Nonlinear complex eikonal approximation: Optical bistability in absorbing media

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A recently developed nonlinear complex eikonal approximation which was previously employed to the study of dispersively nonlinear resonators is applied to absorptively nonlinear resonator configurations. The nonlinear absorption is introduced phenomenologically, encompassing a large variety of intensity-dependent absorption mechanisms. Conditions for obtaining optical bistability are derived. Dynamically increasing absorption was not considered, and optical bistability in free propagation was not obtained.

I. INTRODUCTION

A nonlinearly absorbing medium in an optical resonator was the first configuration for which the existence of optical bistability was theoretically predicted by Szoke *et al.*[1] and by Siedel.[2]

Although difficult to implement experimentally, absorptive bistability has been a major subject of theoretical research involving the elucidation of the mechanisms and the characterization of the relevant microscopic phenomena. The main part of this theoretical work, initiated by Bonifacio and Lugiato,[3] was carried out for a nonlinear medium consisting of an ensemble of two-level systems, thus exhibiting saturation of resonant absorption. In this context, analysis of the steady-state and temporal behavior and of the stability conditions were carried out,[4–6] using, for the most part, the mean-field approximation, which ignores propagation effects. Recent efforts concentrated on media exhibiting dynamically increasing absorption.[7]

In the present paper we analyze the steady-state characteristics of absorptive bistability from a phenomenological point of view rather than assuming particular microscopic models for the optical nonlinearity. The main analytical tool is the complex nonlinear eikonal approximation.[8] In this approximation, we follow the accumulation of the phase of light propagating in a nonlinear medium. The medium, in turn, is stratified due to the variation of the intensity-induced complex refractive index. The first-order coupling term between the phase accumulation and the intensity-induced stratification, is the nonlinear complex eikonal equation.[9]

The application of the eikonal approximation was demonstrated for dispersively nonlinear media[8] and was utilized for the analysis of dispersively coupled configurations.[10,9]

The generality of the eikonal model and its simplicity made it applicable to a very large family of phenomenological nonlinearities, both absorptive and dispersive, and to a very large class of optical configurations. Dynamically increasing absorption is not treated in the present article.

Some of the questions to be addressed in the present paper are as follows: What types of nonlinearity give rise to optically bistable response? Can a "well-behaved" nonlinearity be the source of optical bistability in free propagation? How can specific switching patterns be produced?

In order to study propagation in absorptive media the eikonal approximation is generalized in the present paper to include imaginary phase accumulation. In Sec. II we present this generalization and derive intraresonator intensity terms for several simple resonators. We also consider noncoherent light propagation in nonlinear optical resonators. In Secs. III and IV we analyze the light propagation in media exhibiting intensity-dependent decreasing and increasing absorption, respectively. In Sec. V we demonstrate the case of light propagation in coupled, dispersively and absorptively nonlinear media. In all these cases the generality, the straightforward applicability, and the total equivalence of the treatment with respect to real and imaginary phases are emphasized.

II. PROPAGATION EFFECTS IN ABSORBING MEDIA

The nonlinear complex eikonal approximation is a procedure for the treatment of light propagation in media characterized by a nonlinear refractive index $n(I)=n_D(I)+in_A(I)$. As indicated, in general both the dispersive and the absorptive parts of the refractive index depend on the local intensity $I(x)$. The eikonal approximation is expressed in terms of the phase accumulated upon propagation through a distance x in the medium,

$$\phi(x)=(2\pi/\lambda_0)\int_0^x n(I(x'))dx' \ . \quad (1)$$

The local intensity depends on the local accumulated phase; it also depends, as a consequence of the boundary conditions, on the phase $\phi(L)$ accumulated along the medium length L. This procedure results in the integral equation[8,10]

$$\phi(x)=(2\pi/\lambda_0)\int_0^x n(I(x',\phi(x'),\phi))dx' \ , \quad (2)$$

where

$$\phi(x)=\phi_D(x)+(i/2)\phi_A(x) \ ,$$

$$\phi=\phi_D+(i/2)\phi_A\equiv\phi(L) \ ,$$

and λ_0 is the vacuum wavelength. $\phi_D(x)$ and $\phi_A(x)$ denote the dispersive and absorptive contributions to $\phi(x)$, respectively. The factor $\frac{1}{2}$ is introduced to follow common conventions.

It will be convenient to express the absorptive part of the refractive index in terms of the absorption coefficient $\alpha(I)$,

$$n_A(I)=\lambda_0\alpha(I)/(4\pi) \ . \tag{3}$$

The procedure for obtaining the transmission characteristics $I_{\rm out}$ versus $I_{\rm in}$ of a particular optical configuration consists of the following steps:[8,10] (a) specifying the appropriate form for the nonlinear index of refraction $n(I)$, and the boundary conditions. The latter, in turn, determine the form of $I(x,\phi(x),\phi(L))$; (b) solving the integral equation (2) for $\phi(x)$ or alternatively using its differential form

$$d\phi(x)/n(I(x,\phi(x),\phi))=(2\pi/\lambda_0)dx \tag{4}$$

which is solved by separation of variables and integration; (c) obtaining $\phi(L)$ by imposing self-consistency; (d) calculating $I_{\rm out}$ versus $I_{\rm in}$.

In many situations either the absorptive or the dispersive nonlinearity can be ignored. For a purely dispersive nonlinearity, which was studied in detail in Refs. 8 and 10, the following equation is obtained:

$$d\phi_D(x)/n_D(I(x,\phi_D(x),\phi_D))=(2\pi/\lambda_0)dx \ , \tag{5}$$

whereas in the purely absorptive case

$$d\phi_A(x)/\alpha(I(x,\phi_A(x),\phi_A))=dx \ . \tag{6}$$

When the medium is homogeneous in the limit of low light intensity, the explicit dependence of I on x arises from the formation of standing waves. Therefore, in the systems we discuss, it only occurs when the radiation source is coherent and the configuration involves counterpropagation. In the actual treatment of these systems we average out over this explicit x dependence. This accounts in part for the various mechanisms, such as thermal diffusion, resulting in suppression of the high-frequency spatial modulation, which are not explicitly taken care of in the present study. The boundary conditions, which determine the functional dependence of I on x, depend on the coherence properties of the light source. We shall only investigate in detail the two limiting cases—totally coherent and totally incoherent source. In the former case the steady-state condition is expressed in terms of the complex electric field amplitude, and in the latter, in terms of the field intensity. We now demonstrate these two limiting cases.

A. Absorptive Fabry-Perot resonator (FPR), with coherent incident illumination

The FPR configuration together with the propagating (E^+) and counterpropagating (E^-) field amplitudes is shown in Fig. 1(a). The expressions for E^+ and E^- are

$$E^+(x)=E_{\rm in}t_1\exp[i\phi(x)]+E^-r_1'\exp[2\phi(x)] \ , \tag{7a}$$

$$E^-(x)=E^+(x)r_2\exp\{2i[\phi-\phi(x)]\} \ , \tag{7b}$$

where $E_{\rm in}$ is the incident field amplitude, $r_j(t_j)$ $(j=1,2)$ are the forward amplitude reflectivities (transmitivities) of the mirror M_j, and $r_j'(t_j')$ are the corresponding backward values.

Solving Eqs. (7) for E^+ and E^- in terms of $E_{\rm in}$ we obtain the total local-field amplitude in the form

$$E(x)=E^+(x)+E^-(x)=E_{\rm in}\exp[i\phi(x)](1+r_2\exp\{2i[\phi-\phi(x)]\})[1-r_1'r_2\exp(2i\phi)]^{-1} \ . \tag{8}$$

Thus, the local intensity is

$$\begin{aligned} I(x)&=(c/4\pi)n_0\,|E(x)|^2 \\ &=I_{\rm in}(T_1/n_0)\exp[-\Phi_A(x)](1+2r_2\exp[-\phi_A+\phi_A(x)]\cos\{2[\phi_D-\phi_D(x)]\}+r_2^2\exp\{-2[\phi_A-\phi_A(x)]\})/G_1 \ , \end{aligned} \tag{9}$$

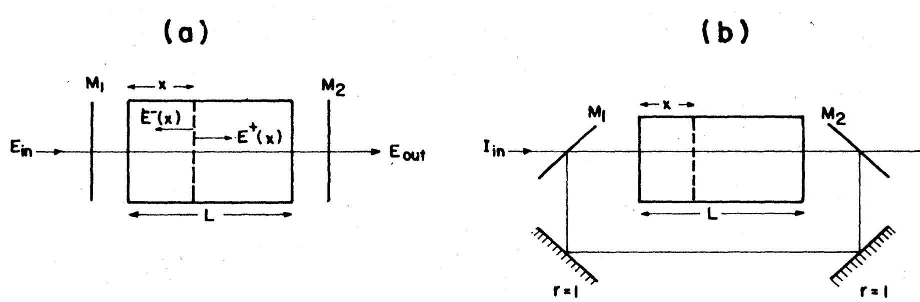

FIG. 1. Optical resonators containing nonlinear media. M_1 and M_2 are the front and the back mirror, respectively. (a) Fabry-Perot resonator. (b) Ring resonator.

where $I_{in}=(c/4\pi)n_0E_{in}^2$ and T_1 is the intensity transmitivity of mirror 1, given by

$$T_1=(n_0/n_{air})t_1^2 .$$

n_0 is the low-intensity limit of $n(I)$ and

$$G_1=1-r_1'r_2\exp(-\phi_A)\cos(2\phi_D)+(r_1'r_2)^2\exp(-2\phi_A) . \quad (10)$$

Equation (8) was obtained previously[10] for the particular case of propagation through a Kerr medium for which $\phi_A=0$ ($\alpha=0$).

B. Absorptive FPR with incoherent incident illumination

When the incident light is totally incoherent the mutual coherence of the interfering waves is zero, thus self-consistency at steady state is maintained for light intensities, rather than amplitudes. The dispersion contribution vanishes, and we obtain

$$I^+(x)=(I_{in}T_1/n_0)\exp[-\phi_A(x)]+I^-(x)R_1'\exp[-2\phi_A(x)] , \quad (11a)$$

$$I^-(x)=I^+(x)R_2\exp\{-2[\phi_A-\phi_A(x)]\} . \quad (11b)$$

Solving Eqs. (11) we obtain the local intensity

$$I(x)=I^+(x)+I^-(x)=(I_{in}T_1/n_0)\exp[-\phi_A(x)]\times(1+R_2\exp\{-2[\phi_A-\phi_A(x)]\})/G_2 , \quad (11c)$$

where

$$G_2=1-R_1'R_2\exp(-2\phi_A) . \quad (12)$$

The reason for considering incoherent illumination in connection with absorptive bistability is that the feedback necessary for maintaining this type of bistability is an intensity feedback.

The same general procedure can be applied to many other resonators. In the present paper we consider the FPR, the ring resonator (RR) (Fig. 1), and free propagation (F_p). In all these cases $I(x)$ is written in the form

$$I(x)=K_iI_{in}\exp[-\phi_A(x)] , \quad (13)$$

where K_i is given in Table I. i is an index denoting the various configurations, as defined in the table.

