{Q_2}{2}\left[\frac{\partial\phi}{\partial r_2}\right]^2 + \frac{Q_1}{2r_1^2}\left[\frac{\partial\phi}{\partial\theta_1}\right]^2 + \frac{Q_2}{2r_2^2}\left[\frac{\partial\phi}{\partial\theta_2}\right]^2 + r_1(\nu_1 - r_1^2 - b r_2^2)\frac{\partial\phi}{\partial r_1}$$
$$- r_2\left[-\nu_1\frac{c-1}{b-d}d - c r_1^2 - d r_2^2\right]\frac{\partial\phi}{\partial r_2} - \epsilon r_1 g_1 \frac{\partial\phi}{\partial r_1} - \epsilon r_2 g_2 \frac{\partial\phi}{\partial r_2} + \omega_1\frac{\partial\phi}{\partial\theta_1} + \omega_2 \frac{\partial\phi}{\partial\theta_2} = 0 . \tag{5.6}$$

It is solved by ϕ, independent of θ_1, θ_2, by making the ansatz

$$\phi = \sum_{n=1}^{\infty} \epsilon^n \phi_n . \tag{5.7}$$

From the equation for ϕ_1 it follows that

$$\phi_1 = F_1(u) , \tag{5.8}$$

where F_1 is an arbitrary function and u is given by

$$u(r_1,r_2) = r_1^\alpha r_2^\beta \left[-\nu_1 + r_1^2 - d\frac{\beta}{\alpha} r_2^2 \right] . \tag{5.9}$$

We use the abbreviations

$$\alpha = 2d\frac{c-1}{A}, \quad \beta = \frac{2(b-d)}{A}, \quad A = d - bc . \tag{5.10}$$

From Eq. (5.8) it follows that the extrema of $u(r_1,r_2)$ will be extrema of ϕ to lowest order in ϵ The extrema of u are

$$P_1\!: \; r_1 = r_2 = 0 ,$$
$$P_2\!: \; r_1 = 0, \; r_2 = \left[\frac{\nu_1(1-c)}{b-d}\right]^{1/2} \quad \left[\frac{\nu_1(1-c)}{b-d} > 0\right] ,$$
$$P_3\!: \; r_1 = \sqrt{\nu_1}, \; r_2 = 0 \;\; (\nu_1 > 0) , \tag{5.11}$$
$$P_4\!: \; r_{10} = \left[-\frac{\nu_1 d}{b-d}\right]^{1/2}, \; r_{20} = \left[\frac{\nu_1}{b-d}\right]^{1/2} \quad \left[\frac{\nu_1}{b-d} > 0, \; d < 0\right] .$$

They correspond to fixed points of Eq. (5.3) to lowest order in ϵ. In the following, we consider the case where the fixed point P_4 exists and corresponds to a minimum or a maximum of u. In this case closed contours $u(r_1,r_2)=$const. must exist in the (r_1,r_2) plane which surround the fixed point P_4. For the fixed point P_4 to be either a minimum or a maximum of u, the determinant of the Hessian of u with respect to r_1,r_2 must be positive at P_4. Evaluating it, we find

$$\left[\frac{\partial^2 u}{\partial r_1^2}\frac{\partial^2 u}{\partial r_2^2} - \left[\frac{\partial^2 u}{\partial r_1 \partial r_2}\right]^2\right]_{P_4} = -\frac{4\nu_1^2\beta^2 d}{(b-d)^2} A r_{10}^{2\alpha-2} r_{20}^{2\beta-2} , \tag{5.12}$$

which is positive if

$$A > 0 \tag{5.13}$$

holds, together with the inequalities necessary for the existence of P_4.

The trace of the Hessian at the point P_4 is evaluated as

$$\frac{\partial^2 u}{\partial r_1^2} + \frac{\partial^2 u}{\partial r_2^2} = -\frac{4\nu_1^2 d r_{10}^{\alpha-2} r_{20}^{\beta-2}}{A}\frac{b-c}{b-d} . \tag{5.14}$$

In the region $bc < 0$ it is positive and u is a minimum at P_4 if $b > |d/c| > 0 > c,d$, or if $c > |d/b| > 0 > d > b$; it is negative and u is a maximum at P_4 if $c > |d/b| > 0 > b > d$. Altogether we have now to distinguish four cases, and it is useful to sketch the relevant contours $u=$const for them. This is done in Figs. 2(a)–2(d).

The equation for ϕ_2 reads

$$\tfrac{1}{2}Q(u,r_2)\left[\frac{dF_1}{du}\right]^2 + g(u,r_2)\frac{dF_1}{du} + \left.\frac{\partial\phi_2}{\partial r_2}\right|_u = 0 \tag{5.15}$$

