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Renyi dimensions from local expansion rates

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A general self-similarity relation is shown to exist, expressing the Renyi-dimension function in terms of local expansion rates both for flows and maps. For the particular case of the information dimension, such an implicit equation yields the well-known Kaplan-Yorke relation. Moreover, it can be explicitly solved in some interesting cases, among which are two-dimensional maps with constant Jacobian. Detailed measurements are performed for the Hénon attractor, with a very accurate estimate of its capacity. Finally, an expansion around the information dimension allows recovery of the Grassberger-Procaccia estimates in an easy way.

I. INTRODUCTION

Several methods have been introduced to characterize strange attractors from a purely geometrical point of view, all relying on concepts developed in the theory of fractal sets.[1] The relevant information is extracted from the scaling properties of the probability distribution, when the observational resolution is increased. Accordingly, an infinity of dimensions D_q [Renyi dimension function (Ref. 2)] is commonly used to describe the geometric and probabilistic features of the attractors. Many different algorithms presently exist to evaluate some particularly meaningful dimensions [capacity D_0,[3] information dimension D_1, correlation exponent D_2 (Ref. 4)] by computing interpoint distances, or counting boxes.

However, strange attractors cannot be viewed simply as geometrical objects; their dynamical properties also require careful analysis. In particular, the Lyapunov characteristic exponents are a standard means to quantify the degree of chaos in nonlinear dynamical systems. As these exponents express the expansion rates of neighboring distances along each direction in the phase space, it has been conjectured that they could be used to estimate fractal dimensions. In fact, Kaplan and Yorke[5] introduced a dimensionlike quantity which has been proved to be an upper bound to the information dimension.[6] Anyway, for the cases usually encountered in the literature, the Kaplan-Yorke relation is found to hold as an equality, within the numerical accuracy. However, typical chaotic attractors are not uniform, i.e, the Renyi dimensions D_q are such that $D_{+\infty} < D_{-\infty}$; therefore it would be desirable to have a relation which yields all the D_q's in terms of the expansion rates. A partial solution to this problem was given by Grassberger and Procaccia,[7] who attributed the nonuniformity to the fluctuations of the Lyapunov exponents along the trajectory. They conjectured that the evolution of the Lyapunov exponents could be described approximately by a set of Langevin equations with a Gaussian random source. With such assumptions they were able to derive upper bounds to all the D_q's which, however, are in general not exact, even when non-Gaussian corrections are approximately taken into account.

In this paper we derive an implicit exact relation yielding all the Renyi dimensions (for the whole q axis) in terms of averages of suitable powers of the local volume-expansion rates. Our approach, based on self-similarity arguments, is first introduced for the paradigmatic example of the generalized baker transformation[8] and then extended to generic flows and maps. For several dynamical systems, for which the volume contraction rate is constant, an explicit equation for the D_q's can be written. This allows easy and fast numerical calculations. As an example, the Hénon map is discussed in detail, and the capacity D_0 is provided. For the same class of models, the Renyi-dimension function is also shown to depend in a simple way on the cumulant generating function for the probability distribution of local Lyapunov exponents.

The Sinai map is then analyzed, as an example of a highly nonuniform attractor,[9] and for its fluctuating Jacobian, which requires an indirect approach.

In Sec. II we derive the main relations, in both the implicit and explicit form. Section III is devoted to numerical applications. In Sec. IV the expansion of the dimension function allows the recovery, in a straightforward manner, of the Grassberger-Procaccia estimates,[7] which are shown to be connected to the uniformity factor.[10] In Sec. V we summarize the main results, commenting on the reliability of the tails of the distribution of Lyapunov exponents. Finally, the relation between expansion rates and metric entropies is also sketched.[11]