In the case of dispersive optical bistability the intensity dependence of the refractive index, i.e., the intensity dependence of the optical length, combined with the resonator feedback, results in a self-matching tendency of the resonator. Using the eikonal method, in which dispersive and absorptive phases are treated on the same footing, the interpretation of bistability in terms of self-matching holds for absorptive bistability as well. We now demonstrate this interpretation for a ring resonator, using a matching criterion in terms of the integrated intensity in the resonator.

For a purely dispersive linear coherently pumped RR (CRR) the integrated intensity is given by

$$S_1=\int_0^{\phi_D(L)} I(\phi_D(x))d\phi_D(x)=2\pi I_{in}(1-R)L/\lambda_0[1-2r_1'r_2\cos(2\pi Ln_D/\lambda_0)+(r_1'r_2)^2] . \quad (14)$$

In Fig. 2(a), S_1 is depicted as a function of the medium length. For each local maximum of S_1, where matching of the resonator is satisfied, a bistable loop can be formed if the linear medium is replaced by a nonlinear one.

For a purely linear absorptive CRR, the integrated intensity is given by

$$S_2=\int_0^{\phi_A} I(\phi_A(x))d\phi_A(x)=I_{in}K_1[1-\exp(-\phi_A)] . \quad (15)$$

S_2 is depicted in Fig. 2(b) as a function of the medium length, keeping the resonator dispersively matched. S_2 has one maximum which corresponds to absorptive matching. Note that if the resonator is dispersively unmatched so that $\pi/2\le\phi_D\le3\pi/2$, there is no absorptive matching (no maximum). Upon replacing the linear absorptive medium by a nonlinear one, the absorptive length

TABLE I. Functional forms of K_i for various resonator-excitation source configurations. In the text we will frequently use the following notations: CRR for $i=1$, IRR for $i=2$, CFPR for $i=4$, and IFPR for $i=5$.

i	Configuration	Source	K_i
1	RR[a]	C[d]	$(1-R)/n_0[1-2r_1'r_2(\cos\phi_D)\exp(-\phi_A/2)+(r_1'r_2)^2\exp(-\phi_A)]$
2	RR	I[e]	$(1-R)/n_0[1-(r_1'r_2)^2\exp(-\phi_A)]$
3	F_p[b]	C,I	1
4	FPR[c]	C	$(1-R)(1-2r_2\cos\{2[\phi_D-\phi_D(x)]\}\exp[-\phi_A+\phi_A(x)]+r_2^2\exp[-2\phi_A+2\phi_A(x)])/G_1n_0$
5	FPR	I	$(1-R)\{1+r_2^2\exp[-2\phi_A+2\phi_A(x)]\}/G_2n_0$

[a]RR, ring resonator.
[b]F_p, free propagation.
[c]FPR, Fabry-Perot resonator.
[d]C, coherent.
[e]I, incoherent.

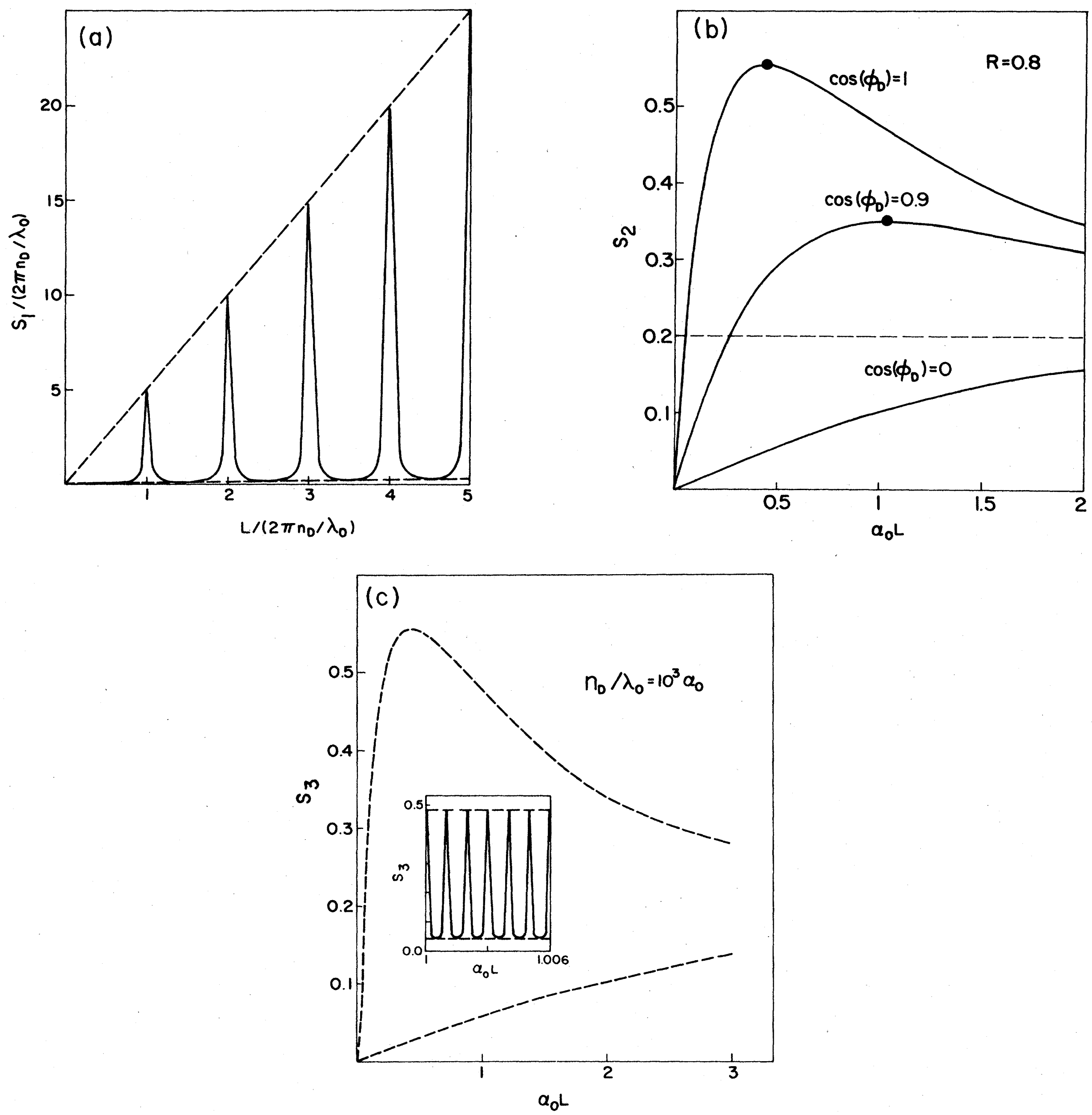

FIG. 2. Matching of a linear ring resonator via integrated intraresonator intensity. The reflectivity of both mirror M_1 and M_2 is 0.8 and the input intensity is $I_{in}=1$. (a) Integrated intensity of a purely dispersive medium. The dashed lines are the envelope, corresponding to a matched resonator (upper line) and to a totally mismatched resonator (lower line). (b) Integrated intensity of a purely absorptive medium. Solid circles denote maxima. (The curves are asymptotically approaching $1-R=0.2$.) (c) Integrated intensity in a medium which is both dispersive and absorptive. The dashed lines are the envelope. An enlarged section is depicted in the central frame.

becomes intensity dependent, giving rise to a self-absorptive-matching tendency, which is a necessary condition for the formation of bistability. S_2 is the first-order integrated intensity. If the nonlinear absorbed has a higher-order intensity dependence, higher-order intensity integrals have to be considered, in order to properly describe the absorptive matching.

The case of a medium which is both linearly dispersive and linearly absorptive in a CRR is depicted in Fig. 2(c). The integrated intensity has a global maximum due to absorptive matching and local maxima due to dispersive matching. This case will be discussed in further detail, in Sec. V, in connection with coupling of absorptive and dispersive nonlinear media.

III. PROPAGATION THROUGH MEDIA WHOSE ABSORPTION DECREASES WITH INTENSITY

In this section we analyze several examples of light propagation through a medium for which $\alpha(I)$ is a decreasing intensity-dependent function. The procedure for solving the eikonal equation, which was discussed in connection with Eq. (4), involves the graphical solution of the equation for the self-consistent phase. This equation can in general be written as

$$\phi_A = P + N(\phi_A) , \tag{16}$$

where P is the linear part and N is the nonlinear part.

In the case of decreasing absorption the solutions of Eq. (16) are the intersections of either a monotonically decreasing N or an N with a single minimum (which corresponds to maximal absorptive matching of the resonator) with a line with a unit slope (Fig. 3). It is obvious that the threshold for bistability is obtained by requiring the maximum of the first derivative to be equal to 1. This condition determines both the minimal input intensity and the minimal medium length. For any bistable loop, the up- and down-switching thresholds are determined by requiring the first derivative of the N to become equal to 1 for the self-consistent phase [Fig. 3(b)]:

$$\begin{aligned} &\partial N/\partial\phi_A \,|_{\phi_{A\,\mathrm{switch}}} = 1 , \\ &\phi_{A\,\mathrm{switch}} = P + N(\phi_{A\,\mathrm{switch}}) . \end{aligned} \tag{17}$$

The following phenomenological functional forms for $\alpha(I)$ are examined.

(a) Polynomial intensity dependence

$$\alpha(I) = \alpha_0[1 - p(I)] , \tag{18a}$$

where $p(I)$ is a monotonically increasing polynomial in I, satisfying $0 \le p(I) \le 1$.

(b) General saturable absorption

$$\alpha(I) = \alpha_0/[1 + p(I/I_s)] . \tag{18b}$$

Here I_s is a characteristic saturation intensity and $p(x)$ is a monotonically increasing positive polynomial. The common case of saturable absorption is associated with $p = I/I_s$.

For each one of the forms (a) and (b) we now investigate the case involving a linear polynomial. This is followed by an investigation of the case involving a quadratic polynomial.

A. Linearly intensity-dependent decreasing absorption

The simplest example of Eq. (18a) is the linearly intensity-dependent decreasing absorption coefficient

$$\alpha = \alpha_0(1 - \alpha_1 I), \quad I < 1/\alpha_1 . \tag{19}$$

This phenomenological expression for α is obtained in many particular cases, such as third-order perturbation treatment of saturable absorption in a two-level system,[11] or for a photochemically reactive disappearing absorber.[12]

For cases 1, 2, and 3 of Table I, for which K_i is independent of x, the solution of Eq. (6) with $\alpha(I)$ given by Eq. (19) is the transcendental equation

$$\begin{aligned} \phi_A &= \alpha_0 L - \ln\{[1 - \alpha_1 I_{\mathrm{in}} K_i \exp(-\phi_A)]/(1 - \alpha_1 I_{\mathrm{in}} K_i)\} \\ &\equiv F_i(\phi_A) . \end{aligned} \tag{20}$$

Equation (20) can be solved graphically for ϕ_A. The output intensity of the resonator is

$$I_{\mathrm{out}} = I(L) T_2 = I_{\mathrm{in}} K_i T_2 \exp(-\phi_A) , \tag{21}$$

where use has been made of Eq. (13).