with

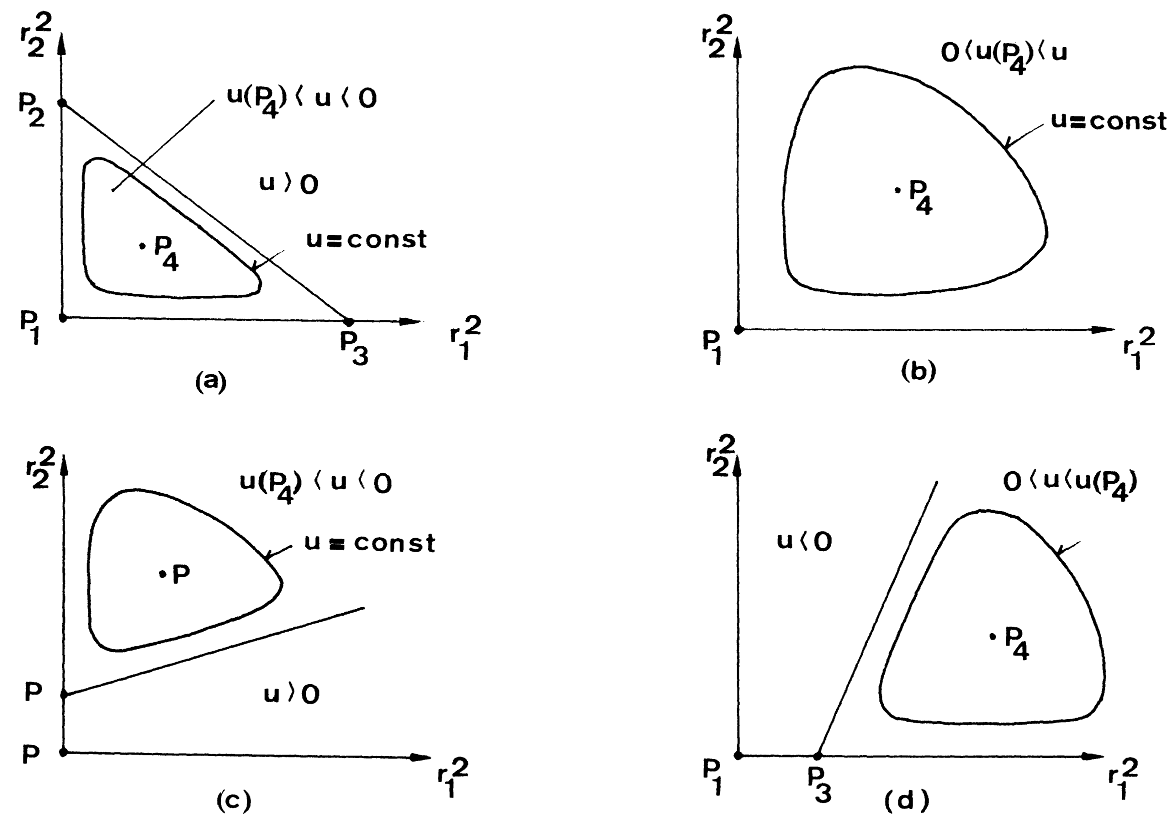

FIG. 2. Level curves of u. (a) $b > |d/c| > c$, $u(P_4)=\min<0$, (b) $c>1>|d/b|>0>d>b$, $u(P_4)=\min>0$, (c) $1>c>|d/b|>0>d>b$, $u(P_4)=\min<0$, (d) $c>|d/b|>0>b>d$, $u(P_4)=\max>0$.

$$g(u,r_2)=\frac{\beta r_1^{\alpha} r_2^{\beta}\left[g_1(r_1,r_2)\left[\frac{\alpha}{\beta}\nu_1+cr_1^2+dr_2^2\right]+g_2(r_1,r_2)(\nu_1-r_1^2-br_2^2)\right]}{r_2\left[-\nu_1\frac{\alpha}{\beta}-cr_1^2-dr_2^2\right]}\,,$$
$$Q(u,r_2)=\frac{\beta^2 r_1^{2\alpha-2} r_2^{2\beta-2}\left[Q_1 r_2^2\left[\frac{\alpha}{\beta}\nu_1+cr_1^2+dr_2^2\right]^2+Q_2 r_1^2(-\nu_1+r_1^2+br_2^2)^2\right]}{r_2\left[-\nu_1\frac{\alpha}{\beta}-cr_1^2-dr_2^2\right]}\,. \tag{5.16}$$

Here and in the following r_1 is always assumed to be given as a function of u and r_2 by (5.9). The expression for $g(u,r_1)$ can be simplified by using the relation

$$\left.\frac{\partial r_1(u,r_2)}{\partial r_2}\right|_u=\frac{r_1(\nu_1-r_1^2-br_2^2)}{r_2\left[-\frac{\alpha}{\beta}\nu_1-cr_1^2-dr_2^2\right]}\,. \tag{5.17}$$

We obtain in this way

$$g(u,r_2)=\beta r_1^{\alpha-1}r_2^{\beta-1}\left[-r_1 g_1(r_1,r_2)+\left.\frac{\partial r_1}{\partial r_2}\right|_u r_2 g_2(r_1,r_2)\right]. \tag{5.18}$$

We may integrate Eq. (5.15) around a closed contour $u(r_1,r_2)=\text{const}$. Then we obtain from

$$\oint d\phi_2=0 \tag{5.19}$$

the solvability condition for $F_1(u)$,

$$F_1'(u)=-2\frac{g(u)}{Q(u)}\,, \tag{5.20}$$

and hence

$$\phi_1(r_1,r_2)=-2\int_{u_0}^{u(r_1,r_2)}d\tilde{u}\frac{g(\tilde{u})}{Q(\tilde{u})}+\phi_1(u_0) \tag{5.21}$$

with

$$g(u)=\beta \oint dt\, r_1^{\alpha-1} r_2^{\beta-1}[r_2 g_2(r_1,r_2)\dot r_1 - r_1 g_1(r_1,r_2)\dot r_2]\ ,$$
$$Q(u)=\beta^2 \oint dt\, r_1^{2\alpha-2} r_2^{\beta-2}(Q_1 \dot r_2^2 + Q_2 \dot r_1^2)\ , \tag{5.22}$$

where we defined the positive integration parameter t around a closed contour $u=$const by

$$dr_2 = r_2\left[-\nu_1\frac{\alpha}{\beta} - cr_1^2 - dr_2^2\right]dt\ . \tag{5.23}$$

The expression for $g(u)$ can be transformed by inserting $g_1(r_1,r_2)$, $g_2(r_1,r_2)$ and trading dr_1 for dr_2 by partial integration. We find

$$g(u)=\beta \oint dr_2\, r_1^{\alpha} r_2^{\beta-1}\left[\frac{\beta}{\alpha}\nu_2 - k_1 r_1^4 - k_2 r_1^2 r_2^2 - k_3 r_2^4\right] \tag{5.24}$$

with

$$k_1=\frac{\beta h+(\alpha+4)e}{\alpha+4}\ ,$$
$$k_2=\frac{(\beta+2)j+(\alpha+2)f}{\alpha+2}\ , \tag{5.25}$$
$$k_3=\frac{(\beta+4)k+\alpha g}{\alpha}\ .$$

In the following it is useful to introduce the function

$$z(r_1,r_2)=-\nu_1\frac{\alpha}{\beta}-cr_1^2-dr_2^2\ . \tag{5.26}$$

We note that $dr_2 = zr_2\,dt$ according to Eq. (5.23), hence z vanishes in the points of minimal and maximal r_2 of the integration contour in (5.22) and (5.24). In Appendix E the expression (5.24) is brought into the form

$$g(u)=\beta \oint dt\, r_1^{\alpha} r_2^{\beta}(Bz^2+Dz+E)z \tag{5.27}$$

with

$$B=\frac{1}{A}\left[\frac{bk_1}{c}-\frac{(\alpha+2)}{c(\alpha+4)}k_2+\frac{\alpha}{d(\alpha+4)}k_3\right]\ ,$$
$$D=\frac{\nu_1}{A(\alpha+2)}\left[\frac{(\alpha+4)[(\alpha+2)d-\alpha bc]}{2c}k_1 - (\alpha+2)k_2+\frac{\alpha c(2-c)}{d}k_3\right]\ , \tag{5.28}$$
$$E=\frac{\beta}{\alpha}\nu_2-\frac{\nu_1^2\alpha^2c^2}{d^2(\alpha+2)^2}k_3\ .$$

We now examine the condition $\partial\phi_1/\partial u_c=0$ for the existence of a limit cycle $u(r_1,r_2)=u_c$. If $Q(u_c)\neq 0$, the condition is equivalent to $g(u_c)=0$. From Eq. (5.27) we obtain

$$E=-D\langle z(u_c)\rangle - B\langle z^2(u_c)\rangle \tag{5.29}$$

with

$$\langle z^n(u)\rangle=\frac{\oint dt\, r_1^{\alpha} r_2^{\beta} z^{n+1}}{\oint dt\, r_1^{\alpha} r_2^{\beta} z}\ . \tag{5.30}$$