II. THEORETICAL RESULTS

We start by discussing the simple, analytically solvable, generalized baker transformation,[8]

$$\begin{aligned} x_{n+1} &= x_1/\alpha_a \ , \\ y_{n+1} &= y_n/p_a, \ \ y_n < p_a \ ; \qquad (2.1\text{a}) \\ x_{n+1} &= x_n/\alpha_b + (1-1/\alpha_b) \ , \end{aligned}$$

$$y_{n+1}=(y_n-p_a)/p_b, \quad y_n>p_a \; ; \tag{2.1b}$$

where $p_a+p_b=1$ and $\alpha_a^{-1}+\alpha_b^{-1}<1$. Its asymptotic attractor can easily be recognized as the product of a continuum by a Cantor set. The generalized dimensions d_q of a horizontal section are rigorously defined through the self-similarity relation[12]

$$p_a^q \alpha_a^{-(1-g)d_q}+p_b^q \alpha_b^{-(1-g)d_q}=1 \; . \tag{2.2}$$

This equation, which was derived from geometrical arguments, can also be interpreted from a dynamical point of view. In fact, the action of the map can be summarized in the following way. The unit square is cut horizontally at a height $y=p_a$ and the two parts contracted in the x direction by a factor μ_2 which can either be equal to α_a^{-1} or to α_b^{-1}, according to the value of y [see Eq. (2.1)], with probability p_a and p_b, respectively. The two resulting rectangles are then stretched along the y direction by a factor μ_1, which assumes the value p_a^{-1} in the lower part, and p_b^{-1} in the upper one. The numbers μ_1 and μ_2 are the local (i.e., position-dependent) multipliers that, in generic systems, can be either positive or negative. As only their magnitude is important for what concerns calculations of fractal dimension and metric entropy, we only consider their absolute values, indicated in the following by m_1 and m_2.

Exploiting these considerations, we can rewrite Eq. (2.2) as a time average along the trajectory,

$$\langle (m_1 m_2^{d_q})^{(1-q)} \rangle = \langle [V(q)]^{(1-q)} \rangle = 1 \; . \tag{2.3}$$

Here, the quantity $V(q)$ stands for a volume-contraction rate in the metric in which we assign a unitary dimension $d=1$ to the stretching direction ($m_1>1$) and a fractal dimension $d=d_q$ to the contracting one ($m_2<1$). If the factor $V(q)$ does not fluctuate [for a choice of d_q such that the equality holds in Eq. (2.3)], then we can eliminate the average and, consequently, the q dependence. In such a case, Eq. (2.3) implies that the overall dimension is $D_q=1+d_q=1+\ln m_1/|\ln m_2|$. The independence of q reflects the uniformity of the set, which is obtained for $p_i \propto \alpha_i^{-d}$ ($i=a,b$).

The previous considerations indicate that the nonuniformity of a fractal set originates from the fluctuations of the product $V(q)$. However, an important condition to be verified by Eq. (2.3) for it to be a meaningful relation is its independence of the number of iterations of the map for which the multipliers are evaluated. In the case of the transformation (2.1), the value of $V(q)$ is completely decorrelated from its value at the previous iteration. This ensures, *a fortiori*, the validity of Eq. (2.3) when interpreting m_1 and m_2 as multipliers over any number of iterations. In general, instead, we expect correlations to exist between $V(q,t)$ and $V(q,t+\tau)$ [where $V(q,t)$ stands for $V(q,\mathbf{x}(t))$]. Hence, it is necessary to compute the multipliers over a time τ, large enough to allow the decay of correlations. In the limit $\tau\to\infty$, we expect only the fluctuations owing to the nonuniformity to remain.

We can now easily extend the previous results to the general case of an E-dimensional phase space, by considering the volume-expansion rate over a time τ, evaluated from the starting point x_i on the trajectory

$$V_i(q,\tau)\equiv \prod_{k=1}^{E} m_k^{d_q^{(k)}}(\mathbf{x}_i,\tau)\equiv \prod_{k=1}^{E} e^{\tau d_q^{(k)}\lambda_k(\mathbf{x}_i)} \; , \tag{2.4}$$