For the case of free propagation ($i=3$) there is obviously only one solution for the self-consistent phase, since

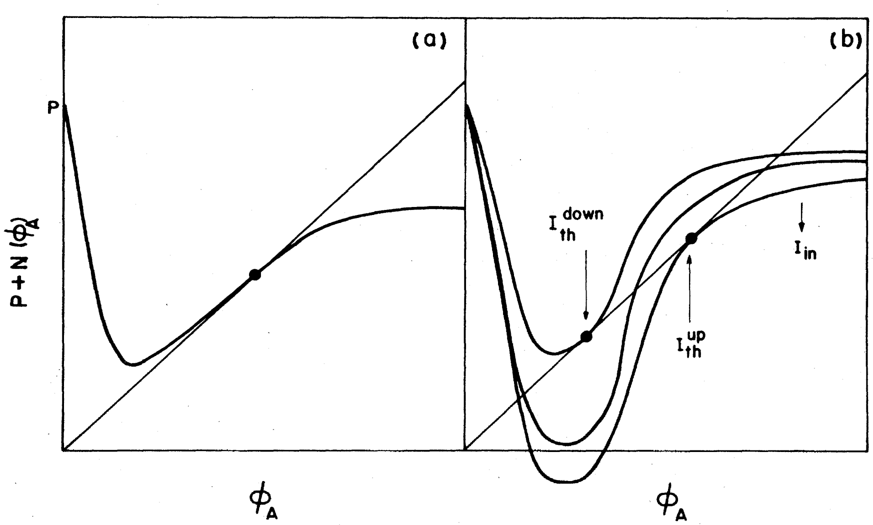

FIG. 3. Schematic representations of the solutions for the self-consistent phase in a medium exhibiting intensity-dependent decreasing absorption. (a) The case of a bistability threshold, i.e., self-consistency of the phase for which the maximum of the first derivative of the N equals one. (b) Up- and down-switching intensities for a typical bistability loop (solid circles).

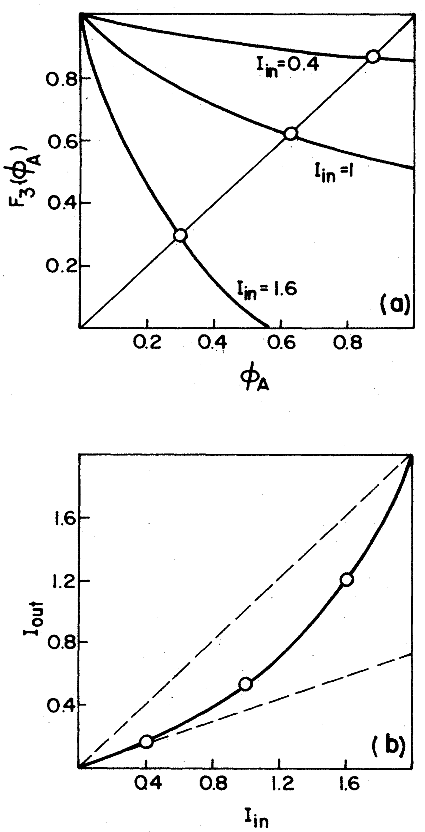

FIG. 4. Free propagation in a medium exhibiting linearly intensity-dependent decreasing absorption. (a) Solutions for the self-consistent phase (open circles). (b) Output vs input intensity. The upper dashed line is for a totally bleached absorber ($\alpha=0$) and the lower dashed line is for $\alpha=\alpha_0$.

$F_3(\phi_A)$ is a monotonically decreasing function of L.

The solutions for ϕ_A and for I_{out} versus I_{in} in this case are shown in Figs. 4(a) and 4(b), respectively.

For a coherent RR configuration $F_1(\phi_A)$ exhibits a minimum (Fig. 5). In the present case the minimum of $F_1(\phi_A)$ is too shallow to generate bistability. The reason is that such a model for nonlinear absorption cannot support intensities greater than (or even equal to) twice the saturation intensity [I_s is defined by $\alpha(I_s)=(\frac{1}{2})\alpha_0$]. Thus, the lower absorbance branch of a bistable solution, which represents, at least for $x=0$, a local intensity greater than $2I_s$, is not formed. A higher-order intensity dependence is required in order to maintain bistability. Yet the minimum in F_1 separates the solutions into two families. Denoting by $\phi_A^{\min}$ the value of ϕ_A at which $F_1(\phi_A)$ obtains its minimal value, we observe that solutions of Eq. (20) for which $\phi_A<\phi_A^{\min}$ form a saturation branch of the resonator, for which the effective absorption coefficient is smaller than that corresponding to the low-intensity absorption coefficient. The more interesting branch, corresponding to $\phi_A>\phi_A^{\min}$, exhibits differential reverse saturation, i.e., for finite intensities the absorption increases with L more steeply than in the low-intensity limit (Fig. 6).

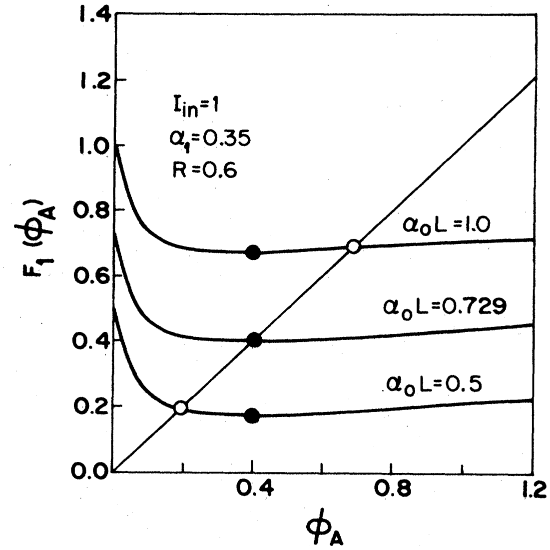

FIG. 5. Solutions for the self-consistent phase in a ring resonator containing a medium which exhibits linearly dependent decreasing absorption (open circles), for several medium lengths. Solid circles denote the minima of $F_1(\phi_A)$ corresponding to maximal absorptive matching.

For the same medium in a coherent FPR the intraresonator intensity depends explicitly on x, because of the formation of standing waves. Invoking the averaging discussed in connection with Eqs. (5) and (6) we obtain

$$d\phi_A(x)/(1-\alpha_1 I_{\text{in}} T_1\{\exp[-\phi_A(x)]+r_2^2\exp[-2\phi_A+\phi_A(x)]\}/(G_1 n_0))=\alpha_0 dx\ . \tag{22}$$

Integration and rearrangement give

$$\phi_A=\alpha_0 L+(1/D)\ln([A\exp(-\phi_A)+1-D](A+1+D)/\{[A\exp(-\phi_A)+1+D](A+1-D)\})+\phi_A\ , \tag{23}$$

where

$$D=\{1-4[\alpha_1 I_{\text{in}} T_1 r_2 \exp(-\phi_A)/G_1 n_0]^2\}^{1/2}$$

and

$$A=-2\alpha_1 I_{\text{in}} T_1/G_1 n_0\ .$$

The solution of Eq. (23) gives similar results to those obtained for the CRR.

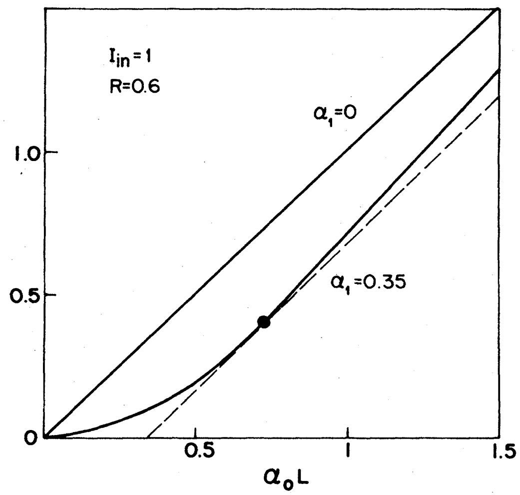

FIG. 6. Single-pass absorbance (ϕ_A) as a function of medium length for a medium exhibiting linearly intensity-dependent decreasing absorption in a ring resonator. The branching point is denoted by a solid circle. The dashed line with unit slope emphasizes the difference between the two branches with respect to their differential absorption.

B. Quadratically decreasing absorption

We now add a quadratic intensity term to $\alpha(I)$ of Eq. (19),

$$\alpha(I)=\alpha_0(1-\alpha_1 I+\alpha_2 I^2) . \tag{24}$$

To ensure that $\alpha(I)$ is a positive, monotonically decreasing function of the intensity, with a positive curvature, we set $\alpha_1,\alpha_2,\alpha_1^2-4\alpha_2>0$ and restrict I to the range $0\leq I\leq(\alpha_1-\Delta)2\alpha_2$, where $\Delta=(\alpha_1^2-4\alpha_2)^{1/2}$. The case of negative curvature was found to be similar to the linearly decreasing absorption. In particular, it exhibits no bistability.

The eikonal equation for this $\alpha(I)$, for the cases denoted by $i=1,2,3$ in Table I, becomes

$$d\phi_A(x)/[1-\alpha_1 I(x)+\alpha_2 I^2(x)]=\alpha_0 dx , \tag{25}$$

where $I(x)$ is given by Eq. (13). Integration and rearrangement yield

$$\begin{aligned}\phi_A &=\alpha_0 L-(\tfrac{1}{2})\ln\{[a\exp(-2\phi_A)-b\exp(-\phi_A)+1]/(a-b+1)\}\\ &\quad+(\alpha_1/2\Delta)\ln([2a\exp(-\phi_A)-b-\Delta](2a-b+\Delta)/\{[2a\exp(-\phi_A)-b+\Delta](2a-b-\Delta)\})\\ &\equiv W_i(\phi_A)\end{aligned} \tag{26}$$

where

$$a=\alpha_2(I_{in}K_i)^2$$

and

$$b=\alpha_1 I_{in}K_i .$$

The graphical solution of Eq. (26) and the resulting transmission curve are shown in Figs. 7(a) and 7(b), respectively. In this case a triple solution is obtained resulting in a typical bistable (hysteresislike) I_{out} versus I_{in}.