Inserting the expression for E from Eq. (5.28) we find

$$\nu_2=\frac{\alpha}{\beta}\left[\frac{\nu_1^2\alpha^2}{d^2\beta^2}k_3 - D\langle z\rangle - B\langle z^2\rangle\right]\ . \tag{5.31}$$

It is easy to show with (5.30) that $\langle z^n\rangle \sim \nu_1^n$. Hence (5.31) is of the form $\nu_2=\text{const}\times\nu_1^2$. The value of the constant depends on all the parameters of the local flow. There may even be several simultaneous solutions of (5.31) for the same value of u_c. A general discussion of all the possibilities is very tedious and will not be carried out here. We merely investigate the vicinity of the Hopf bifurcation which gives rise to the appearance or disappearance of the limit cycle. If $b>|d/c|>0>c$ or $c>|d/b|>0>d>b$, the Hopf bifurcation occurs when u_c has its smallest possible value $u_c=u_{\min}$. If $c>|d/b|>b>d$, the Hopf bifurcation occurs when u_c has its largest possible value $u_c=u_{c\max}$. In both cases the limit cycle collapses to the fixed point P_4 of Eq. (5.11). We note that the variable z defined in Eq. (5.26) vanishes in the fixed point. Hence, according to Eq. (5.30) $\langle z^n\rangle=0$ for $n\geq 1$ and we obtain the bifurcation line from Eq. (5.31),

$$\nu_2=\nu_{21}=-\frac{\alpha^3c^3k_3}{d^2(\alpha+2)^3}\nu_1^2\ . \tag{5.32}$$

The stability condition for the limit cycle is $\partial^2\phi/\partial u_c^2>0$ which, for $Q(u_c)>0$, is equivalent to $\partial g(u_c)/\partial u_c<0$. From Eq. (5.27) it follows that for $E=0$, $g(u)$ increases for increasing $u_c-u_{c\min}$ if $\beta D>0$. Hence stability of the limit cycle is obtained for the case where $u_c=u_{c\min}$ at the bifurcation point (i.e., $b>|d/c|>0>c$ or $c>|d/b|>0>d>b$) if

$$\beta D<0\ . \tag{5.33}$$

The Hopf bifurcation (5.32) is supercritical, if (5.33) is satisfied, otherwise it is subcritical. Similarly, stability of the limit cycle is obtained for the case where $u_c=u_{c\max}$ at the bifurcation point (i.e., $c>|d/b|>b>d$) if

$$D>0\ , \tag{5.34}$$

because in the latter case $g(u)$ decreases with decreasing $u_c-u_{c\max}$. Again, the Hopf bifurcation (5.32) is supercritical if (5.34) holds and subcritical otherwise.

Going now back to Eq. (5.22) in order to check whether $Q(u_c)>0$ is satisfied in the case (5.32), it turns out that actually $Q(u_c)=0$ in this case. Hence $\phi'(u_c)$ does not vanish in this case, after all. Evaluating $\phi'(u_c)$ from Eq. (5.21) by the rule of Bernoulli-Hospital we find that

$$\phi'(u_c)\to +0\quad(\text{for } u_c\to u_{c\min}) \tag{5.35}$$

in the case of Eq. (5.33), and

$$\phi'(u_c)\to -0\quad(\text{for } u_c\to u_{c\max}) \tag{5.36}$$

in the case of Eq. (5.36). Equations (5.35) and (5.36) express the fact that $\phi(u)$ has a boundary minimum at $u_c=u_{c\min}$, and $u_c=u_{c\max}$, respectively, i.e., the conclusion about the stability of the limit cycle in both cases remains valid, but the minimum of the potential has degenerated in a boundary minimum.

In the case $b>c$ the size of the limit cycle is limited by the existence of the fixed points P_2,P_3 of Eq. (5.11). The limit cycle degenerates into a heteroclinic connection of these fixed points for $u_c=0$. In this case the integrals (5.24) can be evaluated explicitly, and we obtain after some algebra the condition

$$v_2=v_{22}=-\frac{v_1^2}{(\alpha+\beta+2)(\alpha+\beta+4)} \times \left[c\,[\beta h+(\alpha+4)e] + \frac{\alpha^2}{\beta d}[(\beta+2)j+(\alpha+2)f] + \frac{\alpha^3 b}{\beta^2 d^3}[(\beta+4)k+\alpha g] \right] . \quad (5.37)$$

(A special case of this condition was derived in Ref. 19. However, the special result given there is incorrect due to algebraic errors.)

Let us now investigate the stability of the fixed point P_4. For $b>0>c$ or $c>0>d>b$ it corresponds to a minimum of u, and therefore also to a minimum of ϕ if

$$\left.\frac{\partial\phi}{\partial u}\right|_{u_{\min}}>0 . \quad (5.38)$$

If (5.38) is satisfied, the fixed point is stable, otherwise it is a maximum of ϕ and unstable. As $Q(u)$ is positive, the sign of $\partial\phi/\partial u$ is opposite to the sign of $g(u)$. It follows from Eq. (5.27) and the fact that $g(u_{\min})=0$ for $E=0$ that $g(u_{\min})<0$ if

$$\beta E \oint dt\, r_1^\alpha r_2^\beta z<0 . \quad (5.39)$$

In order to evaluate this condition the sign of the integral must be determined. To this end we use Eq. (5.26) to write

$$\oint dt\, r_1^\alpha r_2^\beta z=\int_{r_{2\min}}^{r_{2\max}} r_2^{\beta-1}(r_{1>}^\alpha - r_{1<}^\alpha)dr_2 , \quad (5.40)$$

where $r_{1>}$ ($r_{1<}$) denotes the branch of the double-valued function $r_1(u,r_2)$ along which the contour integral has to be taken for increasing (decreasing) r_2 between the boundaries $r_{2\min}<r_{2\max}$ where $z=0$. From Eq. (5.23) and $dt>0$ we conclude that for $c<0$,

$$r_{1>}^2>r_{1<}^2,\quad \alpha>0 ,$$
$$r_{1>}^\alpha>r_{1<}^\alpha , \quad (5.41)$$
$$\oint dt\, r_1^\alpha r_2^\beta z>0 ;$$

for $c>1$,

$$r_{1>}^2<r_{1<}^2,\quad \alpha<0 ,$$
$$r_{1>}^\alpha>r_{1<}^\alpha , \quad (5.42)$$
$$\oint dt\, r_1^\alpha r_2^\beta z>0 ;$$

for $1>c>0$,

$$r_{1>}^2<r_{1<}^2,\quad \alpha>0 ,$$
$$r_{1>}^\alpha<r_{1<}^\alpha , \quad (5.43)$$
$$\oint dt\, r_1^\alpha r_2^\beta z<0 .$$

