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Nonresonant wave-particle interaction in semiclassical quasilinear theory

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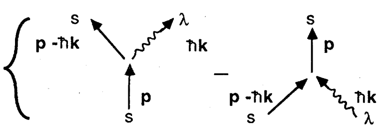
A nonresonant nature of wave-particle interaction is clarified from the viewpoint of quantum mechanics. The interaction of particles and quasiparticles can be described by the use of transition probability which is found to have both resonant and nonresonant contributions. The resonant transition probability is known as Fermi's golden rule, which is now supplemented by the nonresonant contribution, resulting in the proper conservation of energy and momentum in the particle-quasiparticle system.

I. INTRODUCTION

The quasilinear theory of weak plasma turbulence was introduced from the two different points of view: one based on the Vlasov equation[1,2] and the other based on the quantum-mechanical viewpoint.[3,4] In the quantum-mechanical viewpoint, the decay (or growth) of plasma waves is considered as a result of competition between absorption and emission of quasiparticles by plasma particles. The process involves a resonant transfer of energy between plasma particles and quasiparticles. The rate of change of number of quasiparticles $N_\lambda(\hbar\mathbf{k})$ with momentum $\hbar\mathbf{k}$ and energy $\hbar\Omega_\mathbf{k}$ is conveniently given by Feynman diagrams as[4]

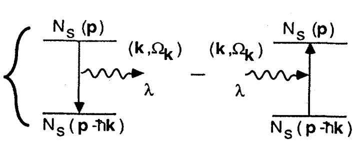
$$\frac{\partial N_\lambda(\hbar\mathbf{k})}{\partial t}=\sum_s\sum_\mathbf{p}\Bigg\{$$


$$\Bigg\}, \tag{1}$$

where summations are taken for the momentum of plasma particles $\mathbf{p}$ and for species of particles s. Based on Fermi's golden rule, the equation leads, in a semiclassical limit, to

$$\frac{\partial N_\lambda(\hbar\mathbf{k})}{\partial t}=\sum_s\sum_\mathbf{p}\frac{8\pi^2e_s^2\Omega_\mathbf{k}N_\lambda(\hbar\mathbf{k})}{Vk^2(\partial\omega\epsilon/\partial\omega)_{\Omega_\mathbf{k}}}\mathbf{k}\cdot\frac{\partial N_s}{\partial\mathbf{p}}\delta(\Omega_\mathbf{k}-\mathbf{k}\cdot\mathbf{p}/m_s)+\sum_s\sum_\mathbf{p}\frac{8\pi^2e_s^2\Omega_\mathbf{k}N_s(\mathbf{p})}{\hbar Vk^2(\partial\omega\epsilon/\partial\omega)_{\Omega_\mathbf{k}}}\delta(\Omega_\mathbf{k}-\mathbf{k}\cdot\mathbf{p}/m_s)\ , \tag{2}$$

where $N_s(\mathbf{p})$ is the number of plasma particles of species s with momentum $\mathbf{p}$ and a mass m_s, V is the volume of the box in which the system is quantized, and ϵ is the dielectric constant. The first and the second terms of Eq. (1) can be interpreted as induced and spontaneous emission in a two-energy-level system, respectively;[5] Eq. (1) may be described by

$$\frac{\partial N_\lambda(\hbar\mathbf{k})}{\partial t}=\sum_s\sum_\mathbf{p}\Bigg\{$$


$$\Bigg\}.$$

Although the quantum-mechanical treatment is straightforward because of its heuristic nature, it was noticed that the quantum derivation for the rate of change of plasma particles does not introduce the nonresonant interaction between particles and wave fields.[6,7] The classical derivation reveals that nonresonant particles do carry energy and momentum attributed to waves.[8] It has indeed been shown that nonresonant interaction plays an essential role in the plasma turbulence such as the high-energy ion tail formation in the ion-acoustic turbulence.[9,10] This was the motivation of the present paper in which the quantum derivation of the quasilinear theory is reconsidered.

In Sec. II a transition probability is formulated for the particle-quasiparticle interaction. A general perturbation method is used to recover Fermi's golden rule, and the extra term which is responsible for the nonresonant interaction is first introduced. In Sec. III the transition probability is used to derive the rate of change of the number of plasma particles in the process of interaction between particles and quasiparticles. In Sec. IV equations of conservation of energy and momentum in the particle-quasiparticle system are derived. Section V concludes with discussions.

II. TRANSITION PROBABILITY

The quantum-mechanical behavior of systems composed of particles and quasiparticles can be described by a wave function or state vector, which can be expanded in a series of eigenstates whose coefficients $b_m(t)$ satisfy the equation

$$\frac{d}{dt}b_m(t)=-\frac{i}{\hbar}\sum_n e^{i\hbar^{-1}(E_m-E_n)t}H_{mn}(t)b_n(t)\ , \tag{3}$$

where $E_{m(n)}$ is the energy of the system in the state $m\,(n)$ and $H_{mn}(t)$ is the matrix element of the Hamiltonian whose contribution comes from the interaction between plasma particles and quasiparticles (plasmons, phonons, etc.). The quantity $|b_m(t)|^2$ is the probability for the system to be in a state m at time t.