Bistability due to quadratic intensity dependent nonlinear absorption is restricted to a very limited domain of

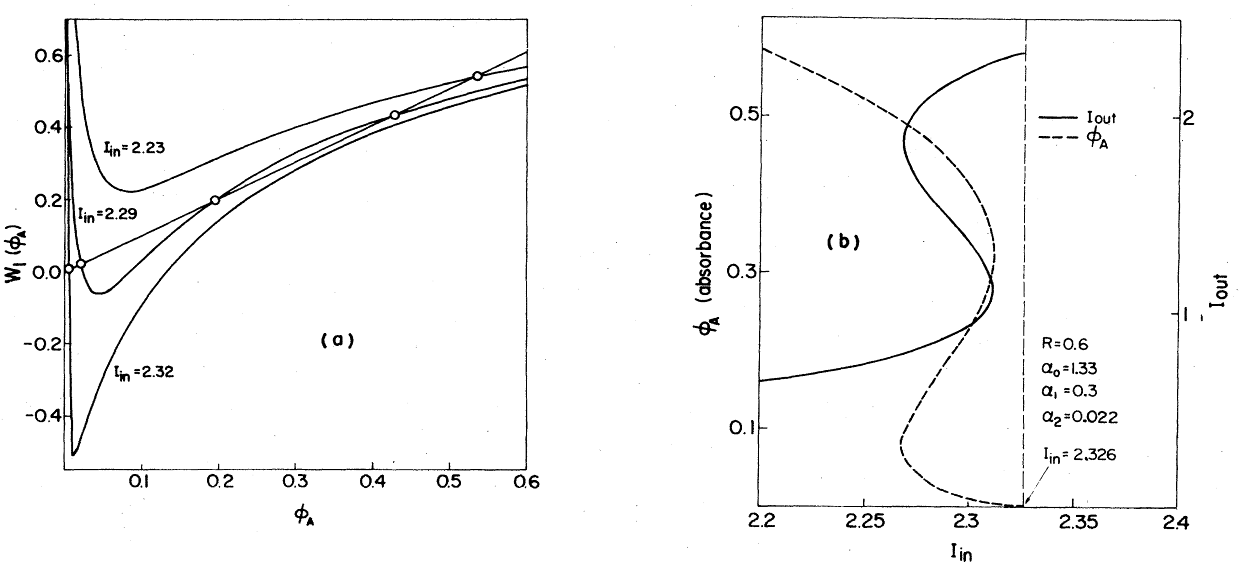

FIG. 7. Bistability in a ring resonator, containing a medium exhibiting quadratically intensity-dependent decreasing absorption. (a) The solution for the self-consistent phase (open circles). (b) Single-pass absorbance and output intensity vs input intensity. For $I_{in}>2.326$ the model has no physical meaning for this particular absorptive nonlinearity.

parameters $(I_{in}, R, \alpha_1, \alpha_2, \alpha_0)$ since this nonlinearity can support only up to 3.4 times the saturation intensity, and only by a very fine adjustment of the parameters can one reach bistability where internal local intensity in the lower absorbance branch does not exceed this limit.

C. Saturable absorber

The most common nonlinear absorber discussed in optical bistability studies is the simple saturable absorber.

For the cases $i=1,2,3$ the eikonal equation (6) for general saturation is

$$d\phi_A(x)\{1+p(I_{in}K_i\exp[-\phi_A(x)]/I_s)\}=\alpha_0 dx\ , \quad (27)$$

which yields the following transcendental equation for ϕ_A:

$$\phi_A=\alpha_0 L+\int_0^{\exp(-\phi_A)} d\xi\, p((I_{in}K_i/I_s)\xi)/\xi\ . \quad (28)$$

For a simple saturation behavior, $p=I/I_s$, we obtain

$$\phi_A=\alpha_0 L+(I_{in}K_i/I_s)[\exp(-\phi_A)-1]\equiv S_i(\phi_A)\ . \quad (29)$$

For a CRR $(i=1)$ this equation, which is equivalent to that obtained by Bonifacio *et al.*,[3] results in bistability. Using the condition for bistability in this case $(\partial S_1/\partial\phi_A)_{max}=1$ we obtain, for a dispersively matched resonator, the threshold intensity and the minimal medium length as functions of the resonator mirror reflectivity R,

$$\begin{aligned}
&\alpha_0 L_{min}=\phi_{A\,min}-(1-R)(I_{in,min}/I_s n_0)[\exp(-\phi_A)-1]/[1-R\exp(-\phi_{A\,min}/2)]^2\ ,\\
&I_{in,min}=I_s n_0[1-R\exp(-\phi_{A\,min}/2)]^3/\{(1-R)[R\exp(-\phi_{A\,min}/2)-\exp(-\phi_{A\,min})]\}\ ,\\
&\phi_{A\,min}=-2\ln\{[R^2-1+(R^4-R^2+1)^{1/2}]/R\}\ .
\end{aligned} \quad (30)$$

The minimal medium length $(\alpha_0 L_{min})$ as a function of R, compared to the results of the mean-field limit,[3] are depicted in Fig. 8.

We now consider switching of the saturable absorber CRR by changing the absorptive medium length $(\alpha_0 L)$, the dispersive length (ϕ_D), or both.

We first change the absorber length continuously, keeping the resonator dispersively matched. For intensities smaller than the threshold, we obtain the formation of a saturation and of a reverse saturation branch, which were discussed for a linearly decreasing absorption (Fig. 9). Here the absorbance at the branch separation $[\min S_1(\phi_A)]$ is determined only by the resonator mirror reflectivity ($\phi_A=-2\ln R$ for a dispersively matched resonator) in contrast to the former nonlinearities where branching is dependent also on the parameters of the medium. For intensities above the threshold we obtain an absorbance hysteresis cycle as a function of medium length (Fig. 9). Output intensities in these cases are depicted in Fig. 10. Changing the dispersive length (changing mirror separation, for example) changes the resonator matching. If $\alpha_0 L$ and I_{in} are set to provide a bistable solution in a matched resonator, changing the dispersive length can cause only up-switching of the absorbance (Fig. 11). If $\alpha_0 L$ and I_{in} are set to maintain the matched resonator in a low-absorbance solution, a full hysteresis cycle can be formed while changing the dispersive length. Changing both the dispersive and the absorptive length

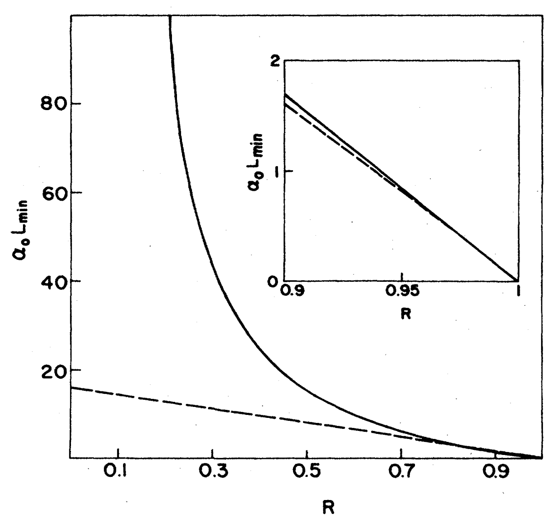

FIG. 8. Minimal medium length as a function of mirror reflectivity for a saturable absorber in a matched ring resonator with reflectivities $R_1=R_2=R$. The dashed line is the mean-field limit $(\alpha_0 L\to 0,\ R\to 1)$, $\alpha_0 L/(1-R)=$const.

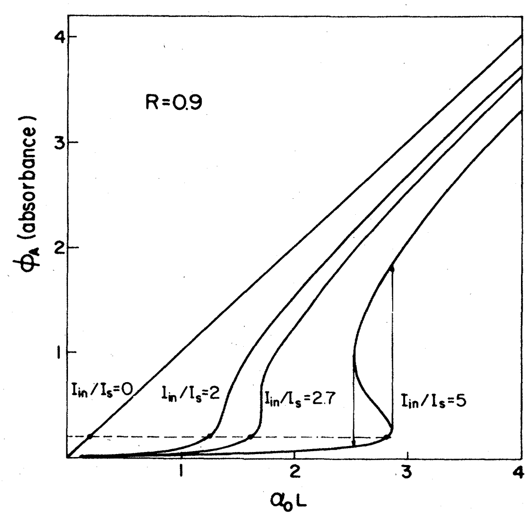

FIG. 9. Single-pass absorbance (ϕ_A) for a matched ring resonator $(R_1=R_2=0.9)$ containing a simple saturable absorber, as a function of the medium length. Absorbance at the branching points (solid circles) does not depend on either the medium parameters or the input intensity. An absorbance hysteresis cycle exists for certain values of the input intensity.

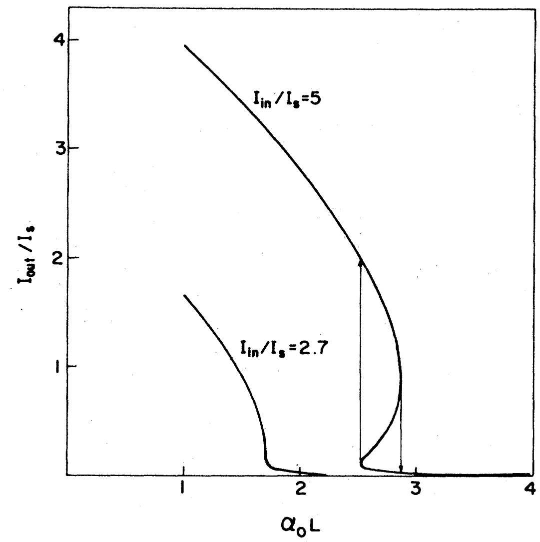

FIG. 10. Output intensity of a ring resonator containing a simple saturable absorber, as a function of the medium length, for several values of the input intensity.

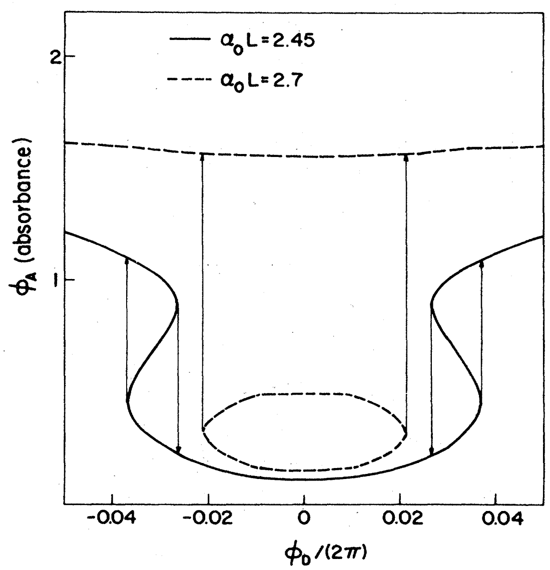

FIG. 11. Switching of a ring resonator containing a simple saturable absorber by changing the dispersive matching of the resonator. The dashed line represents the case where bistability exists in the matched resonator; the solid line represents the case where the resonator is in its low-absorbance branch for a matched resonator.

(for example, changing the medium length without compensation for the resonator dispersive mismatch) we obtain the following (Fig. 12): For small values of $\alpha_0 L$ we obtain continuous oscillations between lower and higher absorbance as a function of medium length due to a dispersive cycle of mismatching. For values of $\alpha_0 L$ close to (but smaller than) the down-switching threshold of Fig. 9, these oscillations become bistable. At exactly the down threshold of Fig. 9, where the upper absorbance branch is just formed in a dispersively matched resonator, the bistable oscillations precipitate to this branch and from this medium length onward there exist only continuous oscillations between the upper absorbance branch and a higher branch corresponding to a totally mismatched resonator.