Hence, we have the following stability condition for the fixed point:

$$E>0 \quad \text{for } c<0;\text{ or } c>1,\ b<d \quad (5.44)$$

$$E<0 \quad \text{for } 1>c>0,\ b<d . \quad (5.45)$$

For $c>|d/b|>0>b>d$ the point P_4 corresponds to a maximum of $u(r_1,r_2)$ and is therefore stable if

$$\left.\frac{\partial\phi}{\partial u}\right|_{u_{\max}}<0 . \quad (5.46)$$

We conclude as before that (5.46) is equivalent to

$$\beta E \oint dt\, r_1^\alpha r_2^\beta z>0 . \quad (5.47)$$

We can use the results (5.42) for $c>1$ to conclude that P_4 is stable if

$$E>0 \quad \text{for } c>1,\ b>d . \quad (5.48)$$

We now combine these results with the results obtained above on the nature of the Hopf bifurcation and summarize the various cases as follows.

(i) The case $b>|d/c|>0>c$. In this case $\alpha>0$, $\beta>0$. A stable fixed point P_4 exists for $E<0$. If $D>0$ it coexists with an unstable limit cycle surrounding it. For $E=0$ a Hopf bifurcation takes place which is subcritical for $D>0$ and supercritical for $D<0$. For $E>0$ the fixed point P_4 is unstable, and, in the case $D<0$ it coexists with a stable limit cycle. The size of the limit cycle is limited by the heteroclinic bifurcation (5.37).

(ii) The case $c>1>|d/b|$, $d>b$. In this case $\alpha<0$, $\beta<0$. A stable fixed point P_4 exists for $E>0$ and coexists there with an unstable limit cycle if $D<0$. The Hopf bifurcation takes place at $E=0$, it is subcritical for $D<0$, and supercritical for $D>0$. Thus, if $D>0$ a stable limit cycle coexists with the unstable fixed point in the region $E<0$.

(iii) The case $1>c>|d/b|$, $d>b$. Now we have $\alpha>0$, $\beta<0$. A stable fixed point P_4 exists for $E<0$, and if $D<0$ it coexists with an unstable limit cycle. The Hopf bifurcation at $E=0$ is subcritical if $D<0$ and supercritical if $D>0$. In the latter case, the unstable fixed point P_4 for $E>0$ coexists with the stable limit cycle.

(iv) The case $c>|d/b|$, $b>d$. Then $\alpha<0$, $\beta>0$. The fixed point P_4 is stable for $E>0$ and unstable for $E<0$. The Hopf bifurcation is supercritical for $D>0$ and subcritical for $D<0$.

VI. CONCLUSIONS

We conclude with a few brief remarks on the results obtained in this paper. In the analysis of the normal forms of codimension-2 bifurcations with weak additive noise it turned out that, after rescaling, all cases could be considered either as weakly perturbed conservative systems, treated in Secs. III—V and Appendix D, or, with suitably chosen noise sources, as purely dissipative systems, treated in Appendix C. We may add that the latter case could be made the starting point of a perturbative analysis, in cases where the strength of the noise sources is chosen in a slightly different way. In fully conservative systems the potential is a constant (there are no attractors and repellors). In this case the prefactor treated in Appendix D dominates the whole stationary probability density. For weak but nonvanishing dissipation a nonconstant potential appears whose equipotential lines are approximated by the trajectories of the conservative system. A normalizable probability density and the potential associated with it via Eq. (1.7) exist only when there are closed contours, i.e., closed trajectories of the conservative system.

As we have shown explicitly for the codimension-2 case, the potential can be used to study bifurcations and stability. The information concerning the deterministic dynamics which can be obtained by the standard analysis of Ref. 19 is contained in ϕ. In addition, ϕ provides a Lyapunov function for the attractors of the deterministic dynamics and generalizes the formalism of thermodynamics to the system under consideration. However, not only are results concerning the deterministic system obtained by other methods thus reproduced in a conceptually satisfying way, but also new information is obtained about the probability density on the center manifold of the stochastic system (1.2).

In previous work it was shown that the coexistence of attractors may lead to the appearance of merely piecewise differentiable potentials. This phenomenon was not observed in the examples of the present paper, due to the simplifying feature that all cases considered were very close either to fully dissipative or to fully conservative dynamics.

APPENDIX A

Our aim here is to construct a nonequilibrium potential for the case of a codimension-1 Hopf bifurcation (2.4) with a general transport matrix of the form (2.8). We also wish to show that for μ sufficiently small, only the trace of the transport matrix (2.8) is important, i.e., the difference between the cases (2.8) and (2.9) disappear.

We have to solve the Hamilton-Jacobi equation (1.4) which, for the case of (2.4) and (2.8), reads

$$\frac{Q_1}{2}\left[\frac{\partial\phi}{\partial x}\right]^2+\frac{Q_2}{2}\left[\frac{\partial\phi}{\partial y}\right]^2+\{-y+x[\mu-(x^2+y^2)]\}\frac{\partial\phi}{\partial x}+\{x+y[\mu-(x^2+y^2)]\}\frac{\partial\phi}{\partial y}=0\ . \tag{A1}$$

It is convenient to rescale the variables and the potential according to

$$x=\sqrt{|\mu|}\,\bar{x},\quad y=\sqrt{|\mu|}\,\bar{y},\quad \phi=|\mu|^2\bar{\phi}\ . \tag{A2}$$

We then find

$$-\bar{y}\frac{\partial\bar{\phi}}{\partial\bar{x}}+\bar{x}\frac{\partial\bar{\phi}}{\partial\bar{y}}+|\mu|\left[\tfrac{1}{2}Q_1\left[\frac{\partial\bar{\phi}}{\partial\bar{x}}\right]^2+\tfrac{1}{2}Q_2\left[\frac{\partial\bar{\phi}}{\partial\bar{y}}\right]^2+\frac{\partial\bar{\phi}}{\partial\bar{x}}\bar{x}[\mathrm{sgn}(\mu)-\bar{x}^2-\bar{y}^2]+\frac{\partial\bar{\phi}}{\partial\bar{y}}\bar{y}[\mathrm{sgn}(\mu)-\bar{x}^2-\bar{y}^2]\right]=0\ . \tag{A3}$$

This equation can be solved by an ansatz in the form of a power series in $|\mu|$,

$$\bar{\phi}=\sum_{n=0}^{\infty}|\mu|^n\bar{\phi}_n\ . \tag{A4}$$

In zeroth order in $|\mu|$ we find with $\bar{r}=(\bar{x}^2+\bar{y}^2)^{1/2}$,

$$\bar{\phi}_0=F_0(\bar{r})\ , \tag{A5}$$

where $F_0(\bar{r})$ is a yet undetermined function. The equation to be solved in first order in $|\mu|$, in polar coordinates, reads

$$\tfrac{1}{2}F_0'^2(Q_1\cos^2\varphi+Q_2\sin^2\varphi)+F_0'\bar{r}[\mathrm{sgn}(\mu)-\bar{r}^2]+\frac{\partial\bar{\phi}_1}{\partial\varphi}=0\ . \tag{A6}$$