First, we shall consider a plasma with no external magnetic field and follow the quantum-mechanical derivation of Fukai and Harris[11] with some modification. We will see where the concept of nonresonant interaction can be introduced. Suppose an electrostatic wave has a potential

$$\phi(\mathbf{x},t)=\phi_{\mathbf{k}}(t)\exp\{i[\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}(t-t_0)]\}\ , \tag{4}$$

where the amplitude $\phi_{\mathbf{k}}(t)$ is slowly time varying. A first-order perturbation theory together with the Born approximation gives the probability amplitude for the transition $i(\mathbf{q}-\mathbf{k})\rightarrow f(\mathbf{q})$ during the time $t-t_0$ as

$$b_f(t)=-\frac{i}{\hbar}\int_{t_0}^{t}dt'\,e^{i\hbar^{-1}(E_{\mathbf{q}}-E_{\mathbf{q}-\mathbf{k}})t'}\times e\phi_{\mathbf{k}}(t')e^{-i\omega_{\mathbf{k}}(t'-t_0)}\ , \tag{5}$$

where we have used

$$b_n(t_0)=\begin{cases}0, & n\neq i\\ 1, & n=i\ .\end{cases}$$

Let us set $s=t-t'$. If the time variation governed by the term $e^{-i\hbar^{-1}(E_{\mathbf{q}}-E_{\mathbf{q}-\mathbf{k}})s}$ is fast compared to the wave evolution time of $\phi_{\mathbf{k}}$, we may expand $\phi_{\mathbf{k}}(t-s)$ as

$$\phi_{\mathbf{k}}(t-s)\simeq\phi_{\mathbf{k}}(t)-s\frac{\partial\phi_{\mathbf{k}}}{\partial t}\ . \tag{6}$$

The transition probability per unit time is then

$$\frac{d}{dt}|b_f|^2=\left[\frac{1}{\hbar}\right]^2 e^2|\phi_{\mathbf{k}}|^2\frac{2\sin(\nu_{\mathbf{q},\mathbf{k}}\tau)}{\nu_{\mathbf{q},\mathbf{k}}}+\left[\frac{1}{\hbar}\right]^2 e^2\frac{d}{dt}|\phi_{\mathbf{k}}|^2\frac{2-2\cos(\nu_{\mathbf{q},\mathbf{k}}\tau)}{\nu_{\mathbf{q},\mathbf{k}}^2}-\left[\frac{1}{\hbar}\right]^2 e^2\left[\phi_{\mathbf{k}}\frac{\partial\phi_{\mathbf{k}}^*}{\partial t}\left[\frac{1-e^{i\nu_{\mathbf{q},\mathbf{k}}\tau}}{\nu_{\mathbf{q},\mathbf{k}}^2}-\frac{\tau e^{-i\nu_{\mathbf{q},\mathbf{k}}\tau}}{i\nu_{\mathbf{q},\mathbf{k}}}\right]+\text{c.c.}\right]\ , \tag{7}$$

where $\tau=t-t_0$ and

$$\nu_{\mathbf{q},\mathbf{k}}=\frac{1}{\hbar}(E_{\mathbf{q}}-E_{\mathbf{q}-\mathbf{k}})-\omega_{\mathbf{k}}\ . \tag{8}$$

The second and third terms of the right-hand side of Eq. (7) was not considered in the derivation by Fukai and Harris. Taking the limit $\tau\rightarrow\infty$ and discarding rapidly oscillating terms with $e^{\pm i\nu_{\mathbf{q},\mathbf{k}}\tau}$, we obtain

$$\frac{d}{dt}|b_f|^2=\frac{2\pi}{\hbar^2}e^2|\phi_{\mathbf{k}}(t)|^2\delta(\nu_{\mathbf{q},\mathbf{k}})+\frac{1}{\hbar^2}e^2\frac{d}{dt}|\phi_{\mathbf{k}}(t)|^2\mathrm{P}\left[\frac{1}{\nu_{\mathbf{q},\mathbf{k}}^2}\right]\ , \tag{9}$$

where the symbol P denotes the principal value. The second term in the right-hand side of Eq. (9) is responsible for the nonresonant interaction between plasma particles and wave fields as we will see later.

Now we turn to general solutions of Eq. (3) involving a time-dependent interaction Hamiltonian. The method we describe here is similar to the one developed by Heitler[12] for the radiation problems involving discrete bound states of electrons.

Let us start with Eq. (3) which gives the time variation of the probability amplitude $b_m(t)$ for the state m. We seek a solution which satisfies an initial condition such that at $t=0$ the system is in a state $|\Psi_i\rangle$ and all other probability amplitudes are zero: $b_n(0)=\delta_{ni}$, where δ_{ni} is the Kronecker delta. Here we note that Eq. (3) is meaningful only when $t>0$. We may extend the solution, simply for the analytical convenience, to the negative t by adding the inhomogeneous term to the right-hand side of Eq. (3),

$$\frac{d}{dt}b_m(t)=-\frac{i}{\hbar}\sum_n e^{i\hbar^{-1}(E_m-E_n)t}H_{mn}(t)b_n(t)+\delta_{mi}\delta(t)\ . \tag{10}$$

Equation (10) is valid for all t and we seek a solution which satisfies conditions

$$\begin{aligned}&b_{m\neq i}(t)=b_i(t)=0\quad\text{for } t<0\ ,\\ &b_i(+0)=1\ ,\\ &b_{m\neq i}(+0)=0\ ,\end{aligned} \tag{11}$$

where $t=+0$ means that t approaches zero from the positive side. The Dirac delta function $\delta(t)$ in Eq. (10) expresses the required jump at $t=0$. By integrating over a small region of t around $t=0$, we get $b_i(+0)-b_i(-0)=1$, while $b_{m\neq i}$ is continuous at $t=0$.