For a CFPR, using the averaging procedure over I, we get the following eikonal equation:

$$d\phi_A(x)\{1+I_{\rm in}(1-R)(\exp[-\phi_A(x)]+r_2^2\exp[-2\phi_A]\exp[\phi_A(x)])/n_0 I_s G_1\}=\alpha_0 dx \ . \tag{31}$$

Integrating and rearranging we obtain

$$\phi_A=\alpha_0 L-(I_{\rm in}(1-R)/n_0 I_s G_1)[1+(r_2^2-1)\exp(-\phi_A)-r_2^2\exp(-2\phi_A)] \ , \tag{32}$$

which gives similar results to the CRR configuration, but with smaller threshold intensities.

1. Quadratically saturable absorber

Adding a quadratic term in the intensity $p=I/I_s+a(I/I_s)^2$, substituting and integrating the eikonal equation (for cases 1—3), we obtain the following equation for the phase:

$$\phi_A=\alpha_0 L+(I_{\rm in}K_i/I_s)[\exp(-\phi_A)-1]+[a(I_{\rm in}K_i/I_s)^2/2][\exp(-2\phi_A)-1] \ . \tag{33}$$

This equation does not yield bistability in free propagation but yields bistability in a CRR configuration. The most interesting result is that this nonlinearity induces bistability in incoherent configurations. This cannot happen for any of the formerly discussed nonlinearities. The explanation of this observation is that the integrated intensity of an incoherent RR (IRR) has no maximum and thus has no absorptive matching condition for first-order saturation. However, the integrated squared intensity of the IRR has a maximum. Therefore, quadratic nonlinearities such as the one presently discussed can give rise to an absorptive self-matching tendency and thus yield bistability in this configuration. This result has a practical importance, since the matching of coherent resonators is a difficult task.

The threshold conditions for a purely quadratically saturable absorber $[p=(I/I_s)^2]$ in an IRR are derived using the procedure formerly discussed, yielding

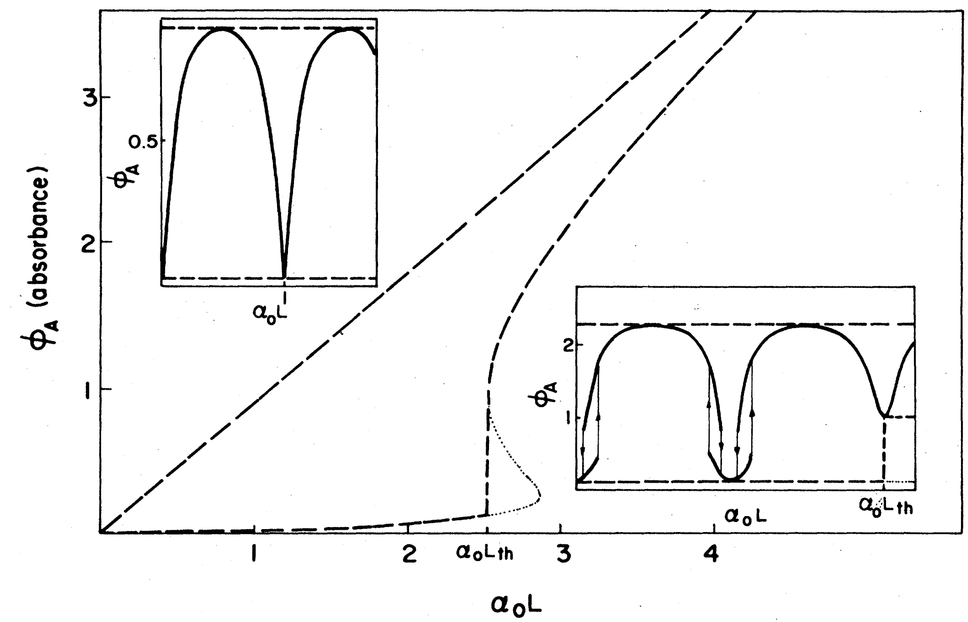

FIG. 12. Switching of a ring resonator containing a simple saturable absorber by changing both the dispersive and the absorptive matching of the resonator. The dashed line is the envelope of the single-pass absorbance as a function of medium length. The left frame is an enlargement of the length scale near $\alpha_0 L = 1$, where continuous oscillations between high and low absorbance exist. The right frame is an enlargement of the length scale near $\alpha_0 L_{\text{th}}$. Bistable loops exist up to the threshold length ($\alpha_0 L_{\text{th}}$), where the solutions precipitate to the higher absorbance branch.

$$
\begin{aligned}
&\alpha_0 L_{\min} = \phi_{A\,\min} - [(1-R)^2 I_{\min}^2 / 2 I_s^2 n_0^2][\exp(-2\phi_{\min}) - 1]/[1 - R^2 \exp(-\phi_{A\,\min})]^2 \ , \\
&I_{\min} = I_s n_0 [1 - R^2 \exp(-\phi_{A\,\min})]^{3/2}]\{(1-R)[R^2 \exp(-\phi_{A\,\min}) - \exp(-2\phi_{a\,\min})]\}^{1/2} \ , \\
&\phi_{A\,\min} = -\ln\{[R^4 - 1 + (R^8 - R^4 + 1)^{1/2}]/R^2\} \ .
\end{aligned}
\tag{34}
$$

Threshold intensities and medium lengths are depicted in Fig. 13. Note that in contrast to a simple saturable absorber in a CRR, we have here a finite threshold intensity even for $R \rightarrow 1$.

IV. PROPAGATION THROUGH MEDIA WHOSE ABSORPTION INCREASES WITH INTENSITY

In this section we analyze several examples of light propagation through a medium for which $\alpha(I)$ is an increasing intensity-dependent function. Note that we do not mean by "increasing absorption" the self- or dynamically increasing absorption discussed in Ref. 7.

The phenomenological functional forms of $\alpha(I)$ are chosen to model several realistic absorption coefficients. We examine the following.

(i) Polynomial intensity dependence

$$\alpha = \alpha_0[1 + p(I)] \ , \tag{35a}$$

where $p(I)$ is a monotonically increasing polynomial in I.

(ii) Bounded increasing absorption

$$\alpha(I) = \alpha_0 f(I)/g(I) \ , \tag{35b}$$

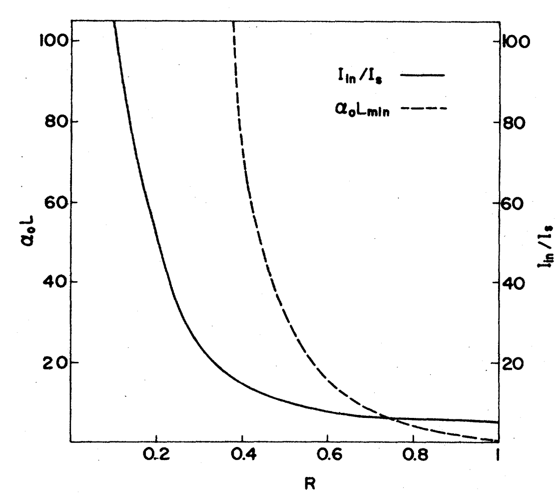

FIG. 13. Threshold intensity and minimal medium length for a ring resonator containing a quadratically intensity-dependent saturable absorber. The resonator is illuminated by a totally incoherent light source. Both mirror reflectivities are R.

where $\alpha(I)$ is a monotonically increasing rational function of I, so the $f(I)$ and $g(I)$ have the same degree.

(iii) Absorption going through a maximum

$$\alpha(I)=\alpha_0/(1-aI+bI^2),\quad a,b>0;\ a^2<4b\ . \tag{35c}$$

In most of these cases the graphical solution for the self-consistent phase is the intersection of either a monotonically increasing function (with a monotonically decreasing first derivative) or of a function with a single maximum, with a line with unit slope (Fig. 14). Therefore, it is obvious that in these cases the existence of bistability is prohibited. The physical reason is that feedback in resonators enhances the branches of high-energy content. Here, however, the absorption coefficient increases with the intensity, which in turn reduces the feedback intensity, so that the feedback loop is broken.

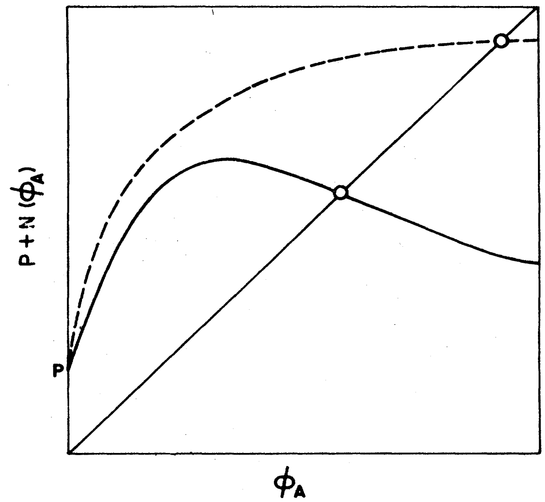

FIG. 14. Schematic representation of solutions for the self-consistent phase in a medium exhibiting intensity-dependent increasing absorption. The solutions are denoted by circles. The solid curve is for configurations where absorptive matching is possible, thus the curve exhibits a maximum. The dashed line is for configurations which do not give rise to absorptive matching.

A. Linearly increasing absorption

This case is the simplest approximation for common physical models of increasing absorption. $\alpha(I)$ is written as

$$\alpha(I)=\alpha_0(1+\alpha_1 I)\ . \tag{36}$$

Solving the eikonal equation in this case (cases 1, 2, and 3 of Table I) yields

$$\phi_A=\alpha_0 L-\ln\{[1+\alpha_1 I_{\text{in}}K_i\exp(-\phi_A)]/(1+\alpha_1 I_{\text{in}}K_i)\}$$
$$\equiv Q_i(\phi_A)\ . \tag{37}$$

For free propagation ($i=3$), we obtain the expected results: superexponential intensity decrease as a function of medium length. For a CRR, we obtain two families of solutions. For solutions of Eq. (37) satisfying $\phi_A<\max[Q_1(\phi_A)]$, we obtain an inverse saturation branch (differential absorbance greater than 1), which is the normally expected branch. For solutions greater than $\phi_A^{\max}$ we obtain a second saturation branch induced by the resonator (Fig. 15). No bistability is obtainable in this configuration. For a IRR ($i=2$) configuration, the solution is very similar to the free-propagation case (no branching). The solutions for a FPR are similar.

B. Quadratically increasing absorption

For a quadratic nonlinearity $\alpha(I)=\alpha_0(1+\alpha_1 I+\alpha_2 I^2)$, where $\alpha(I)$ is monotonically increasing, we obtain the following solution to the eikonal equation:

$$\phi_A=\alpha_0 L-(\tfrac{1}{2})\ln\{[a\exp(-2\phi_A)+b\exp(-\phi_A)+1]/(a+b+1)\}$$
$$+(b/2d)\ln\{[2a\exp(-\phi_A)+b-d](2a+b+d)/[2a\exp(-\phi_A)+b+d](2a+b-d)\}\ , \tag{38}$$

where

$$a=\alpha_2(I_{\text{in}}K_i)^2\ ,$$

$$b=\alpha_1 I_{\text{in}}K_i\ ,$$

$$d=I_{\text{in}}K_i(\alpha_1^2-4\alpha_2)^{1/2}\ .$$

For F_p and for a CRR, the results are similar to those obtained in the linearly increasing case. For an IRR configuration, however, the quadratic nonlinearity induces branching of the solutions, similarly to the formerly discussed case. The branching is due to the formation of absorptive matching of the incoherent resonator, which can be realized from the maximum of Eq. (38), when substituting K_2 from Table I.