where $m_k(\mathbf{x}_i,\tau)$ are the multipliers (in absolute value) over a time τ, computed by evolving the linearized equations; the $d_q^{(k)}$ are the "partial dimensions"[13] and $\lambda_k(\mathbf{x}_i)=(1/\tau)\ln m_k(\mathbf{x}_i,\tau)$ are the Lyapunov exponents computed over the finite time τ. The partial dimensions $d_q^{(k)}$ are a useful tool to characterize the local structure of the attractor, and can be roughly interpreted as the dimensions along the eigendirections of the linearized flow. In general, it is not known how to compute them directly, however, they can be used to yield upper bounds to the Renyi dimensions D_q. In the case of axiom-A systems, all $d_q^{(k)}$'s corresponding to expanding directions are rigorously equal to 1.[13(b)] Furthermore, the $d_q^{(k)}$'s, while satisfying, for $q>0$, the obvious constraint $0\le d_q^{(k)}\le 1$, for $q<0$, can be larger than one, as will be seen in the following. The Renyi dimensions D_q are then given by

$$D_q=\sum_{k=1}^{E} d_q^{(k)} \; . \tag{2.5}$$

Finally, Eq. (2.3) can be rewritten as

$$\lim_{\tau\to\infty} \langle V(q,\tau)^{(1-q)} \rangle = 1 \; . \tag{2.6}$$

This relation can be further justified by noting that it is a constraint for the dimension list[13] $d_q^{(k)}$, determining the proper metric which keeps the volume-expansion rate $[V(q,\tau)]^{(1-q)}$ equal to 1, on the average. In particular, taking the limit $q\to 1$, we obtain the general condition

$$\sum_{k=1}^{E} d_1^{(k)} \langle \lambda_k \rangle = 0 \tag{2.7}$$

for the information dimension D_1, which can be satisfied by various choices of the $d_1^{(k)}$'s. The Kaplan-Yorke choice is recovered by defining j as the largest integer for which $\sum_{k=1}^{j}\langle\lambda_k\rangle\ge 0$, and choosing $d_1^{(k)}=1$ for $k\le j$, $d_1^{(j+1)}=\sum_{k=1}^{j}\langle\lambda_k\rangle/|\langle\lambda_{j+1}\rangle|$, and $d_q^{(k)}=0$ for $k>j+1$. This is the choice that maximizes the estimate of the information dimension D_1 and it is equivalent to assuming that the attractor is Cantor along the $(j+1)$th direction only. It has been proven[14] that such a choice is exact for two-dimensional diffeomorphisms. For higher-dimensional sets, one needs additional information on the d_q's to satisfy Eq. (2.6), as shown by Shtern[15] for a particular version of the baker map. However, the Kaplan-Yorke relation seems to hold as an equality for all the attractors usually studied in the literature. To obtain analogous upper bounds for the case $q\ge 0$, we can proceed as follows. Starting with the largest possible value for D_q (i.e., $d_q^k=1$, $k=1,2,\ldots,E$) it is easily seen that the average volume-contraction rate is smaller than 1, so that D_q must be decreased to meet the requirement (2.6). The smallest possible decrement is accomplished via a gradual decrease of the dimensions $d_q^{(k)}$ starting from the last one ($d_q^{(E)}$). As a result, defining the generalized index j_q as the largest integer for which

$$\lim_{\tau\to\infty} \left\langle \prod_{k=1}^{j_q} e^{\tau(1-q)\lambda_k(\mathbf{x}_i)} \right\rangle^{1/\tau(1-q)} \ge 1 \; , \tag{2.8}$$

the upper bound for D_q corresponds to $d_q^{(k)}=1$, for $k \le j_q$, $d_q^{(k)}=0$ for $k > j_q+1$, and $d_q^{(j_q+1)}$ determined by relation (2.6). To clarify the meaning of Eqs. (2.6) and (2.8), note that $\lambda_k(\mathbf{x}_i)$, computed over a time τ, is a random variable whose probability distribution depends on τ itself. Therefore, a nontrivial dependence of j_q on q is expected, indicating that the Renyi dimension function D_q may cross one (or more) integer values. In such a case, the attractor could not be simply schematized as the product of a continuum and a Cantor set. A generic example of this behavior is provided by filtered chaotic signals.[16] More caution is necessary for $q<0$, since in principle, the Cantor direction can yield $d_q^{(k)}>1$.[17] However, in case j_q does not change for $q \to -\infty$, the procedure described above yields unambiguously an upper bound to D_q.