Let us write a solution of Eq. (10) as

$$b_m(t)=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}dE\,G_{mi}(E,t)e^{-i\hbar^{-1}(E-E_m)t}\ , \tag{12}$$

then the function $G_{mi}(E,t)$ is required to satisfy

$$(E-E_m)G_{mi}(E,t)+i\hbar\frac{\partial G_{mi}}{\partial t}=\sum_n H_{mn}(t)G_{ni}(E,t)+\delta_{mi}\ , \tag{13}$$

where we used the relation

$$\delta(t)=\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dE\,e^{-i\hbar^{-1}(E-E_m)t}\ . \tag{14}$$

Equation (13) can now be written for $m\neq i$ as

$$G_{mi}(E,t)=\left[\sum_n H_{mn}(t)G_{ni}(E,t)-i\hbar\frac{\partial G_{mi}}{\partial t}\right]\zeta(E-E_m)\ , \tag{15}$$

where the singular function $\zeta(x)$ is defined as

$$\zeta(x)=\lim_{\eta\rightarrow+0}\left[\frac{1}{x+i\eta}\right]=\mathrm{P}\left[\frac{1}{x}\right]-i\pi\delta(x) \tag{16}$$

and has properties

$$x\zeta(x)=1\ , \tag{17}$$

$$\zeta(x)e^{ixt}=\begin{cases}0 & (t\rightarrow\infty)\\ -2\pi i\delta(x) & (t\rightarrow-\infty)\end{cases} \tag{18}$$

$$\int_{-\infty}^{\infty} \zeta(x) e^{ixt} dx = \begin{cases} 0 & (t>0) \\ -2\pi i & (t<0) \, . \end{cases} \tag{19}$$

For the purpose of the present analysis we assume that the function $G_{mi}(E,t)$ is slowly time varying and we use the approximation $G_{mi} \approx \sum_n H_{mn} G_{ni} \zeta(E-E_m)$ in the iteration procedure. The summation in Eq. (15) can split off a factor G_{ii} as

$$\sum_n H_{mn} G_{ni} = H_{mi} G_{ii} + \sum_{n\,(\neq i)} H_{mn} G_{ni} \, , \tag{20}$$

then Eq. (15), through the iteration, becomes

$$\begin{aligned} G_{mi}(E,t) = {} & G_{ii}(E)\zeta(E-E_m)U_{mi}(E,t) \\ & - i\hbar\zeta(E-E_m)V_{mi}(E,t) \end{aligned} \tag{21}$$

for $m \neq i$, where

$$\begin{aligned} U_{mi}(E,t) = {} & H_{mi}(t) \\ & + \sum_{n\,(\neq i)} H_{mn}(t)\zeta(E-E_n)U_{ni}(E,t) \, , \end{aligned} \tag{22}$$

$$\begin{aligned} V_{mi}(E,t) = {} & G_{ii}(E)\zeta(E-E_m)\frac{\partial H_{mi}}{\partial t} \\ & + \sum_{n\,(\neq i)} V_{mi}(E,t)\zeta(E-E_n)H_{mn}(t) \, . \end{aligned} \tag{23}$$

The function G_{ii} may be obtained directly from Eq. (13), by setting $\partial G_{ii}/\partial t \approx 0$, as

$$G_{ii}(E) = \frac{1}{E - E_i + i\hbar\Gamma_i(E)} \, , \tag{24}$$

where

$$\Gamma_i(E) = \frac{i}{\hbar} \left[H_{ii} + \sum_{n\,(\neq i)} H_{in}\zeta(E-E_n)U_{ni}(E) \right] . \tag{25}$$

The assumption $\partial G_{ii}/\partial t \approx 0$ corresponds to the time independence of the wave frequency. Since G_{mi} is given by Eq. (21), we are now in a position to calculate $b_m(t)$ by substituting G_{mi} into Eq. (12). We are interested in the time at which a system is found in the state m with the probability $|b_m(t)|^2$. The time should be long enough so that at least some transitions from state i can occur; $t \to \infty$. We assume that $\Gamma_i = \mathrm{Re}\Gamma_i + i\,\mathrm{Im}\Gamma_i$ is independent of E. Then

$$b_i(t) = \exp(-i\,\mathrm{Im}\Gamma_i t - \mathrm{Re}\Gamma_i t) \, . \tag{26}$$

Since $|b_i(t)|^2 = e^{-2\mathrm{Re}\Gamma_i t}$, $1/2\mathrm{Re}\Gamma_i$ is the lifetime of the initial state. When we consider $\mathrm{Re}\Gamma_i \to 0$,

$$G_{ii}(E) = \frac{1}{E - E_{i'} + i\hbar\mathrm{Re}\Gamma_i} \to \zeta_\Gamma(E - E_{i'}) \, , \tag{27}$$

where $E_{i'} = E_i + \Delta E$, $\Delta E = \hbar\,\mathrm{Im}\Gamma_i$, and $\zeta_\Gamma(E-E_{i'}) = \lim_{\Gamma\to+0}[1/(E-E_{i'}+i\Gamma)]$. We express $\zeta(E-E_m)$ in Eq. (21) as $\zeta_\sigma(E-E_m) = \lim_{\sigma\to+0}[1/\,(E-E_m+i\sigma)]$, and use the relation

$$\begin{aligned} & \zeta_\Gamma(E-E_{i'})\zeta_\sigma(E-E_m) \\ & \quad = \zeta_{\sigma-\Gamma}(E_{i'}-E_m)[\zeta_\Gamma(E-E_{i'}) - \zeta_\sigma(E-E_m)] \end{aligned} \tag{28}$$

for $\sigma > \Gamma$ (assumption $\sigma < \Gamma$ will result in the same transition probability to find) to get $b_m(t)$ from Eq. (12),

$$\begin{aligned} b_m(t) = {} & \zeta_{\sigma-\Gamma}(E_{i'}-E_m)[U_{mi}(E_{i'},t)e^{-i\hbar^{-1}(E_{i'}-E_m)t} - U_{mi}(E_m,t)] \\ & - i\hbar\zeta_{\sigma-\Gamma}(E_{i'}-E_m)\left[\left[\zeta_\Gamma(E_m-E_{i'}) + \frac{i}{\hbar}t \right] \frac{\partial H_{mi}}{\partial t} + \zeta_\sigma(E_{i'}-E_m)\frac{\partial H_{mi}}{\partial t} e^{-i\hbar^{-1}(E_{i'}-E_m)t} \right] . \end{aligned} \tag{29}$$