C. Bounded increasing absorption

This form of nonlinearity is the counterpart of saturation in decreasing absorption. It arises in several absorbing media, such as in a three-level system with consecutive two-photon absorption.[12]

In this case $\alpha(I)$ is written as

$$\alpha=\alpha_0(1+\alpha_1 I)/(1+I),\quad \text{where } \alpha_1>1\ . \tag{39}$$

For cases 1—3 of Table I, the solution of the eikonal equation yields

$$\phi_A=\alpha_0 L-[(1-\alpha_1)/\alpha_1]\ln\left[\frac{1+\alpha_1 I_{\text{in}}K_i\exp(-\phi_A)}{1+\alpha_1 I_{\text{in}}K_i}\right]\ . \tag{40}$$

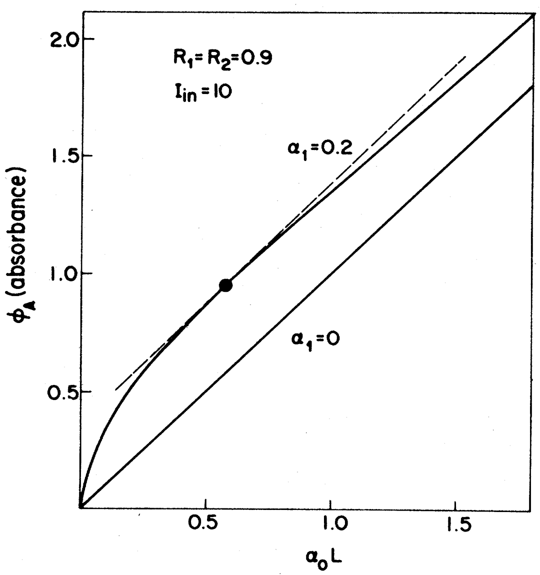

FIG. 15. Single-pass absorbance (ϕ_A) as a function of medium length for a medium exhibiting linearly intensity increasing absorption in a ring resonator. The branching point is denoted by a full circle. The dashed line with the unit slope emphasizes the difference between the two branches with respect to their differential absorption.

The solutions are similar to those obtained for a linearly increasing absorption.

D. Absorption going through a maximum

This nonlinearity characterizes a medium whose increasing absorption saturates. $\alpha(I)$ is thus written as

$$\alpha(I)=\alpha_0/(1-aI+bI^2)\ , \tag{41}$$

where

$$a,b>0,\quad a^2<4b\ .$$

Note that $\alpha(I)$ obtains a maximum at $I_{\max}=a/(2b)$.

Solving the eikonal equation in this case we obtain for $i=1,2,3$ of Table I

$$\phi_A=\alpha_0L-aI_{\rm in}K_i[\exp(-\phi_A)-1] +0.5b(I_{\rm in}K_i)^2[\exp(-2\phi_A)-1]\ . \tag{42}$$

Note that for this type of nonlinearity there is an absorptive matching tendency even in free propagation ($i=3$). Thus, the two families of solutions, corresponding to the increasing ($I<I_{\max}$) and saturation ($I>I_{\max}$) parts of the absorption coefficient, are obtained for F_p as well. In contrast to the formerly discussed nonlinearity, this branching has a threshold ($I_{\rm in}>I_{\max}$). The absorption coefficient and the absorbance as a function of the propagation distance in the medium are depicted in Fig. 16 for various input intensities. Optical bistability is not obtained.

For a CRR we obtain branching of the solution and the existence of bistability. This is due to the fact that for high intensities the absorption coefficient saturates, which is enhanced by the resonator.

Bistability also exists for IRR but with much higher threshold intensities. The threshold intensities can be derived using the procedure presented in Sec. III.

For a FPR, using the averaging procedure for I (removing the explicit x dependence), and solving the eikonal equation, we obtain

$$\begin{aligned}\phi_A=\alpha_0L&-aI'[\exp(-\phi_A)-1]+aI'r_2^2[\exp(-\phi_A)-\exp(-2\phi_A)]+\tfrac{1}{2}bI'^2[\exp(-2\phi_A)-1]\\&-\tfrac{1}{2}bI'^2r_2^4[\exp(-2\phi_A)-\exp(-4\phi_A)]-2bI'r_2^2\exp(-2\phi_A)\phi_A\ ,\end{aligned} \tag{43}$$

where $I'=I_{\rm in}T_1/(n_0G_1)$ for CFPR and $I'=I_{\rm in}T_1/(n_0G_2)$ for IFPR. This solution yields bistability both for a coherent and for an incoherent FPR.

V. RESONATORS WITH NONLINEARITIES IN BOTH ABSORPTION AND DISPERSION

In order to analyze the coupling of absorptive and dispersive nonlinear media in an optical resonator, we have to solve the full-fledged complex eikonal equation.

In this section we examine "external" coupling—where the absorptive and dispersive nonlinearities are not correlated (different media)—and "internal" coupling—where the absorptive and dispersive nonlinearities are in the same medium and are caused by the same physical mechanism.

The complex eikonal equation can be written as

$$\begin{aligned}\phi_A(x)&=\int_0^x\alpha(I(x',\phi_A(x'),\phi_D(x'),\phi_A,\phi_D))dx'\ ,\\ \phi_D(y)&=(2\pi/\lambda_0)\int_0^y n_D(I(y',\phi_A(y'),\phi_D(y'),\phi_A,\phi_D))dy'\ .\end{aligned} \tag{44}$$

When the media are locally coupled, i.e., the medium is homogeneous with both absorptive and dispersive nonlinearities ($x=y$), we use

$$\begin{aligned}\partial\phi_D(x)/\partial x&=[\partial\phi_D(x)/\partial\phi_A(x)][\partial\phi_A(x)/\partial x]\\&=[\partial\phi_D(x)/\partial\phi_A(x)]\alpha(I)\end{aligned} \tag{45}$$

in order to obtain

$$\begin{aligned}d\phi_A(x)/\alpha(I)&=dx\ ,\\ d\phi_D(x)&=(2\pi/\lambda_0)[n_D(I)/\alpha(I)]d\phi_A\ .\end{aligned} \tag{46}$$

This form of the eikonal equation is convenient, since

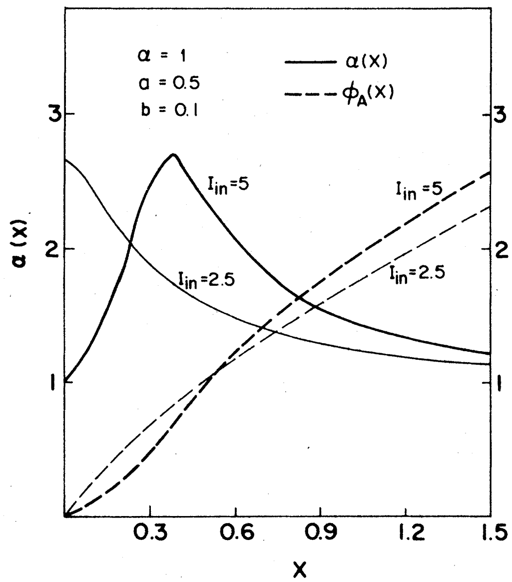

FIG. 16. Absorption coefficient and the total absorbance for a free propagation of light in a medium exhibiting absorption going through a maximum, for two input intensities. Here, in contrast to all the formerly discussed absorptive nonlinearities, the absorption coefficient is not monotonic. Branching point of the absorbance (denoted by a solid circle) exists only for $I_{in} > I_{max}$.

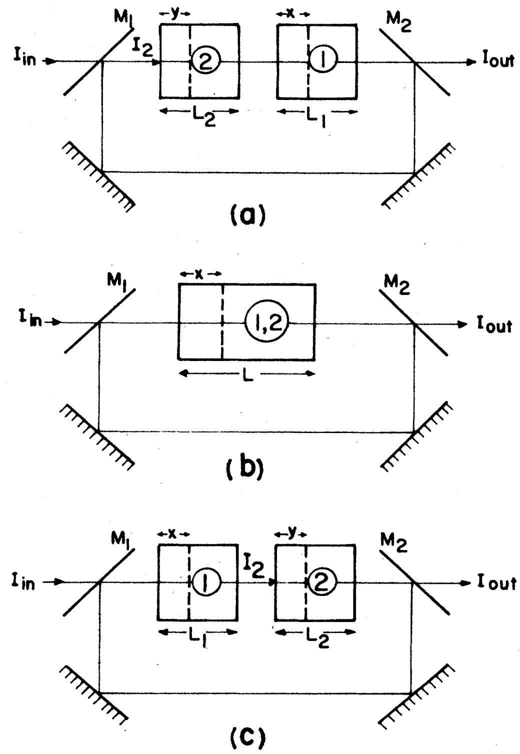

FIG. 17. Configurations of coupled nonlinear media in a ring resonator. Medium 1 is a saturable absorber and 2 is a Kerr-like medium. In all cases the resonator is dispersively matched for low input intensity, and has mirror reflectivities ($R_1 = R_2 = R$). (a) The absorber is closer to the input port. (b) The absorber and the nonlinear Kerr-like medium are in the same location. (c) The nonlinear Kerr-like medium is closer to the input port.

in all optical configurations which do not involve the formation of standing waves I is independent of $\phi_D(x)$, although it still depends on $\phi_D \equiv \phi_D(L)$. In this case the two coupled equations (46) can be integrated explicitly. For configurations involving standing waves the explicit dependence of the intensity in $\phi_D(x)$ can be removed using averaging over the standing waves, and then again we can integrate the two equations explicitly.

A. External coupling of absorptive and dispersive nonlinearities in a resonator

We consider the coupling of a low-intensity Kerr-like medium $[n_D = n_{D0}(1+n_2 I)]$ and a purely absorptive saturable absorber in an optical resonator. These configurations are discussed. The configurations and definition of coordinates are specified in Figs. 17(a)–17(c).

For the configurations a,c the coupling is such that each medium is controlling the input intensity to the other by a partial attenuation of light intensity (the nonlinear absorber) or by changing the matching between the input and the feedback fields (the nonlinear Kerr medium). For these configurations we can write the complex eikonal approximation as follows:

$$d\phi_A(x)\{1+(I_{in}H/I_s)\exp[-\phi_A(x)]\} = \alpha_0 dx \ ,$$
$$d\phi_D(y) = (2\pi n_0/\lambda_0)(1+n_2 I_2 H)dy \ , \quad (47)$$

where

$$H = (1-R)/n_0[1-2R\cos(\phi_D)\exp(-\phi_A/2) + R^2\exp(-\phi_A)] \quad (48)$$

and $\phi_A \equiv \phi_A(L_1); \phi_D \equiv \phi_D(L_2)$ and

$$I_2 = I_{in}\exp(-\phi_A)$$

in configuration (a), (49)

$$I_2 = I_{in}$$

in configuration (c).