In the following, for simplicity, we consider j_q constant and one single Cantor direction, so that, from here on, we indicate $d_q^{(j+1)}$ simply with d_q. For the fulfillment of the self-similarity relation (2.6), one needs to assign several trial values of d_q, determining the right one by interpolation. Fortunately, such a lengthy procedure is not necessary for some dynamical systems with constant Jacobian or divergence. In fact, for an E-dimensional map, we have

$$\prod_{k=1}^{E} m_k(\tau) = |J|^{\tau} . \tag{2.9}$$

If $E=j+1$, we solve Eq. (2.9) for $m_{j+1}(\tau)$ and substitute into Eq. (2.6) obtaining

$$\langle [m_1(\tau)m_2(\tau)\cdots m_j(\tau)]^{(1-d_q)(1-q)} \rangle = |J|^{-\tau d_q(1-q)} . \tag{2.10}$$

Hence, introducing the new variable $\Theta=(1-q)(1-d_q)$, we have

$$D(\Theta)=j+\left[1+|\ln|J||\Big/\lim_{\tau\to\infty}\left[\ln\left\langle\prod_{k=1}^{j} m_k^{\Theta}(\tau)\right\rangle\Big/\Theta\tau\right]\right]^{-1} , \tag{2.11}$$

where $D(\Theta)$ is another dimension function[17] obtained through the following transformation of the q axis:

$$D[\Theta=(1-q)(1-d_q)]=D_q=j+d_q . \tag{2.12}$$

The simultaneous evaluation of $D(\Theta)$ for some Θ values yields a fairly good estimate of the whole dimension function. In particular, for $\Theta \to 0$, the Kaplan-Yorke relation is recovered while, for $\Theta=1-d_0$, the capacity D_0 is found.

III. NUMERICAL RESULTS

In this section we perform calculations of the dimension function for the Hénon attractor and the Sinai map.[18] The first model is given by

$$\begin{aligned} x_{n+1}&=1-ax_n^2+y_n , \\ y_{n+1}&=bx_n , \end{aligned} \tag{3.1}$$

with $a=1.4$ and $b=|J|=0.3$. Therefore, Eq. (2.11) reads

$$D(\Theta)=1+\left[1+\frac{|\ln b|}{\lim_{\tau\to\infty}\left[\dfrac{\ln\langle e^{\lambda_1\Theta\tau}\rangle}{\Theta\tau}\right]}\right]^{-1} , \tag{3.2}$$

where $\lambda_1(\tau)=1/\tau \ln m_1(\tau)$. Note that the function $G(\eta) = \langle e^{\lambda_1 \eta}\rangle$ is the characteristic function of the probability distribution $P(\lambda_1)$ of the (first) Lyapunov exponent. Its logarithm, entering the definition of dimension function (3.2), is the generating function for the cumulants. This suggests that an expansion around $\Theta=0$ (i.e., around the information dimension) will provide a measure of the nonuniformity in terms of the deviations of the distribution $P(\lambda_1)$ from a Gaussian. This point is further treated in Sec. IV.

In Fig. 1 we report the measured values of d_q versus q, in the range $-6<q<6$, for $\tau=50$ iterations. The averages in (3.2) have been performed over 2×10^6 independent samples of length τ. The central part of the curve $(-2<q<2)$ is very reliable, yielding, for instance, an estimated value for D_0 of $1.2755\pm5\times10^{-4}$. This value is in fairly good agreement with Grassberger's result[19] $D_0=1.28\pm0.01$, while all other values in the literature are quite a bit lower. In particular, all box-counting methods, as shown in Ref. 19, suffer from a lack of statistical convergence. In fact, it has been proven that the number of points necessary to cover a fixed fraction of the support of the attractor diverges faster than the number of boxes itself, for nonuniform attractors.[20] On the other hand, for large $|q|$, the uncertainty on d_q rapidly increases because of lack of statistical convergence. This difficulty