Since the probability of finding a system in a state m at time t is given by $|b_m(t)|^2$, the transition probability from the initial state i to the state m per unit time w_{mi} can be found by

$$\begin{aligned} w_{mi} & = \frac{d}{dt}|b_m(t)|^2 \\ & = \dot{b}_m b_m^* + \dot{b}_m^* b_m \, , \end{aligned} \tag{30}$$

where the overdot denotes the time derivative and the asterisk denotes the complex conjugate. We can evaluate $\dot{b}_m$ from Eq. (29) as

$$\begin{aligned} \dot{b}_m = {} & -\frac{i}{\hbar}U_{mi}(E_{i'},t)e^{-i\hbar^{-1}(E_{i'}-E_m)t} \\ & + \zeta_{\sigma-\Gamma}(E_{i'}-E_m)\left[\frac{\partial U_{mi}(E_{i'})}{\partial t} - \frac{\partial H_{mi}}{\partial t} \right] \\ & \times (e^{-i\hbar^{-1}(E_{i'}-E_m)t} - 1) + O(\partial^2 H_{mi}/\partial t^2) \, , \end{aligned} \tag{31}$$

where the relation (17) was used. Since we have assumed that $G_{mi}(E,t)$ is slowly time varying, we can set $\partial U_{mi}(E_{i'})/\partial t \approx \partial H_{mi}/\partial t$ and $\partial^2 H_{mi}/\partial t^2 \approx 0$. Therefore, $\dot{b}_m$ may be approximated by

$$\dot{b}_m = -\frac{i}{\hbar}U_{mi}(E_{i'},t)e^{-i\hbar^{-1}(E_{i'}-E_m)t} \, . \tag{32}$$

We now calculate a transition probability per unit time from Eq. (30) with Eqs. (29) and (32),

$$\begin{aligned} w_{mi} = {} & -\frac{i}{\hbar}U_{mi}(E_{i'},t)e^{-i\hbar^{-1}(E_{i'}-E_m)t} \\ & \times \Bigg\{ \zeta^*_{\sigma-\Gamma}(E_{i'}-E_m)[U^*_{mi}(E_{i'},t)e^{i\hbar^{-1}(E_{i'}-E_m)t} - U^*_{mi}(E_m,t)] \\ & + i\hbar\zeta^*_{\sigma-\Gamma}(E_{i'}-E_m)\left[\left[\zeta^*_\Gamma(E_m-E_{i'}) - \frac{i}{\hbar}t \right] \frac{\partial H^*_{mi}}{\partial t} + \zeta^*_\sigma(E_{i'}-E_m)\frac{\partial H^*_{mi}}{\partial t}e^{i\hbar^{-1}(E_{i'}-E_m)t} \right] \Bigg\} + \mathrm{c.c.} \end{aligned} \tag{33}$$

The most interesting situation is that where the transition from the initial state i to the final state f takes place after an elapse of long time. Setting $t \to \infty$ in Eq. (33) and using Eq. (18), we obtain for w_{fi},

$$w_{fi} = \left[-\frac{i}{\hbar} \, | U_{fi}(E_{i'},t) |^2 \zeta^*_{\sigma - \Gamma}(E_{i'} - E_f) + U_{fi}(E_{i'},t) \frac{\partial H^*_{fi}}{\partial t} \zeta^*_{\sigma - \Gamma}(E_{i'} - E_f) \zeta^*_{\sigma}(E_{i'} - E_f) \right] + \text{c.c.} \tag{34}$$

Approximating $U_{mi}(E_{i'},t)$ by the first term of Eq. (22) and using the definition of the ζ function, we finally obtain

$$w_{fi} = \frac{2\pi}{\hbar} \, | H_{fi}(t) |^2 \delta(E_i - E_f) + \mathrm{P} \left[\frac{1}{(E_i - E_f)^2} \right] \frac{d}{dt} \, | H_{fi}(t) |^2 , \tag{35}$$

where $E_{i'}$ $(= E_i + \hbar \, \mathrm{Im} \Gamma_i)$ is replaced by E_i assuming $E_i \gg \hbar \, \mathrm{Im} \Gamma_i$. If we neglect the time dependence of the interaction Hamiltonian, we recover the transition probability known as Fermi's golden rule,

$$w_{fi} = \frac{2\pi}{\hbar} \, | H_{fi} |^2 \delta(E_i - E_f) \; . \tag{36}$$

III. RESONANT AND NONRESONANT PARTICLE-QUASIPARTICLE INTERACTION

Our goal in this section is to derive a rate of change of plasma particles resulting from the interaction between particles and quasiparticles in the lowest-order process. The transition probability derived in Sec. II plays a major role in the analysis.

For simplicity, we consider an electrostatic oscillation in a plasma in the absence of magnetic field. Let us introduce the dielectric constant $\epsilon(\mathbf{k},\omega)$ which has a real part ϵ_r and an imaginary part ϵ_i defined by

$$\epsilon_r = 1 + \sum_s \sum_{\mathbf{p}} \frac{4\pi e_s^2}{V k^2} \mathrm{P} \left[\frac{\mathbf{k} \cdot \dfrac{\partial N_s}{\partial \mathbf{p}}}{\omega - \mathbf{k} \cdot \mathbf{p} / m_s} \right] , \tag{37}$$