It has the general solution

$$\bar{\phi}_1=F_1(\bar{r})+\left[\tfrac{1}{2}F_0'^2\left[\frac{Q_1+Q_2}{2}\right]+F_0'\bar{r}[\mathrm{sgn}(\mu)-\bar{r}^2]\right]\varphi+\tfrac{1}{4}F_0'^2\left[\frac{Q_1-Q_2}{2}\right]\sin(2\varphi)\ , \tag{A7}$$

where $F_1(\bar{r})$ is again undetermined. The periodicity of $\bar{\phi}_1$ in φ requires that we choose F_0' as a nontrivial solution of

$$\frac{Q_1+Q_2}{4}F_0'^2+F_0'\bar{r}[\mathrm{sgn}(\mu)-\bar{r}^2]=0\ . \tag{A8}$$

We obtain

$$\bar{\phi}_0=F_0(\bar{r})=\frac{-\bar{r}^2[\mathrm{sgn}(\mu)-\frac{1}{2}\bar{r}^2]}{\frac{1}{2}(Q_1+Q_2)}\ . \tag{A9}$$

Hence, in leading order in the small parameter $|\mu|$ the nonequilibrium potential is only dependent on the trace Q_1+Q_2 of the transport matrix.

In the original variables we obtain to this order

$$\phi=\frac{-\mu r^2+\frac{1}{2}r^4}{\frac{1}{2}(Q_1+Q_2)}\ . \tag{A10}$$

In order to determine the correction $\bar{\phi}_1$ we have to fix $F_1(\bar{r})$. This makes it necessary to study the solvability condition for $\bar{\phi}_2$. It then turns out that the periodicity of $\bar{\phi}_2$ in φ makes it necessary to set $F_1(\bar{r})=\text{const}$. We are left with the result

$$\bar{\phi}_1=\frac{Q_1-Q_2}{\frac{1}{2}(Q_1+Q_2)^2}\bar{r}^2[\mathrm{sgn}(\mu)-\bar{r}^2]^2\sin(2\varphi) \tag{A11}$$

and find to this order

$$\phi=\frac{-\mu r^2+\frac{1}{2}r^4}{\frac{1}{2}(Q_1+Q_2)}+\frac{(Q_1-Q_2)}{\frac{1}{2}(Q_1+Q_2)^2}r^2(\mu-r^2)^2\sin(2\varphi)\ . \tag{A12}$$

Higher-order terms could be calculated by repeating this procedure. Several points are noteworthy. First we recall that already the correction $\bar{\phi}_1$ which we have determined actually goes beyond the accuracy of the local normal form (2.4), which is valid only in leading order in μ. Hence, it is consistent to work with the expression (A10) sufficiently close to the bifurcation point. Second, the expansion in $|\mu|$ which we have carried out turns out to be equivalent to an expansion in the ratio $(Q_1-Q_2)/(Q_1+Q_2)$ as can be seen from Eq. (A12). Sufficiently close to the bifurcation point we may therefore put $Q_1=Q_2$. Physically, this is due to the fact that the oscillation at the Hopf bifurcation is very rapid on the time scale set by $|\mu|$ and tends to average over the two orthogonal directions in state space preferred by the noise sources. Finally we note from Eq. (A12) that the extrema of ϕ at $r=0$ and $r=\sqrt{\mu}$ for $\mu>0$ are not changed by the correction $\bar{\phi}_1$ and remain being determined by the attractors and repellors of the local dynamical system (2.4).

APPENDIX B

Here we give a more explicit expression for the potential than in the main text for the symmetric case of Sec. III C 1. For $\nu_1<0$, and for $\nu_1>0$ if the energy is positive, the x coordinate may vary in the interval $-|x_1(E)|\le x\le +|x_1(E)|$, where

$$x_1^2(E)=\nu_1+(\nu_1^2+4E)^{1/2} . \tag{B1}$$

The integrals $\bar{v}_n(E)$ can be expressed in terms of hypergeometric functions[36] and we obtain

$$\frac{\bar{v}_2(E)}{\bar{v}(E)}=\frac{x_1^2(E)}{4}\frac{F(-\frac{1}{2},\frac{3}{2},3,x_1^2(E)/[2\nu_1-x_1^2(E)])}{F(-\frac{1}{2},\frac{1}{2},2,x_1^2(E)/[2\nu_1-x_1^2(E)])} . \tag{B2}$$

For $\nu_1>0$ in the energy range $-\nu_1^2/4\le E\le 0$ there are two accessible regions $-|x_1(E)|\le x\le -|x_2(E)|$; $|x_2(E)|\le x\le |x_1(E)|$ where $x_1(E)$ is given by (B1), and

$$x_2^2(E)=\nu_1-(\nu_1^2+4E)^{1/2}>0 . \tag{B3}$$

Since the system is symmetric one finds the same averages $\bar{v}_n(E)$ in both intervals. The evaluation of integrals yields

$$\frac{\bar{v}_2(E)}{\bar{v}(E)}=x_2^2(E)\frac{F(\frac{3}{2},-\frac{1}{2},3,1-x_1^2(E)/x_2^2(E))}{F(\frac{3}{2},\frac{1}{2},3,1-x_1^2(E)/x_2^2(E))} , \tag{B4}$$

which joins continuously to (B2) at $E=0$.

In the case of a vanishing noise in the first equation of (3.30) ($Q_1=0$) the potential is given as

$$\phi(x,v)=\epsilon\phi_1(E)=-\frac{2\epsilon\nu_2}{Q_2}(E-E_0)-\frac{2\epsilon b}{Q_2}\int_{E_0}^{E}\frac{\bar{v}_2(\tilde{E})}{\bar{v}(\tilde{E})}d\tilde{E} , \tag{B5}$$

where (3.26) has been used. This example illustrates that the potential is typically nonpolynomial in spite of the fact that the drift [given by (3.30)] is a simple polynomial expression.