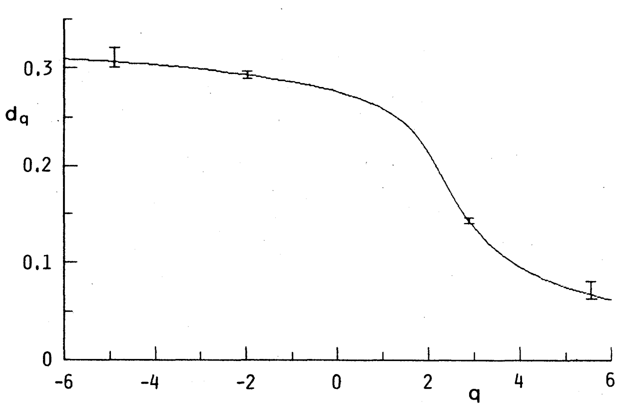

FIG. 1. Fractional part of the Renyi-dimension function d_q for the Hénon attractor. The error bar on d_q rapidly grows for increasing $|q|$.

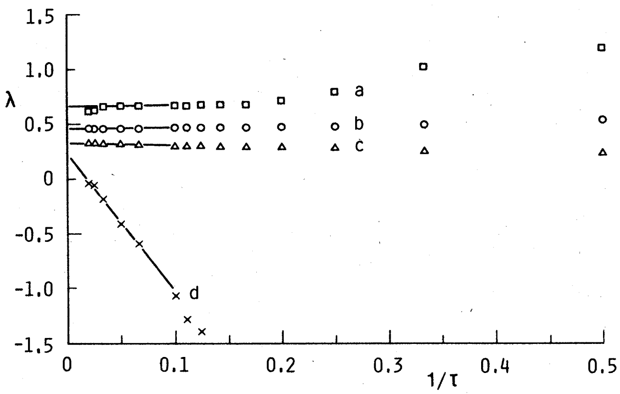

FIG. 2. The "Lyapunov exponents" $\ln G(\Theta\tau)/\Theta\tau$ are plotted for $\Theta=+\infty$, 0.725, -0.785, and $-\infty$ (curves a, b, c, and d, respectively) versus $1/\tau$. The first and the last curves correspond to the maximum and minimum value of the Lyapunov exponent, while curves b and c practically give the dependence of the correlation exponent and of the capacity on the delay time τ.

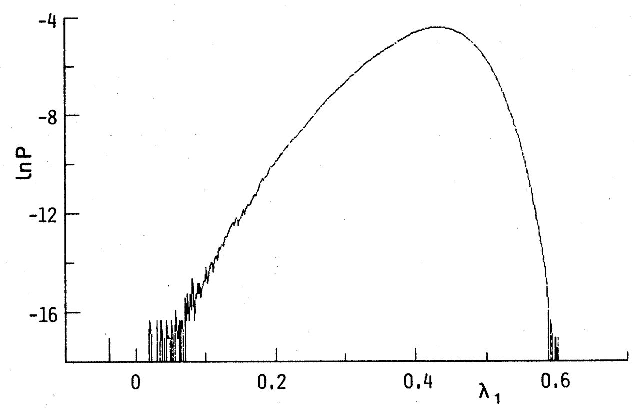

FIG. 3. The logarithm of the probability distribution of λ_1 is plotted vs the Lyapunov exponent λ_1 calculated over a time $\tau=50$. The number of computed points is 2×10^6. Deviations from the parabolic (Gaussian for the probability itself) shape are self-evident. The difficulty of a reliable estimate of P on the tails is also clear.

can be discussed with reference to Fig. 2 where $\min_\tau\{\lambda_1(\tau)\}$, $\max_\tau\{\lambda_1(\tau)\}$, and $[\ln G(\Theta\tau)]/\Theta\tau$ [for $D(\Theta)=D_0$ and D_2] are plotted versus $1/\tau$. The two central curves b and c (corresponding to the capacity and the correlation exponent, respectively) already converge for $\tau=10$. The upper curve, instead, after a plateau for τ between 5 and 25, exhibits a slow decrease with increasing τ. This is due to the broadening of the probability distribution of the values $m_1(\tau)$, which requires a prohibitively large number of samples for the estimate. This effect is more evident for low values of λ_1 and the fast increase in curve d, with $\tau\to\infty$, is mainly due to the relatively small number of samples (2×10^6). Moreover, extension of the range of values allowed for the first Lyapunov exponent λ_1 to negative values is reasonably connected to the presence of homoclinic tangencies in the Hénon attractor.[21] In such points, the unstable manifold, having the same direction as the stable one, is characterized by a contracting multiplier ($m_1<1$). A possible implication of this fact is that the global dimension D_q can become smaller than one, for q larger than a certain critical value. Finally, in Fig. 3, we report the probability distribution of the largest Lyapunov exponent λ_1, for $\tau=50$, from which can be seen the lack of definiteness of the tails, notwithstanding a number of points as large as 2×10^6. This explains the difficulty encountered in estimating the two asymptotes of the curve d_q.