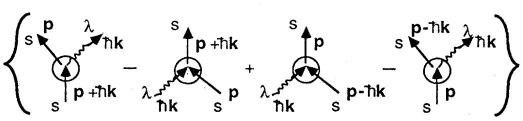
$$\epsilon_i = - \sum_s \sum_{\mathbf{p}} \frac{4\pi^2 e_s^2}{V k^2} \mathbf{k} \cdot \frac{\partial N_s}{\partial \mathbf{p}} \delta \left[\omega - \frac{\mathbf{k} \cdot \mathbf{p}}{m_s} \right] , \tag{38}$$

where $N_s = N_s(\mathbf{p})$ is the number of plasma particles of momentum $\mathbf{p}$. The relation

$$\epsilon_r(\mathbf{k}, \Omega_{\mathbf{k}}) = 0 \tag{39}$$

gives the dispersion characteristics of quasiparticles with momentum $\hbar \mathbf{k}$ and energy $\hbar \Omega_{\mathbf{k}}$. Following Harris,[4] the rate of change of $N_s(\mathbf{p})$ can be written by the Feynman diagram as

$$\frac{\partial N_s(\mathbf{p})}{\partial t} = \sum_{\mathbf{k}}$$



(40)

where plasma particles of species s with momentum $\mathbf{p}$ are either created or destroyed by the interaction with quasiparticles denoted by λ with momentum $\hbar \mathbf{k}$. Different from the scattering process between particles,[13] where only "resonant particles" interact with each other, the scattering process between particle and quasiparticle involves "nonresonant particles" which carry some of the energy and momentum of quasiparticles. To emphasize the nonresonant interaction, circles are added at the interaction region in Eq. (40). Temporal evolution of the number of quasiparticles $N_\lambda(\hbar \mathbf{k})$ should be incorporated in the process of interaction. Equation (40) is now expressed by the mathematical equation, using the transition probability Eq. (35)

$$\begin{aligned}
\frac{\partial N_s(\mathbf{p})}{\partial t} = \sum_{\mathbf{k}} \left[\frac{4\pi e_s^2 \hbar \Omega_{\mathbf{k}}}{V k^2} \frac{1}{(\partial \omega \epsilon / \partial \omega)_{\Omega_{\mathbf{k}}}} \right.
& \times \left\{ \left[\frac{2\pi}{\hbar} \delta \left[\frac{1}{2m_s} \, | \mathbf{p} + \hbar \mathbf{k} |^2 - \frac{1}{2m_s} p^2 - \hbar \Omega_{\mathbf{k}} \right] + \mathrm{P} \left[\frac{1}{\left[\frac{1}{2m_s} \, | \mathbf{p} + \hbar \mathbf{k} |^2 - \frac{1}{2m_s} p^2 - \hbar \Omega_{\mathbf{k}} \right]^2} \right] \frac{\partial}{\partial t} \right] \right. \\
& \times \{ N_s(\mathbf{p} + \hbar \mathbf{k}) [1 - N_s(\mathbf{p})] [1 + N_\lambda(\hbar \mathbf{k})] - [1 - N_s(\mathbf{p} + \hbar \mathbf{k})] N_s(\mathbf{p}) N_\lambda(\hbar \mathbf{k}) \} \\
& + \left[\frac{2\pi}{\hbar} \delta \left[\frac{1}{2m_s} \, | \mathbf{p} - \hbar \mathbf{k} |^2 + \hbar \Omega_{\mathbf{k}} - \frac{1}{2m_s} p^2 \right] + \mathrm{P} \left[\frac{1}{\left[\frac{1}{2m_s} \, | \mathbf{p} - \hbar \mathbf{k} |^2 + \hbar \Omega_{\mathbf{k}} - \frac{1}{2m_s} p^2 \right]^2} \right] \frac{\partial}{\partial t} \right] \\
& \left. \left. \times \{ N_s(\mathbf{p} - \hbar \mathbf{k}) N_\lambda(\hbar \mathbf{k}) [1 - N_s(\mathbf{p})] - N_s(\mathbf{p}) [1 - N_s(\mathbf{p} - \hbar \mathbf{k})] [1 + N_\lambda(\hbar \mathbf{k})] \} \right\} \right] .
\end{aligned} \tag{41}$$

To make the classical contact, we set $\hbar \to 0$, $N_s(\mathbf{p}) \ll 1$, and $\hbar\Omega_{\mathbf{k}} N_\lambda(\hbar\mathbf{k})$ finite. Then Eq. (41) becomes

$$\frac{\partial N_s(\mathbf{p})}{\partial t} = \sum_{\mathbf{k}} \frac{8\pi^2 e_s^2}{Vk^2(\partial\omega\epsilon/\partial\omega)_{\Omega_{\mathbf{k}}}} \left\{ \hbar\Omega_{\mathbf{k}} N_\lambda(\hbar\mathbf{k})\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}}\left[\delta\left[\Omega_{\mathbf{k}} - \frac{\mathbf{k}\cdot\mathbf{p}}{m_s}\right]\mathbf{k}\cdot\frac{\partial N_s}{\partial\mathbf{p}}\right] + \Omega_{\mathbf{k}}\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}}\left[\delta\left[\Omega_{\mathbf{k}} - \frac{\mathbf{k}\cdot\mathbf{p}}{m_s}\right]N_s(\mathbf{p})\right]\right\}$$
$$+ \sum_{\mathbf{k}} \frac{4\pi e_s^2\hbar\Omega_{\mathbf{k}}}{Vk^2(\partial\omega\epsilon/\partial\omega)_{\Omega_{\mathbf{k}}}}\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}}\left[P\left[\frac{1}{(\Omega_{\mathbf{k}} - \mathbf{k}\cdot\mathbf{p}/m_s)^2}\right]\mathbf{k}\cdot\frac{\partial N_s}{\partial\mathbf{p}}\right]\frac{\partial N_\lambda(\hbar\mathbf{k})}{\partial t} . \quad (42)$$

The first term in the right-hand side was derived previously[3,4] and contributes to the resonant interaction. On the other hand, the second term in the right-hand side contributes to the nonresonant interaction which will be shown to be responsible for the carriage of energy and momentum in the system.

IV. CONSERVATION OF ENERGY AND MOMENTUM

In this section we derive equations of conservation of energy and momentum based on the semiclassical equations (2) and (42).