Integrating we get

$$\phi_A = \alpha_0 L_1 + (I_{in}H/I_s)[\exp(-\phi_A)-1] \ , \quad (50a)$$

$$\phi_D = (2\pi n_{D0} L_2/\lambda_0)(1+n_2 I_2 H) \ . \quad (50b)$$

ϕ_D can be written explicitly from Eq. (50a) as a function of ϕ_A, thus we obtain a single transcendental equation for ϕ_A,

$$\phi_A = \alpha_0 L_1 + [I_{in}H(\phi_A, \phi_D)/I_s][\exp(-\phi_A)-1] \ , \quad (51)$$

where

$$\phi_D=(2\pi n_{D0}L_2/\lambda_0)\{1+(n_2I_sI_2/I_{\text{in}})(\phi_A-\alpha_0L_1)/[\exp(-\phi_A)-1]\}\ . \tag{52}$$

In configuration (b) [Fig. 17(b)], the coupling of the absorptive and dispersive nonlinearities is locally external, i.e., the medium is homogeneous with both absorptive and dispersive nonlinearities but the nonlinearities are not correlated. Thus, Eq. (46) becomes

$$d\phi_A(x)\{1+(I_{\text{in}}H/I_s)\exp[-\phi_A(x)]\}=\alpha_0dx\ , \tag{53a}$$

$$d\phi_D(x)=(2\pi n_{D0}/\lambda_0\alpha_0)\{1+n_2I_{\text{in}}H\exp[-\phi_A(x)]\}\{1+(I_{\text{in}}H/I_s)\exp[-\phi_A(x)]\}d\phi_A(x)\ . \tag{53b}$$

Integrating Eq. (53a) we obtain the same equation as (51a). Integrating Eq. (53b) we obtain

$$\phi_D=(2\pi n_{D0}/\lambda_0\alpha_0)\{\phi_A+I_{\text{in}}H(n_2+1/I_s)[1-\exp(-\phi_A)]+(n_2I_{\text{in}}^2H^2/2I_s)[1-\exp(2\phi_A)]\}\ , \tag{54}$$

where $L_1=L_2=L$.

ϕ_D can be written explicitly in terms of ϕ_A from Eq. (51a), thus using Eq. (54) we obtain a single transcendental equation for ϕ_A,

$$\phi_A=\alpha_0L+[I_{\text{in}}H(\phi_D,\phi_A)/I_s][\exp(-\phi_A)-1]\ , \tag{55}$$

where

$$\begin{aligned}\phi_D&=(2\pi n_{D0}/\lambda_0\alpha_0)\{-n_2I_s\phi_A+(n_2I_s+1)\alpha_0L+\tfrac{1}{2}n_2I_s(\phi_A-\alpha_0L)2[\exp(-\phi_A)+1]/[1-\exp(-\phi_A)]\}\\&=V(\phi_A)\ .\end{aligned} \tag{56}$$

We describe the coupling between the nonlinear absorber and the nonlinear dispersive medium as follows: We fix the parameters of the absorber ($\alpha_0;I_s$), and examine the coupling for different values of n_2. Note that the dispersive phase accumulation (ϕ_D) is a decreasing function of the absorptive phase (ϕ_A) (Fig. 18). For low values of n_2, dispersive bistability occurs only in the lower absorbance branch and for very high intensities. For parameters in which absorptive bistability exists for the pure absorber ($n_2=0$), rather low values of n_2 are sufficient to destroy this bistability as a consequence of the resonator mismatching (Fig. 19). For higher n_2, the density of bistable loops in the lower absorbance branch increases and their up-switching threshold intensities decrease. For very high values of n_2 dispersive bistability loops occur in the upper absorbance branch as well. In this case we obtain a bistability threshold lower than the thresholds obtained for a purely absorptive nonlinearity ($n_2=0$). For high enough values of n_2 and for intensities higher than the purely absorptive up-switching threshold, the loops are continuously going from the lower to the upper absorbance range. For intensities within the purely absorptive bistable region there is a separation between the two families of dispersive bistable loops (Figs. 18 and 20). The down-switching thresholds of the dispersive bistability loops follow closely the intensity curve obtained for the purely absorptive case in a dispersively matched resonator. The up-switching thresholds are limited by the intensity curve for the purely absorptive case in a dispersively totally mismatched resonator. This switching pattern is depicted schematically in Fig. 21. Note that within the eikonal approximation the results are the same for both a positive and a negative Kerr medium.

There is no fundamental difference between cases a, b, and c of Fig. 16, except for the coupling strength. In case c the intensity in the Kerr medium is maximal, thus the nonlinear part of ϕ_D is the largest. For solutions in the high-absorbance branch the difference between the three cases is appreciable, while in the low-absorbance branch, in which the absorber is almost bleached, the difference is minimal. Thus, for the same n_2 the bistable loops are denser, i.e., closest in output intensity for case a, in which they are higher in output intensity since a higher degree of bleaching of the absorber is required.

B. Internal coupling of dispersive and absorptive media

In the case of internal coupling the same mechanism is the driving force for both the absorptive and the dispersive nonlinearities. Thus, in general there is a functional dependence of the type $F(n_D(I),\alpha(I))=0$. In this section we discuss the case of a homogeneously broadened two-level medium while the other functional dependences such as $n_D=n_{D1}N_1(I)+n_{D2}N_2(I)$; $\alpha=\alpha_1N_1(I)+\alpha_2N_2(I)$, where N_1,N_2 are intensity-dependent populations of two species, will be discussed elsewhere[12] in connection with a molecular absorber.

1. Two-level medium

For a two-level medium, the steady-state nonlinear refractive index can be written as

$$\begin{aligned}n_D&=[1+\Delta\alpha_0/(1+I/I_s)]^{1/2}\ ,\\\alpha&=\alpha_0/(1+I/I_s)\ ,\end{aligned} \tag{57}$$

where the detuning parameter Δ is expressed in terms of the resonance frequency ω_A and the transverse relaxation rate $\gamma_\perp$, by[11]

$$\Delta=(\omega_A-\omega)c/\omega\gamma_\perp\ .$$

For a CRR with the medium specified by Eq. (57) the eikonal equation (46) yields the two coupled equations

$$d\phi_A(x)\{1+I(\phi_A(x))/I_s\}=\alpha_0dx\ , \tag{58a}$$

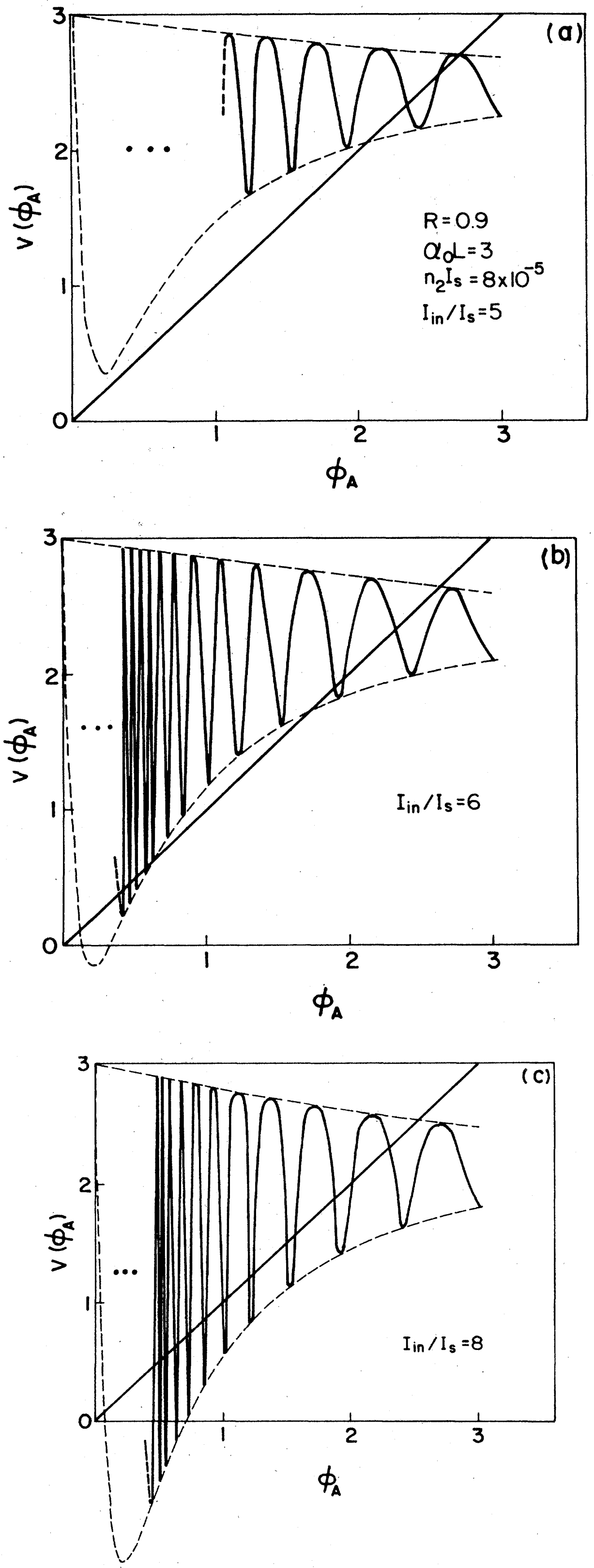

FIG. 18. Graphical solution for the self-consistent phase in case (c) of Fig. 17. The dashed lines are the envelope of V. The oscillation of V for small ϕ_A (in the region denoted by $\cdots$) are too dense to draw.

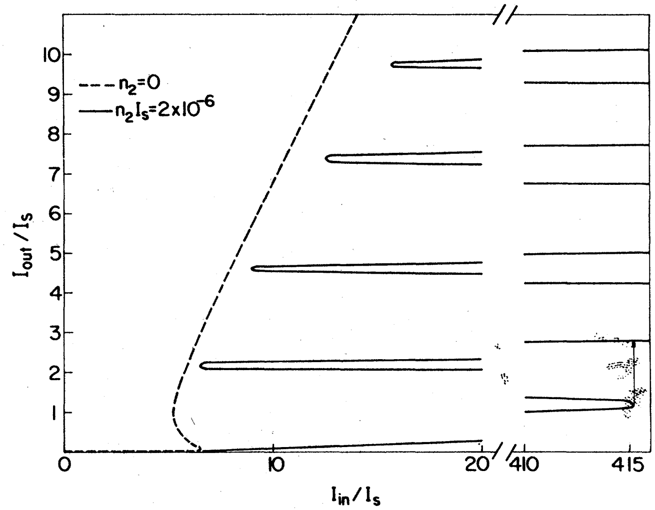

FIG. 19. Output intensity vs input intensity in the case of coupling between a saturable absorber and a Kerr-like medium with a small nonlinear coefficient n_2. The dashed curve is the output intensity for a pure saturable absorber.

$$d\phi_D(x)=(2\pi/\lambda_0\alpha_0)\times\{(1+\Delta\alpha_0+I(\phi_A(x))/I_s)\times(1+I(\phi_A(x))/I_s)\}^{1/2}\,, \tag{58b}$$

where $I(\phi_A(x))$ is given by Eq. (13) for the case $i=1$ of Table I.