The second map studied is the Sinai transformation defined by

$$x_{n+1}=x_n+y_n+g\cos(2\pi y_n) \pmod 1 ,$$
$$y_{n+1}=x_n+2y_n \pmod 1 , \tag{3.3}$$

which has a nonconstant Jacobian. Sinai proved[18] that, for small nonlinearity g, the attractor is the whole unit square, i.e., $D_0=2$. This attractor shows an increasing nonuniformity when g is increased from 0 to $1/2\pi$, when the map becomes noninvertible.[9] Here, we have chosen $g=0.15\lesssim 1/2\pi$ in order to test our method in the presence of strong fluctuations of the Lyapunov exponents. In Fig. 4 we report the average rate $M=\langle\{m_1(\tau)[m_2(\tau)]^d\}^{(1-q)}\rangle^{1/\tau}$ versus d, for $\tau=40$ and $q=0$ and 2. The intersections with the horizontal straight line with height 1 give the estimated values of the capacity and of the correlation exponent. While the expected value of $d_0=1$ is confirmed with a four-figure accuracy, the correlation exponent is $d_2=0.575\pm1\times10^{-3}$. The major source for the error is given by the residual dependence of the volume multipliers on the time τ. It is worthwhile anticipating that in the present case of a highly nonuniform set, the direct application of the exact Eq. (2.6) is by far much better than any perturbative approach, which would require a large number of terms. Finally, note that, as a consequence of the nonuniformity,

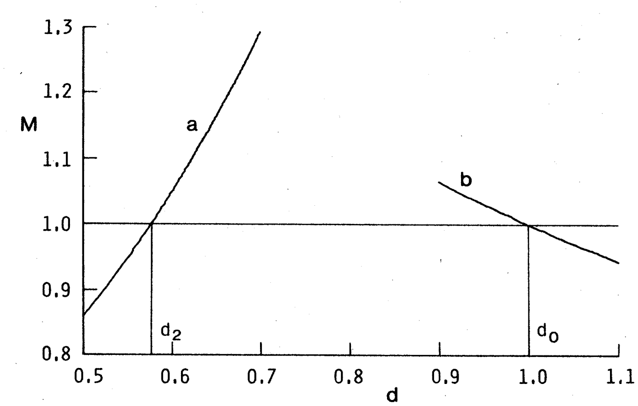

FIG. 4. Average value of the volume multiplier $M=\langle(m_1m_2^d)^{(1-q)}\rangle^{1/\tau}$ vs d for $q=0$ and 2, for the Sinai map [Eq. (3.3)]. The multipliers are evaluated over a number $\tau=40$ iterations. The intersections of the two curves with the horizontal straight line at height 1 indicate the actual value of the fractional part of the correlation exponent and of the capacity (curves a and b, respectively).

d_q is larger than one for $q<0$, yielding an overall dimension D_q exceeding the phase-space dimension. This effect is caused by the presence of regions of the attractor rarely visited.[17]