APPENDIX C

We consider here the case that the coefficients a and b in Eq. (4.1) and b and c in Eq. (5.1) have equal sign, respectively. Rescaling the variables and parameters of Eq. (4.1) by

$$r=\epsilon\bar{r},\ z=\epsilon\bar{z},\ \mu_1=\epsilon\nu_1,\ \mu_2=\epsilon^2\nu_2 ,\quad t=\bar{t}/\epsilon,\ \omega=\epsilon\bar{\omega} , \tag{C1}$$

and omitting the bars henceforth, for simplicity of the notation, we obtain

$$\dot{r}=arz+\nu_1 r+O(\epsilon) ,\quad \dot{z}=\nu_2+br^2-z^2+O(\epsilon) ,\quad \dot{\theta}=\omega . \tag{C2}$$

As long as a and b have equal sign it is always possible to *choose* a transport matrix $Q^{\nu\mu}$ in such a way that the dynamical system (B2) is derivable from a potential

$$Q^{\nu\mu}=\begin{bmatrix} Q_1 & 0 & 0 \\ 0 & 2\frac{b}{a}Q_1 & 0 \\ 0 & 0 & \frac{Q_1}{r^2} \end{bmatrix} . \tag{C3}$$

Then Eq. (B2) takes the form

$$\dot{r}=-\frac{Q_1}{2}\frac{\partial\phi}{\partial r} ,\quad \dot{z}=-\frac{bQ_1}{a}\frac{\partial\phi}{\partial z},\quad \dot{\theta}=\omega \tag{C4}$$

with

$$\phi=-\frac{\nu_1}{Q_1}r^2-\frac{a\nu_2}{bQ_1}z+\frac{a}{3bQ_1}z^3-\frac{ar^2z}{Q_1} . \tag{C5}$$

The bifurcations of the system now follow from a discussion of the extrema of this potential in a straightforward manner. In the case of Eq. (5.1) we proceed in a similar way and choose

$$Q^{\nu\mu}=\begin{bmatrix} Q_1 & 0 & 0 & 0 \\ 0 & \frac{cQ_1}{b} & 0 & 0 \\ 0 & 0 & \frac{Q_1}{r_1^2} & 0 \\ 0 & 0 & 0 & \frac{cQ_1}{br_2^2} \end{bmatrix} . \tag{C6}$$

The potential obtained in this case is

$$\phi=-\frac{\mu_1}{Q_1}r_1^2-\frac{b\mu_2}{cQ_1}r_2^2+\frac{1}{2Q_1}\left[r_1^4+\frac{bd}{c}r_2^4+2br_1^2r_2^2\right] . \tag{C7}$$

The attractors of the system and their bifurcations now again follow from a discussion of the extrema.

APPENDIX D

Here we wish to compute for the cases of Secs. III and IV the leading correction in η to Eq. (1.7). We set

$$W(q,\eta)=Z(q)\exp\left[-\frac{\phi(q)}{\eta}+O(\eta)\right] \quad \text{(D1)}$$

and determine $Z(q)$ as a solution of Eq. (3),

$$\left[K^\nu+Q^{\nu\mu}\frac{\partial\phi}{\partial q^\mu}\right]\frac{\partial Z}{\partial q^\nu}+\left[\frac{\partial K^\nu}{\partial q^\nu}+\frac{1}{2}Q^{\nu\mu}\frac{\partial^2\phi}{\partial q^\nu \partial q^\mu}\right]Z=0\ . \quad \text{(D2)}$$

1. General case of Sec. III

With Eq. (3.1) we obtain from Eq. (D2)

$$\left[v+Q_1\frac{\partial\phi}{\partial x}\right]\frac{\partial Z}{\partial x}+\left[\mu v g-V'+Q_2\frac{\partial\phi}{\partial v}\right]\frac{\partial Z}{\partial v}+\left[\mu g+\frac{Q_1}{2}\frac{\partial^2\phi}{\partial x^2}+\frac{Q_2}{2}\frac{\partial^2\phi}{\partial v^2}\right]Z=0\ . \quad \text{(D3)}$$

Changing the variables x,v into x,E according to Eq. (3.5) and using

$$\phi(x,\pm v(x,E))\equiv\phi^\pm(x,E)\ ,$$
$$Z(x,\pm v(x,E))\equiv Z^\pm(x,E)\ , \quad \text{(D4)}$$

we obtain for $Z^\pm$,

$$\frac{\partial Z^\pm}{\partial x}\left[\pm v+Q_1\left[\frac{\partial\phi^\pm}{\partial x}+V'\frac{\partial\phi^\pm}{\partial E}\right]\right]+\frac{\partial Z^\pm}{\partial E}\left[\mu v^2 g+Q_2 v^2\frac{\partial\phi^\pm}{\partial E}+Q_1 V'\left[\frac{\partial\phi^\pm}{\partial x}+V'\frac{\partial\phi^\pm}{\partial E}\right]\right]$$
$$+Z^\pm\left[\mu g+\frac{Q_1}{2}\left[\frac{\partial^2\phi^\pm}{\partial x^2}+2V'\frac{\partial^2\phi^\pm}{\partial x\,\partial E}+V'^2\frac{\partial^2\phi^\pm}{\partial E^2}+V''\frac{\partial\phi^\pm}{\partial E}\right]+\frac{Q_2}{2}\left[v^2\frac{\partial^2\phi^\pm}{\partial E^2}+\frac{\partial\phi^\pm}{\partial E}\right]\right]=0\ . \quad \text{(D5)}$$

We solve Eq. (D5) perturbatively in μ putting

$$\phi^\pm=\mu F_1(E)+\mu^2\phi_2^\pm+\cdots\ ,$$
$$Z^\pm=Z_0+\mu Z_1^\pm+\cdots\ . \quad \text{(D6)}$$

In zeroth order of μ we obtain with yet arbitrary $G_0(E)$,

$$\ln Z_0^\pm=G_0(E)\ . \quad \text{(D7)}$$

In first order of μ we have to solve

$$\frac{\partial \ln Z_1^\pm}{\partial x}=\mp\left[vg+\left[Q_2 v+Q_1\frac{V'^2}{v}\right]F_1'\right]G_0'$$
$$\mp\left[\frac{g}{v}+\frac{1}{2}\left[Q_1\frac{V'^2}{v}+Q_2 v\right]F_1''+\frac{1}{2}\left[Q_1\frac{V''}{v}+Q_2\frac{1}{v}\right]F_1'\right]\ . \quad \text{(D8)}$$

Integrating (D8) along a closed contour $E=$const over x and using the single-valuedness of $Z_1^\pm$ we obtain

$$G_0'=-\frac{\bar v_g^{(-)}+\frac{1}{2}(Q_2\bar v+Q_1\bar w)F_1''+\frac{1}{2}(Q_1\bar u+Q_2\bar v^{(-)})F_1'}{\bar v_g+(Q_2\bar v+Q_1\bar w)F_1'}\ , \quad \text{(D9)}$$

where $\bar w$ was defined in Eq. (3.20) and

$$\bar v_g^{(-)}=\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\frac{g}{v}dx\ , \quad \text{(D10)}$$

$$\bar v^{(-)}=\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\frac{dx}{v}\ , \quad \text{(D11)}$$

$$\bar u=\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\frac{V''}{v}dx\ . \quad \text{(D12)}$$

Using now Eq. (3.18) and its consequence

$$F_1''=-2\frac{\bar v_g^{(-)}}{Q_1\bar w+Q_2\bar v}+2\frac{\bar v_g}{(Q_1\bar w+Q_2\bar v)^2}(Q_1\bar u+Q_2\bar v^{(-)})\ , \quad \text{(D13)}$$

where $\bar v_g$, $\bar v$ were defined in Eqs. (3.19) and (3.21), Eq. (D9) reduces to

$$G_0'=0\ , \quad \text{(D14)}$$

i.e.,

$$G_0=\text{const}\ . \quad \text{(D15)}$$

Thus, to lowest order in μ, the prefactor Z in Eq. (4.1) is independent of x and v. The expansion can now be carried to higher orders, if desired. In the limit of vanishing dissipation our results agree with recent results of Reibold[37] who assumed exactly vanishing dissipation.