The time change of the plasmon energy can be found by multiplying $\hbar\Omega_{\mathbf{k}}$ on both sides of Eq. (2) as

$$\frac{\partial W_{\mathbf{k}}}{\partial t} = 2\gamma_{\mathbf{k}} W_{\mathbf{k}} + S_{\mathbf{k}} , \quad (43)$$

where

$$W_{\mathbf{k}} = \hbar\Omega_{\mathbf{k}} N_\lambda(\hbar\mathbf{k}) , \quad (44)$$

$$\gamma_{\mathbf{k}} = \sum_s \sum_{\mathbf{p}} \frac{4\pi^2 e_s^2 \Omega_{\mathbf{k}}}{Vk^2(\partial\omega\epsilon/\partial\omega)_{\Omega_{\mathbf{k}}}}\mathbf{k}\cdot\frac{\partial N_s}{\partial\mathbf{p}}\delta\left[\Omega_k - \frac{\mathbf{k}\cdot\mathbf{p}}{m_s}\right] , \quad (45)$$

$$S_{\mathbf{k}} = \sum_s \sum_{\mathbf{p}} \frac{8\pi^2 e_s^2 \Omega_{\mathbf{k}}^2}{Vk^2(\partial\omega\epsilon/\partial\omega)_{\Omega_{\mathbf{k}}}} N_s(\mathbf{p})\delta\left[\Omega_k - \frac{\mathbf{k}\cdot\mathbf{p}}{m_s}\right] . \quad (46)$$

We introduce the approximation

$$\left[\frac{\partial}{\partial\omega}\omega\epsilon\right]_{\Omega_{\mathbf{k}}} \approx \left[\frac{\partial}{\partial\omega}\omega\epsilon_r\right]_{\Omega_{\mathbf{k}}} = \Omega_{\mathbf{k}}\frac{\partial\epsilon_r}{\partial\omega} , \quad (47)$$

where Eq. (39) was used. Then the growth rate $\gamma_{\mathbf{k}}$ given by Eq. (45) can be expressed as

$$\gamma_{\mathbf{k}} = -\frac{\epsilon_i(\mathbf{k},\Omega_{\mathbf{k}})}{[\partial\epsilon_r(\mathbf{k},\omega)/\partial\omega]_{\Omega_{\mathbf{k}}}} . \quad (48)$$

Next, the time change of particle energy in the system can be evaluated from Eq. (42) as

$$\sum_s \sum_{\mathbf{p}} \frac{p^2}{2m_s}\frac{\partial N_s}{\partial t} = \sum_{\mathbf{k}} \left[\frac{2\Omega_{\mathbf{k}}\epsilon_i}{(\partial\omega\epsilon/\partial\omega)_{\Omega_{\mathbf{k}}}}\hbar\Omega_{\mathbf{k}} N_\lambda(\hbar\mathbf{k}) - S_{\mathbf{k}} + \frac{\epsilon_r - 1}{(\partial\omega\epsilon/\partial\omega)_{\Omega_{\mathbf{k}}}}\hbar\Omega_{\mathbf{k}}\frac{\partial N_\lambda(\hbar\mathbf{k})}{\partial t} + \frac{\Omega_{\mathbf{k}}(\partial\epsilon_r/\partial\omega)_{\Omega_{\mathbf{k}}}}{(\partial\omega\epsilon/\partial\omega)_{\Omega_{\mathbf{k}}}}\hbar\Omega_{\mathbf{k}}\frac{\partial N_\lambda(\hbar\mathbf{k})}{\partial t}\right] . \quad (49)$$

In deriving Eq. (49), we carried out a partial integration over $\mathbf{p}$ keeping in mind that $\sum_{\mathbf{p}} \to V\int d^3v$ in the classical limit. With the help of Eqs. (39) and (47), Eq. (49) can be further simplified as

$$\sum_s \sum_{\mathbf{p}} \frac{p^2}{2m_s}\frac{\partial N_s}{\partial t} = \sum_{\mathbf{k}} \left[-2\gamma_{\mathbf{k}} W_{\mathbf{k}} - S_{\mathbf{k}} - \frac{1}{(\partial\omega\epsilon_r/\partial\omega)_{\Omega_{\mathbf{k}}}}\frac{\partial W_{\mathbf{k}}}{\partial t} + \frac{\partial W_{\mathbf{k}}}{\partial t}\right] , \quad (50)$$

where the first term in the bracket of the right-hand side comes from the resonant interaction, the second term is from the spontaneous emission, and the third and the fourth terms are related to the nonresonant interaction. With the help of Eq. (43), Eq. (50) becomes

$$\frac{\partial}{\partial t}\left[\sum_s \sum_{\mathbf{p}} \frac{p^2}{2m_s} N_s(\mathbf{p}) + \sum_{\mathbf{k}} \frac{W_{\mathbf{k}}}{\left[\frac{\partial}{\partial\omega}\omega\epsilon_r\right]_{\Omega_{\mathbf{k}}}}\right] = 0 . \quad (51)$$

Since quasiparticles, whose energy is $\hbar\Omega_{\mathbf{k}} N_\lambda(\hbar\mathbf{k})$, carry the energy not only of the electric field but also of the kinetic energy of oscillating particles in the electric field, the quasiparticle energy with a correction factor $(\partial\omega\epsilon_r/\partial\omega)_{\Omega_{\mathbf{k}}}$ should be used in the conservation of energy. The quantity

$$\mathscr{E}_{\mathbf{k}} = \frac{W_{\mathbf{k}}}{\left[\frac{\partial}{\partial\omega}\omega\epsilon_r\right]_{\Omega_{\mathbf{k}}}} = \frac{\hbar\Omega_{\mathbf{k}} N_\lambda(\hbar\mathbf{k})}{\left[\frac{\partial}{\partial\omega}\omega\epsilon_r\right]_{\Omega_{\mathbf{k}}}} \quad (52)$$

is the energy density of the electric field carried by quasiparticles. Thus, Eq. (51) shows the conservation of energy in the system.