IV. PERTURBATIVE EXPANSION

Because of the implicit nature of Eq. (2.6), it is useful to expand it in perturbative series to yield an explicit expression. The dimension around which it is most convenient to expand is, of course, the information dimension d_1, explicitly given by the Kaplan-Yorke relation. Therefore, we write

$$d_q = d_1 + d'(q-1) + d''(q-1)^2/2 + \cdots , \tag{4.1}$$

where d' and d'' are the derivatives of d_q, computed at $q=1$. The first derivative d' is related to the uniformity factor[10] $\tilde{\lambda}$ ($d' = -\tilde{\lambda} d_1$), once we refer $\tilde{\lambda}$ to the information dimension. By substituting Eq. (4.1) into Eq. (2.6), and introducing $\sigma = \sum_{k=1}^{j} \lambda_k + d_1 \lambda_{j+1}$, we can write

$$1 = \left\langle \exp\left[\sigma(1-q) - \lambda_{j+1} d'(1-q)^2 + \lambda_{j+1} d'' \frac{(1-q)^3}{2} + \cdots \right] \right\rangle . \tag{4.2}$$

Expanding the exponential and equating terms with the same powers of $(1-q)$, we obtain

$$\begin{aligned} &\langle \sigma \rangle = 0 , \\ &d' = \langle \sigma^2 \rangle / (2\langle \lambda_{j+1} \rangle) , \\ &d'' = 2(\langle \sigma \lambda_{j+1} \rangle d' - \langle \sigma^3 \rangle /6) / \langle \lambda_{j+1} \rangle . \end{aligned} \tag{4.3}$$

The first-order term provides the Kaplan-Yorke relation, the second one represents the estimate of the fluctuations in the Gaussian approximation. The last equation, yielding the concavity of d_q in d_1, gives the information on the position of the inflection point where the dependence of d_q on q is strongest.

Since, in most cases, the uniformity factor $\tilde{\lambda}$ is very small,[17] we expect the expansion (4.1) to be sufficiently accurate to the second order in $(1-q)$, even for $|q| \sim 5,6$.

This expansion is equivalent to that developed by Grassberger and Procaccia in Ref. 7. However, the present derivation, having started from the simple implicit relation (2.6), turns out to be more transparent, and the meaning of the correction terms, as coefficients of a series expansion, clearer.

V. CONCLUSIONS

An analytic extension of the Kaplan-Yorke conjecture to the whole Renyi-dimension function has been introduced. This allowed the giving of accurate estimates of the capacity for the Hénon map, chosen as a typical example of a two-dimensional diffeomorphism. Furthermore, the dimension function D_q could be evaluated in a sizable range of q values ($-6 \lesssim q \lesssim 6$). However, the convergence to the correct asymptotic values, $|q| \gg 1$, is slowed down by lack of statistical convergence on the tails of the probability distribution. Also, the existence of homoclinic tangencies (Hénon, Duffing attractors) is reflected in the appearance of sequences of negative values for the most expanding Lyapunov exponents. The possible implications of this fact have been briefly discussed and are currently under investigation.

An important class of related quantities is that of the metric entropies K_q defined by

$$K_q = \lim_{\epsilon \to 0} \lim_{\tau \to \infty} \left[\frac{1}{\tau(1-q)} \ln \langle P^{(q-1)}(x(t), x(t+\tau); \epsilon) \rangle \right] , \tag{5.1}$$

where $P(\mathbf{x}(t), \mathbf{x}(t+\tau); \epsilon)$ is the probability of finding an orbit remaining for a time τ within a distance ϵ from the trajectory joining $\mathbf{x}(t)$ with $\mathbf{x}(t+\tau)$. Noting that P is simply the inverse of the product $V_+(\tau)$ of all expanding multipliers (over the time τ), the metric entropies K_q are readily computed as

$$K_q = \lim_{\tau \to \infty} \left[\frac{1}{\tau(1-q)} \ln \langle V_+^{(1-q)}(\tau) \rangle \right] , \tag{5.2}$$

where $V_+(\tau) = \prod_{k=1}^{j_+} m_k(\tau)$, and j_+ is the largest integer for which $m_{j_+} > 1$. This result, recently derived by Paladin and Vulpiani,[11] is an extension of the expansions given in Ref. 7.

Finally, let us stress that a direct application of this method to experimental data is also possible by implementing one of the recently developed techniques to compute Lyapunov exponents from a time series of a single variable.[22]

After having completed our paper, we became aware of work by Grassberger[23] where a relation equivalent to (2.6) was proposed.

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