2. Case of Section IV

With Eq. (4.3) we obtain from Eq. (D2)

$$\left[arz+\epsilon g_r+Q_1\frac{\partial\phi}{\partial r}\right]\frac{\partial Z}{\partial r}+\left[v_2+br^2-z^2+\epsilon z g_z+Q_2\frac{\partial\phi}{\partial z}\right]\frac{\partial Z}{\partial z}$$
$$+\left[(a-2)z+\epsilon\frac{\partial g_r}{\partial r}+\epsilon\frac{\partial z g_2}{\partial z}+\frac{Q_1}{2}\frac{\partial^2\phi}{\partial r^2}+\frac{Q_2}{2}\frac{\partial^2\phi}{\partial z^2}\right]Z=0\ . \quad \text{(D16)}$$

Introducing the independent variables r,u according to Eq. (4.8) and using with (4.9)

$$\phi(r,\pm z(r,u))\equiv\phi^{\pm}(r,u)\ ,\quad Z(r,\pm z(r,u))\equiv Z^{\pm}(r,u)\ , \tag{D17}$$

we obtain for $Z^{\pm}(r,u)$,

$$\left[\frac{\partial Z^{\pm}}{\partial r}-r^{2/a}\frac{\partial z^2}{\partial r}\frac{\partial Z^{\pm}}{\partial u}\right]\left[\pm arz+\epsilon g_r+Q_1\left[\frac{\partial\phi^{\pm}}{\partial r}-r^{2/a}\frac{\partial z^2}{\partial r}\frac{\partial\phi^{\pm}}{\partial u}\right]\right]\pm 2zr^{2/a}\frac{\partial Z^{\pm}}{\partial u}\left[\nu_2+br^2-z^2\pm\epsilon zg_z\pm 2zr^{2/a}Q_2\frac{\partial\phi^{\pm}}{\partial u}\right]$$
$$+Z^{\pm}\left\{\pm(a-2)z+\epsilon\frac{\partial g_r}{\partial z}+\epsilon\frac{\partial zg_z}{\partial z}\right.$$
$$+\frac{Q_1}{2}\left[\frac{\partial^2\phi^{\pm}}{\partial r^2}-2r^{2/a}\frac{\partial z^2}{\partial r}\frac{\partial^2\phi^{\pm}}{\partial r\,\partial u}+r^{4/a}\left[\frac{\partial z^2}{\partial r}\right]^2\frac{\partial^2\phi^{\pm}}{\partial u^2}-\frac{\partial\phi^{\pm}}{\partial u}\left[\frac{4}{a}r^{2/a-1}\frac{\partial z^2}{\partial r}+r^{2/a}\frac{\partial^2 z^2}{\partial r^2}\right]\right]$$
$$\left.+\frac{Q_2}{2}\left[4z^2r^{4/a}\frac{\partial^2\phi^{\pm}}{\partial u^2}+2r^{2/a}\frac{\partial\phi^{\pm}}{\partial u}\right]\right\}=0\ . \tag{D18}$$

We solve Eq. (4.18) perturbatively in ϵ setting

$$\phi^{\pm}=\epsilon F_1(u)+\epsilon^2F_2^{\pm}+\cdots\ ,\quad Z^{\pm}=Z_0+\epsilon Z_1^{\pm}+\cdots\ . \tag{D19}$$

In zeroth order in ϵ we obtain with yet arbitrary $G_0(u)$,

$$Z_0=r^{2/a-1}G_0(u)\ . \tag{D20}$$

In first order in ϵ we set

$$Z_1^{\pm}=r^{2/a-1}G_1^{\pm}(r,u) \tag{D21}$$

and have to solve the equation

$$\frac{\partial G_1^{\pm}}{\partial r}=\pm[G_0(u)A(r,u)+G_0'(u)B(r,u)] \tag{D22}$$

with

$$A(r,u)=\frac{1}{az}\left[\frac{2}{a}-1\right]r^{-2}g_r+\frac{1}{az}r^{-1}\left[\frac{\partial g_r}{\partial r}+\frac{\partial zg_z}{\partial z}\right]$$
$$+\frac{r^{2/a-2}}{az}F_1'\left[\left[1-\frac{2}{a}\right]\frac{\partial z^2}{\partial r}Q_1\right.$$
$$\left.-\frac{1}{2}\left[r\frac{\partial^2 z^2}{\partial r^2}+\frac{4}{a}\frac{\partial z^2}{\partial r}\right]Q_1+rQ_2\right]$$
$$+\frac{r^{4/a-1}}{az}\left[\frac{Q_1}{2}\left[\frac{\partial z^2}{\partial r}\right]^2+2Q_2z^2\right]F_1''\ , \tag{D23}$$

$$B(r,u)=-\frac{2}{a}r^{2/a-1}\frac{\partial z}{\partial r}g_r+\frac{2z}{a}r^{2/a-1}g_z$$
$$+F_1'\frac{r^{4/a-1}}{az}\left[Q_1\left[\frac{\partial z^2}{\partial r}\right]^2+4Q_2z^2\right]\ . \tag{D24}$$

Integrating Eq. (D22) along a closed contour $u=$const over r and using the single valuedness of $Z_1^{\pm}$ we obtain

$$\frac{d\ln G_0(u)}{du}=-\frac{\int_{r_1}^{r_2}A(r,u)dr}{\int_{r_1}^{r_2}B(r,u)dr}\ , \tag{D25}$$

where r_1,r_2 are as in Eq. (4.16).