The time change of the momentum carried by particles in the system can be calculated similarly. Multiply $\mathbf{p}$ on both sides of Eq. (42) and take the summation over $\mathbf{p}$ and plasma species s; then we find

$$\frac{\partial}{\partial t}\sum_s \sum_{\mathbf{p}} \mathbf{p} N_s(\mathbf{p}) = \sum_{\mathbf{k}} 2\mathbf{k}\hbar\Omega_{\mathbf{k}}\frac{\epsilon_i}{(\partial\omega\epsilon/\partial\omega)_{\Omega_{\mathbf{k}}}} N_\lambda(\hbar\mathbf{k}) - \sum_s \sum_{\mathbf{p}} \sum_{\mathbf{k}} \frac{8\pi^2 e_s^2 \Omega_{\mathbf{k}}}{Vk^2(\partial\omega\epsilon/\partial\omega)_{\Omega_{\mathbf{k}}}}\mathbf{k}\delta\left[\Omega_{\mathbf{k}} - \frac{\mathbf{k}\cdot\mathbf{p}}{m_s}\right]N_s(\mathbf{p}) + \sum_{\mathbf{k}} \frac{(\partial\epsilon_r/\partial\omega)_{\Omega_{\mathbf{k}}}}{(\partial\omega\epsilon/\partial\omega)_{\Omega_{\mathbf{k}}}}\mathbf{k}\hbar\Omega_{\mathbf{k}}\frac{\partial N_\lambda(\hbar\mathbf{k})}{\partial t} , \quad (53)$$

where we carried out a partial integration over $\mathbf{p}$, as was done to derive Eq. (49). Equation (53) can be further simplified as

$$\frac{\partial}{\partial t}\sum_s\sum_{\mathbf{p}}\mathbf{p}N_s(\mathbf{p})=-\sum_{\mathbf{k}}2\gamma_{\mathbf{k}}\hbar\mathbf{k}N_\lambda(\hbar\mathbf{k})-\sum_{\mathbf{k}}\frac{\mathbf{k}}{\Omega_{\mathbf{k}}}S_{\mathbf{k}}+\sum_{\mathbf{k}}\hbar\mathbf{k}\frac{\partial N_\lambda(\hbar\mathbf{k})}{\partial t}\ . \tag{54}$$

The time change of momentum carried by quasiparticles can be found from Eq. (2) by multiplying $\hbar\mathbf{k}$ as

$$\frac{\partial}{\partial t}\hbar\mathbf{k}N_\lambda(\hbar\mathbf{k})=2\gamma_{\mathbf{k}}\hbar\mathbf{k}N_\lambda(\hbar\mathbf{k})+\frac{\mathbf{k}}{\Omega_{\mathbf{k}}}S_{\mathbf{k}}\ , \tag{55}$$

or if we define the momentum of the plasmon as

$$\mathbf{P}_{\mathbf{k}}=\hbar\mathbf{k}N_\lambda(\hbar\mathbf{k})\ , \tag{56}$$

we get

$$\frac{\partial\mathbf{P}_{\mathbf{k}}}{\partial t}=2\gamma_{\mathbf{k}}\mathbf{P}_{\mathbf{k}}+\frac{\mathbf{k}}{\Omega_{\mathbf{k}}}S_{\mathbf{k}}\ . \tag{57}$$

Equation (54) together with Eq. (55) gives the equation of conservation of momentum in the system,

$$\frac{\partial}{\partial t}\left[\sum_s\sum_{\mathbf{p}}\mathbf{p}\mathbf{N}_s(\mathbf{p})\right]=0\ . \tag{58}$$

It is clear from Eq. (54) that nonresonant particles are responsible for canceling the momentum carried by quasiparticles resulting in the conservation of momentum described by Eq. (58).

V. DISCUSSION AND CONCLUSION

We have derived equations of conservation of energy and momentum in the particle-quasiparticle system. The system involves the interaction between particles and quasiparticles in resonance and in nonresonance. Nonresonant interaction has been shown to appear as a term of time-dependent Hamiltonian. The transition probability has thus been modified to include nonresonant nature of particle-quasiparticle interaction. The proper treatment of the transition process allows us to obtain the semiclassical quasilinear equations which advocate conservation of energy and momentum.

For a simple example, consider a plasma (Langmuir) oscillation which is characterized by $\epsilon\approx\epsilon_r=1-\omega_p^2/\omega^2$, where ω_p is an electron plasma oscillation. Conservation of energy in an electron-plasmon system is, from Eq. (51), given by

$$\frac{\partial}{\partial t}\left[\sum_{\mathbf{p}}\frac{p^2}{2m_e}N_e(\mathbf{p})+\sum_{\mathbf{k}}\tfrac{1}{2}\hbar\Omega_{\mathbf{k}}N_\lambda(\hbar\mathbf{k})\right]=0\ . \tag{59}$$

Since plasmons carry electric field energy of $\sum_{\mathbf{k}}\hbar\Omega_{\mathbf{k}}N_\lambda(\hbar\mathbf{k})/2$ and oscillating particle energy of $\sum_{\mathbf{k}}\hbar\Omega_{\mathbf{k}}N_\lambda(\hbar\mathbf{k})/2$, the total energy of the system should be given by

$$\sum_{\mathbf{p}}\frac{p^2}{2m_e}N_e(\mathbf{p})+\sum_{\mathbf{k}}\tfrac{1}{2}\hbar\Omega_{\mathbf{k}}N_\lambda(\hbar\mathbf{k})\ ,$$

not by

$$\sum_{\mathbf{p}}\frac{p^2}{2m_e}N_e(\mathbf{p})+\sum_{\mathbf{k}}\hbar\Omega_{\mathbf{k}}N_\lambda(\hbar\mathbf{k})\ ,$$

which was used in conventional quantum-mechanical derivation.[4,7] Conservation of momentum in the electron-plasmon system is, from Eq. (58), given by

$$\frac{\partial}{\partial t}\left[\sum_{\mathbf{p}}\mathbf{p}N_e(\mathbf{p})\right]=0\ . \tag{60}$$