Equation (58a) is the same as the one obtained for simple saturation, as discussed in Sec. III.

Equation (58b) is integrated to yield

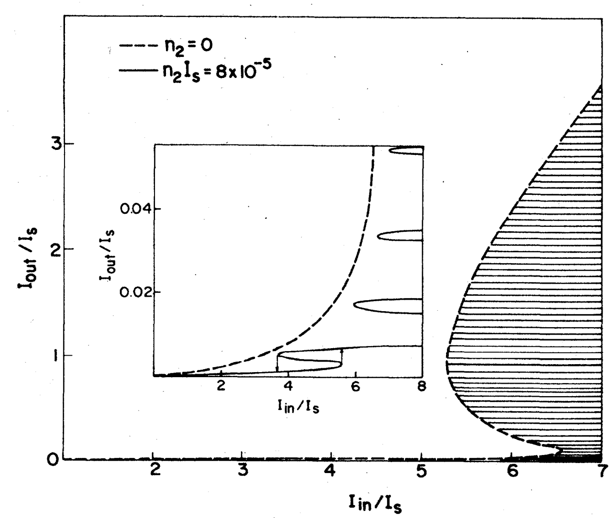

FIG. 20. Output intensity vs input intensity in the case of coupling between a saturable absorber and a Kerr-like medium, with a large nonlinear coefficient n_2. The dashed curve is the output intensity for a pure saturable absorber. Each horizontal line represents a bistable loop. The inset frame presents an enlargement of the lower intensity branch.

$$\phi_D=(2\pi/\lambda_0\alpha_0)(A(0)-A(\phi_A)+B^{1/2}\ln\{[2B+CD\exp(-\phi_A)+2B^{1/2}A(\phi_A)]/[\exp(-\phi_A)(2B+DC+2B^{1/2})A(0)]\}$$
$$+(C/2)\ln\{[C+2D+2A(0)]/[C+2D\exp(-\phi_A)+2A(\phi_A)]\})\ , \qquad (59)$$

where

$$A(\phi_A)=[B+DC\exp(-\phi_A)+D^2\exp(-2\phi_A)]^{1/2}\ ,$$
$$B=1+\Delta\alpha_0\ ,$$
$$C=2+\Delta\alpha_0\ ,$$
$$D=K_1I_{\text{in}}/I_s$$

(K_1 is given in Table I). As in the previous case, ϕ_D is written explicitly as a function of ϕ_A, yielding

$$\phi_A=\alpha_0L+[I_{\text{in}}K_1(\phi_A,\phi_D)/I_s]\times[\exp(-\phi_A)-1]\ . \qquad (60)$$

To obtain ϕ_D as an explicit function of ϕ_A we substitute the expression

$$(I_{\text{in}}K_1/I_s)=(\phi_A-\alpha_0L)/[\exp(-\phi_A)-1] \qquad (61)$$

in Eq. (59).

This solution is equivalent to the analytical solution obtained by Gronchi and Lugiato[13] using the slowly-varying-envelope (SVE) approximation to the Maxwell equation, since in the present case the SVE and eikonal approximations can be shown to coincide.

In this case the nonlinear part of ϕ_D is an increasing function of ϕ_A, in contrast to the former case of a Kerr medium coupled to a saturable absorber. Thus, drastically different results are obtained. A small dispersive nonlinearity (small Δ) can result in lower bistability threshold intensities. Increasing the detuning, we obtain an increased number of bistable loops (in the former case for any finite N_2 the number of dispersive bistable loops was unlimited). In the present case the dispersive bistability loops are formed first in the higher absorbance branch of the purely absorptive bistability, in contrast to the former case. Appearance of bistability loops in the lower absorbance branch requires a larger detuning. Related results were obtained by Gronchi and Lugiato,[13] Bonifacio and Lugiato,[14] and Ikeda.[15]

For a CFPR, we can use the average form of the intensity to solve the eikonal equation. However, in the interesting case of large detuning ($\Delta \gg 0$), we can solve the eikonal equation in closed form without any averaging.

Note that the treatment of the CFPR containing a Kerr medium by Marburger and Felber[16] is the only case in which a fully analytical solution was obtained for a standing-wave configuration. Using the eikonal method, we have obtained the first-order approximation to this analytical solution.[8]

The eikonal equation for propagation in a FPR containing a medium consisting of two-level systems in the large detuning limit, is thus written as

$$d\phi_D(x)=(2\pi/\lambda_0)\{1+\Delta\alpha_0/[1+I(x)/I_s]\}^{1/2}dx\ . \qquad (62)$$

Substituting $I(x)$ from Eq. (9), taking $\phi_A,\phi_A(x)=0$, we obtain

$$d\phi_D(x)=(2\pi/\lambda_0)\{1+\Delta\alpha_0/[1+I_{\text{in}}(1-R)(1+2r_2\cos\{2[\phi_D-\phi_D(x)]\}+r_2^2)/n_0G_1]\}^{-1/2}dx$$

or, by rearrangement,

$$d\Phi\{[a+b\cos(\Phi)]/[c+d\cos(\Phi)]\}^{1/2}=-(4\pi/\lambda_0)dx\ , \qquad (63)$$

where

$$a=1+(1-R)I_{\text{in}}(1+r_2^2)/n_0G_1\ ,$$
$$b,d=(1-R)I_{\text{in}}2r_2/n_0G_1\ ,$$
$$c=a+\Delta\alpha_0\ ,$$
$$\phi_A=0\ ,$$
$$\Phi=2[\phi_D-\phi_D(x)]\ .$$

Equation (63) can be integrated to yield a transcendental equation for ϕ_D, in terms of first- and third-order elliptic functions

$$(1/\sqrt{BD})\{[(B-A)/\alpha(\beta^2-1)]\times[-\beta^2\Pi(\nu;(1-\beta^2);q)+F(\nu;q)]+(B/\alpha)F(\nu;q)\}\ , \qquad (64)$$

where

$$A=1+I_{\text{in}}(1-R)(1+r_2^2+2r_2)/n_0G_1I_s\ ,$$
$$B=1+I_{\text{in}}(1-R)(1+r_2^2-2r_2)/n_0G_1I_s\ ,$$

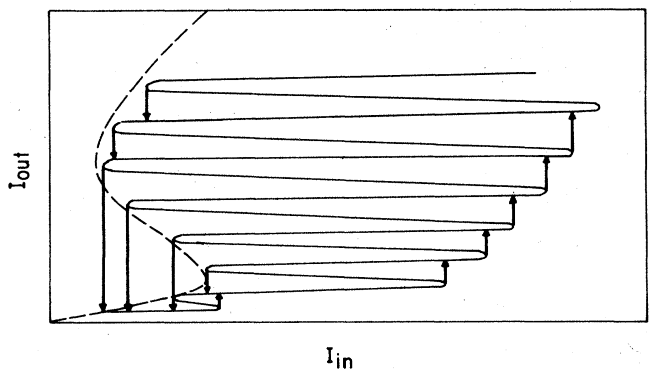

FIG. 21. Schematic drawing of the switching pattern in the coupled configuration of a saturable absorber and a Kerr-like medium in a ring resonator. The anomalous down switching, i.e., direct switching from certain high-intensity bistable loop to the lower output intensity, is the consequence of the following of the down-switching threshold intensities after the output intensity curve of the pure absorber, in the region of negative slope.

$$C=A+\Delta\alpha_0\ ,$$
$$D=B+\Delta\alpha_0\ ,$$
$$\alpha^2=A/B\ ,$$
$$\beta^2=C/D\ ,$$
$$\nu=\arctan[\tan(\phi_D)/\beta]\ ,$$
$$F(\nu;q)=\int_0^\nu dx/[1-q^2\sin^2(x)]^{1/2}\ ,$$
$$\Pi(\nu;\eta;q)=\int_0^\nu dx/[1+\eta\sin^2(x)][1-q^2\sin^2(x)]^{1/2}\ .$$

Equation (64) can be solved graphically for ϕ_D and the output intensity is thus obtained by

$$I_{\text{out}}=I_{\text{in}}(1-R)^2/G_1(\phi_D;\phi_A=0)\ ,$$

exhibiting multistability.

VI. CONCLUSIONS

The nonlinear eikonal approximation has now been established as a powerful tool in determining useful combinations of resonators and optical nonlinear media exhibiting optical bistability. We have demonstrated that nonlinear absorption can be incorporated in a variety of resonators in order to achieve bistable behavior. Our main observations are as follows.

(i) For media characterized by an intensity-dependent absorption exhibiting a maximum, optical bistability can be obtained, using an appropriate resonator, for both coherent and incoherent illumination.

(ii) For media characterized by intensity-dependent decreasing absorption, bistable behavior may be observed only at high intensities where higher-order terms in the absorption coefficient become important. For saturable absorbers incorporated in a CRR and in a CFPR optical bistability can be obtained. For incoherent illumination optical bistability is achieved only for high-order intensity dependence of the saturation.

(iii) Coupling between a Kerr-type dispersive nonlinearity and a nonlinear saturable absorber in the same resonator enables control of switching patterns of the optical bistability depending on the interrelations of the particular media parameters.

(iv) Of particular interest are situations in which optical bistability is observed in free propagation. For a locally multivalued absorption coefficient optical bistability in free propagation may be obtained. However, we show that it is not achieved in any of the nonlinear media discussed in the present paper. This is a general result which is proved here for any absorption coefficient of the type

$$\alpha=\alpha_0 f(I)\ , \tag{65}$$

where $f(I)$ is a well-behaved single-valued function. Since in free propagation $I(x)=I_{\text{in}}\exp[-\phi_A(x)]$,

$$\alpha=\alpha_0 f(I_{\text{in}}\exp[-\phi_A(x)])=g(\phi_A(x))\ . \tag{66}$$

The eikonal equation thus becomes

$$d\phi_A(x)/g(\phi_A(x))=\alpha_0 dx\ . \tag{67}$$

By integration we obtain

$$\phi_A=\alpha_0 L+\int_0^{\phi_A} d\xi[g(\xi)-1]/g(\xi)=M(\phi_A)\ . \tag{68}$$

The necessary condition for obtaining optical bistability is

$$\partial M(\phi_A)/\partial\phi_A=[g(\phi_A)-1]/g(\phi_A)\geq 1 \tag{69}$$

which of course cannot be fulfilled. Thus, we do not expect to obtain bistability in free propagation for any pure absorber. The cases in which optical bistability can be achieved in free propagation require a particular form of the absorption coefficient,[12] as is the case in dynamically increasing absorption.[7]

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