We now use the abbreviations (4.16) and their consequences

$$g'(u)=\tfrac{1}{2}\int_{r_1}^{r_2}r^{-1}\left[\frac{1}{z}\frac{\partial g_r}{\partial r}+\frac{1}{z}\left[\frac{2}{a}-1\right]\frac{g_r}{r}+\frac{1}{z}g_z+\frac{\partial g_z}{\partial z}\right.$$
$$\left.+\frac{\left[\frac{2}{a}-1\right]}{r}\frac{\partial g_r}{\partial z}+\frac{\partial^2 g_r}{\partial r\,\partial z}\right]\ , \tag{D26}$$

$$Q'(u)=\int_{r_1}^{r_2}dr\,r^{2/a-1}\left[\left[\frac{1}{a}-1\right]\left[\frac{6}{a}-1\right]Q_1r^{-2}z\right.$$
$$\left.-\frac{Q_1}{4z}\frac{\partial^2 z^2}{\partial r^2}+\frac{Q_2}{2z}\right]\ . \tag{D27}$$

We remark that some care must be exercised when deriving (D26) and (D27), to bring the integrands of (4.16) into a form, using partial integration, which vanishes at the boundaries of the integral, in order not to pick up contributions from the boundaries of the integrals, when differentiating the integral with respect to u. We can now use these expressions, after some algebra, to rewrite the nominator and denominator of Eq. (D25) as

$$\int_{r_1}^{r_2}B(r,u)dr=\frac{2}{a}g(u)+\frac{4}{a}Q(u)F_1'(u)\ , \tag{D28}$$

$$\int_{r_1}^{r_2} A(r,u)dr$$
$$=\frac{2}{a}Q(u)F_1''(u)+\frac{2}{a}Q'(u)F_1'(u)+\frac{2}{a}g'(u)$$
$$+\frac{1}{a}\left[\frac{2}{a}-1\right]\left[\frac{2}{a}-2\right]\int_{r_1}^{r_2} dr\, r^{2/a-3}zF_1'(u)$$
$$-\frac{4d}{a^2}\int_{r_1}^{r_2}\frac{z}{r}dr\ . \tag{D29}$$

Using, finally, Eq. (4.15) and its derivative in (D28) and (D29), we find from Eq. (D25),

$$\frac{d\ln G_0(u)}{du}=-\frac{1}{2}\frac{Q_1}{Q(u)}\left[\frac{2}{a}-1\right]\left[\frac{2}{a}-2\right]\int_{r_1}^{r_2} dr\, r^{2/a-3}z$$
$$-\frac{2d}{ag(u)}\int_{r_1}^{r_2}\frac{z}{r}dr\ , \tag{D30}$$

from which $G_0(u)$ now follows by quadrature. Thus, for Q_1 and d not both vanishing the prefactor Z, even in lowest order of ϵ, is not a constant but depends on u.

APPENDIX E

Here we present the algebra leading from Eq. (5.24) to Eq. (5.27). Let us define the integrals

$$\mathscr{I}_{nm}=\oint dr_2\, r_1^\alpha r_2^{\beta-1} r_1^n r_2^m\ . \tag{E1}$$

First we reduce the integral

$$\mathscr{I}_{40}=\oint dr_2\, r_1^\alpha r_2^{\beta-1} r_1^4=-\frac{\alpha+4}{\beta}\oint r_2^\beta r_1^{\alpha+3}\,dr_1 \tag{E2}$$

by using Eq. (5.9) and the fact that

$$\oint u r_1^n\,dr_1=\oint u r_2^n\,dr_2=0\ . \tag{E3}$$

We obtain

$$\mathscr{I}_{40}=-\frac{\alpha+4}{\beta}\oint r_2^\beta r_1^{\alpha+1}\left[\nu_1+\frac{d\beta}{\alpha}r_2^2\right]dr_1 \tag{E4}$$

and upon partial integration

$$\mathscr{I}_{40}=\frac{\alpha+4}{\alpha+2}(\nu_1\mathscr{I}_{20}-b\mathscr{I}_{22})\ . \tag{E5}$$

The integral $\mathscr{I}_{20}$ can be reduced directly by using (5.9) and we find

$$\mathscr{I}_{20}=\nu_1\mathscr{I}_{00}+\frac{d\beta}{\alpha}\mathscr{I}_{02}\ . \tag{E6}$$

Similarly, we obtain for $\mathscr{I}_{22}$ by the use of Eq. (5.9)

$$\mathscr{I}_{22}=\nu_1\mathscr{I}_{02}+\frac{d\beta}{\alpha}\mathscr{I}_{04}\ . \tag{E7}$$

Let us now use these relations to express

$$g(u)=\beta\left[\frac{\beta}{\alpha}\nu_2\mathscr{I}_{00}-k_1\mathscr{I}_{40}-k_2\mathscr{I}_{22}-k_3\mathscr{I}_{04}\right] \tag{E8}$$

in the form

$$g(u)=\beta(B\mathscr{I}_2+D\mathscr{I}_1+E\mathscr{I}_{00}) \tag{E9}$$

with the notation

$$\mathscr{I}_n=\oint r_1^\alpha r_2^{\beta-1}z^n dr_2\ , \tag{E10}$$

where z was defined in Eq. (5.26).

This can be done by adding to Eq. (E8) the identically vanishing terms [cf. Eqs. (E5)—(E7)]

$$\beta\lambda_3[(\alpha+2)\mathscr{I}_{40}-(\alpha+4)\nu_1\mathscr{I}_{20}+(\alpha+4)b\mathscr{I}_{22}]$$
$$+\beta\lambda_2\left[\mathscr{I}_{22}-\nu_1\mathscr{I}_{02}-\frac{d\beta}{\alpha}\mathscr{I}_{04}\right]$$
$$+\beta\lambda_1\left[\mathscr{I}_{20}-\nu_1\mathscr{I}_{00}-d\frac{\beta}{\alpha}\mathscr{I}_{02}\right] \tag{E11}$$

with arbitrary parameters $\lambda_1,\lambda_2,\lambda_3$. This extended expression can then be compared with Eq. (E9) noting the relations

$$\begin{aligned}\mathscr{I}_1&=-\nu_1\frac{\alpha}{\beta}\mathscr{I}_{00}-c\mathscr{I}_{20}-d\mathscr{I}_{02}\ ,\\ \mathscr{I}_2&=\nu_1^2\frac{\alpha^2}{\beta^2}\mathscr{I}_{00}+\frac{2\alpha c}{\beta}\nu_1\mathscr{I}_{20}+\frac{2\alpha d}{\beta}\nu_1\mathscr{I}_{02}\\ &\quad+2cd\mathscr{I}_{22}+c^2\mathscr{I}_{40}+d^2\mathscr{I}_{04}\ .\end{aligned} \tag{E12}$$

We find after some algebra

$$\begin{aligned}\lambda_1&=-\frac{\alpha\nu_1}{2(\alpha+2)}\left[4Bc^2+(\alpha+4)k_1-\frac{\alpha c^2}{d^2}k_3\right]\ ,\\ \lambda_2&=\frac{\alpha c}{d}\frac{(k_3+bd^2)}{\alpha+2}\ ,\\ \lambda_3&=\frac{Bc^2+k_1}{\alpha+2}\ ,\end{aligned} \tag{E13}$$

and the values of B,D,E given in Eq. (5.28).

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