Since the momentum of plasmon $\mathbf{P}_{\mathbf{k}}$ is physically carried by nonresonant particles, the total momentum in the system should be given by

$$\sum_{\mathbf{p}}\mathbf{p}N_e(\mathbf{p})\ ,$$

not by

$$\sum_{\mathbf{p}}\mathbf{p}N_e(\mathbf{p})+\sum_{\mathbf{k}}\hbar\mathbf{k}N_\lambda(\mathbf{k})\ ,$$

which was used in conventional quantum-mechanical derivation. The plasmon momentum is related to its energy by

$$\mathbf{P}_{\mathbf{k}}/W_{\mathbf{k}}=\mathbf{k}/\Omega_{\mathbf{k}}\ , \tag{61}$$

which was equivalently shown in the relation between wave momentum and energy in the classical theory.[14]

The classical quasilinear equations may be derived in a straightforward manner with the help of prescriptions

$$\sum_{\mathbf{k}}\to V\int\frac{d^3k}{(2\pi)^3}\ , \tag{62}$$

$$\sum_{\mathbf{p}}N_s(\mathbf{p})\to V\int d^3v\,f_s(\mathbf{v}),\quad \mathbf{p}=m_s\mathbf{v} \tag{63}$$

$$\hbar\Omega_{\mathbf{k}}N_\lambda(\hbar\mathbf{k})=W_{\mathbf{k}}=\left[\frac{\partial}{\partial\omega}\omega\epsilon_r\right]_{\Omega_{\mathbf{k}}}\frac{|\mathbf{E}_{\mathbf{k}}|^2}{8\pi}\ , \tag{64}$$

$$\hbar\mathbf{k}N_\lambda(\hbar\mathbf{k})=\mathbf{P}_{\mathbf{k}}=\mathbf{k}\left[\frac{\partial\epsilon_r}{\partial\omega}\right]_{\Omega_{\mathbf{k}}}\frac{|\mathbf{E}_{\mathbf{k}}|^2}{8\pi}\ , \tag{65}$$

$$\gamma_{\mathbf{k}}=\sum_s\frac{4\pi^2e_s^2}{m_sk^2(\partial\epsilon_r/\partial\omega)_{\Omega_{\mathbf{k}}}}\int d^3v\,\mathbf{k}\cdot\frac{\partial f_s}{\partial\mathbf{v}}\delta(\Omega_{\mathbf{k}}-\mathbf{k}\cdot\mathbf{v})\ , \tag{66}$$

$$S_{\mathbf{k}}=\sum_s\frac{8\pi^2e_s^2\Omega_{\mathbf{k}}}{k^2(\partial\epsilon_r/\partial\omega)_{\Omega_{\mathbf{k}}}}\int d^3v\,f_s(\mathbf{v})\delta(\Omega_{\mathbf{k}}-\mathbf{k}\cdot\mathbf{v})\ . \tag{67}$$

In the classical limit, Eq. (42) becomes

$$\frac{\partial f_s(v)}{\partial t}=\frac{\partial}{\partial\mathbf{v}}\cdot\left[\overleftrightarrow{\mathbf{D}}_s(\mathbf{v})\cdot\frac{\partial f_s(\mathbf{v})}{\partial\mathbf{v}}\right]+\frac{\partial}{\partial\mathbf{v}}\cdot[\mathbf{A}_s(\mathbf{v})f_s(\mathbf{v})]\ , \tag{68}$$

where

$$\overleftrightarrow{\mathbf{D}}_s(\mathbf{v})=\left[\frac{e_s}{m_s}\right]^2\int\frac{d^3k}{(2\pi)^3}\frac{\mathbf{k}\mathbf{k}}{k^2}\times\left[\pi|\mathbf{E}_{\mathbf{k}}|^2\delta(\Omega_{\mathbf{k}}-\mathbf{k}\cdot\mathbf{v})+\frac{1}{2}P\left[\frac{1}{(\Omega_{\mathbf{k}}-\mathbf{k}\cdot\mathbf{v})^2}\right]\frac{\partial|\mathbf{E}_{\mathbf{k}}|^2}{\partial t}\right], \tag{69}$$

$$\mathbf{A}_s(\mathbf{v})=\frac{8\pi^2 e_s^2}{m_s}\int\frac{d^3k}{(2\pi)^3}\frac{\mathbf{k}}{k^2}\frac{1}{(\partial\epsilon_r/\partial\omega)_{\Omega_\mathbf{k}}}\delta(\Omega_\mathbf{k}-\mathbf{k}\cdot\mathbf{v})\ . \tag{70}$$

This is the classical equation governing the time evolution of the particle-distribution function.[15] Equation (43) remains the same in its notation, or

$$\frac{\partial W_\mathbf{k}}{\partial t}=2\gamma_\mathbf{k}W_\mathbf{k}+S_\mathbf{k}\ , \tag{71}$$

where $W_\mathbf{k}$, $\gamma_\mathbf{k}$, and $S_\mathbf{k}$ are now given by Eqs. (64), (66), and (67), respectively. Equations governing conservation of energy and momentum given by Eqs. (51) and (58) become

$$\frac{\partial}{\partial t}\left[\sum_s\int d^3v\,\tfrac{1}{2}m_s v^2 f_s(\mathbf{v})+\int\frac{d^3k}{(2\pi)^3}\frac{|\mathbf{E}_\mathbf{k}|^2}{8\pi}\right]=0 \tag{72}$$

and

$$\frac{\partial}{\partial t}\left[\sum_s\int d^3v\,m_s\mathbf{v}f_s(\mathbf{v})\right]=0\ , \tag{73}$$

in agreement with the classical derivation[14,16] where $f_s(\mathbf{v})$ includes the second-order perturbation.

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