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Noise-induced switching of photonic logic elements

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We study the stability of optical bistable elements against holding-field noise by considering the specific case of a purely dispersive Kerr medium. For small devices, i.e., those for which the round-trip time is small compared to the material-response time, we show that operation of an optical switch with a holding-field intensity a few percent away from the switching value will not be subject to problems due to the noise in the holding laser. This is due to the extremely rapid increase of the average time between noise-induced switching events as the distance, in intensity, from the switching point increases. White noise or Ornstein-Uhlenbeck noise models are not sufficiently smooth for a correct description of this problem, and so we use a more elaborate colored-noise model to evaluate the diffusion constant in the Fokker-Planck equation for the nonlinear phase shift. The time between switching events is then obtained by solving a first-passage-time problem.

I. INTRODUCTION

The proposed use of optical bistable elements (OBE's) in photonic logic applications[1] raises the question of stability against noise.[2] We consider specifically an OBE driven by a cw laser beam with an intensity slightly below the critical value required for up-switching. The practical utility of such a device clearly requires that the mean time between noise-induced switching events should be large compared to the time between switching instructions. Thus we are interested in investigating the stability of deterministic states of the device against small fluctuations in frequency and amplitude of the holding beam. For this purpose we first derive a Langevin equation for the response of the OBE, in the small-cavity limit, to a noisy laser field. In the vicinity of a given deterministic solution, the equation is expanded up to second order in the small fluctuations and the resulting approximate model is treated by Fokker-Planck theory to estimate the average time for noise-induced switching.

This paper is organized as follows. In Sec. II we develop a phenomenological model of dispersive bistability in a ring resonator, including frequency and amplitude fluctuations of the holding beam. We show that even for a deterministic susceptibility χ_{nl}, frequency fluctuations lead to stochastic dynamics for the nonlinear phase shift experienced by the field.[3] Section III specializes these results to the small-cavity limit and derives a Langevin-type equation for the nonlinear phase shift. In Sec. IV we expand this equation up to second order in the small-noise limit, and construct the corresponding Fokker-Planck equation in Sec. V. The explicit noise model is defined in Sec. VI, which shows that simple Weiner-Levy and Ornstein-Uhlenbeck processes are insufficient to properly analyze this problem. Section VII evaluates the time between noise-induced switching events by relating the problem at hand to the first-passage time for a particle escaping from a potential well. Finally, Sec. VIII is a summary and conclusion.

II. FORMALISM

We consider a ring resonator whose upper arm is filled by a purely dispersive Kerr-type medium, see Fig. 1. The total length of the resonator is $2(L+l)$, and the material nonlinear (nl) response is given in terms of a susceptibility χ_{nl} assumed to obey the Debye relaxation equation

$$\frac{\partial \chi_{nl}}{\partial t}+\Gamma\chi_{nl}=\beta\,|\,E\,|^2\,, \qquad (2.1)$$

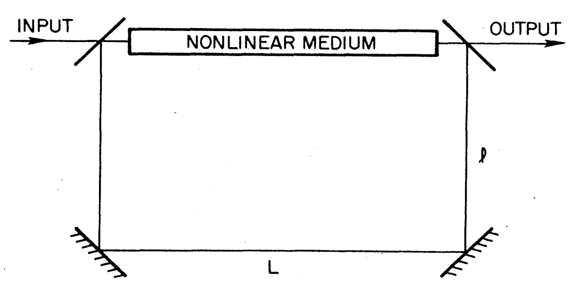

FIG. I. Ring cavity of length $2(L+l)$, with the upper arm of length L filled by a Kerr nonlinear medium of nonlinear susceptibility χ_{nl}. The input and output mirror have intensity reflectivity (transmission) coefficient $R(T)$, with $R+T=1$. The other two mirrors have unit reflection.

where E is the slowly varying envelope of the real intracavity field $\mathbf{E}(z,t)$:

$$\mathbf{E}(z,t)=\tfrac{1}{2}E(z,t)e^{i\Phi(z,t)}+\text{c.c.} \tag{2.2}$$

The instantaneous frequency $\Omega(z,t)$ of the intracavity field is

$$\Omega(z,t)=\frac{\partial\Phi(z,t)}{\partial t} . \tag{2.3}$$

For convenience, we choose the phase $\Phi(0,t)$ at the input port of the resonator ($z=0$) to be equal to the phase of the incident field $\mathbf{E}_i(t)$:

$$\Phi(0,t)=\Phi_i(t) , \tag{2.4}$$

where

$$\mathbf{E}_i(0,t)=\tfrac{1}{2}E_i(t)e^{\Phi_i(t)}+\text{c.c.} \tag{2.5}$$

and the amplitude $E_i(t)$ is taken without loss of generality to be real. As usual, we decompose the polarization accordingly as

$$\mathbf{P}(z,t)=\tfrac{1}{2}\mathscr{P}(z,t)e^{i\Phi(z,t)}+\text{c.c.} \tag{2.6}$$

with

$$\mathscr{P}(z,t)=(\chi_0+\chi_{nl})E(z,t) . \tag{2.7}$$

Here χ_0 is the linear part of the susceptibility. The wave equation for $\mathbf{E}(z,t)$ is

$$\left[\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathbf{E}=\frac{4\pi}{c^2}\frac{\partial^2\mathbf{P}}{\partial t^2} . \tag{2.8}$$

which yields, with

$$n_0^2=1+4\pi\chi_0 , \tag{2.9}$$

$$\left[\nabla^2-\frac{n_0^2}{c^2}\frac{\partial^2}{\partial t^2}\right]Ee^{i\Phi}=\frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}\chi_{nl}Ee^{i\Phi} . \tag{2.10}$$

As is often the case in propagation problems, it is convenient to introduce the retarded time

$$\mu=t-n_0z/c . \tag{2.11}$$

In the variables (μ,z), the dynamics of the field is performed in a frame following it, and the instantaneous frequency $\Omega(z,t)$ is a function of μ only:

$$\Omega(z,t)=\Omega(\mu) . \tag{2.12}$$

With the slowly varying amplitude and phase approximation

$$\left|\frac{1}{E}\frac{\partial E}{\partial z}\right| \ll \frac{\Omega}{c} , \tag{2.13a}$$

$$\left|\frac{1}{E}\frac{\partial E}{\partial t}\right| \ll \Omega , \tag{2.13b}$$

$$\left|\frac{1}{\Omega}\frac{\partial \Omega}{\partial t}\right| \ll \Omega , \tag{21.3c}$$

$$\left|\frac{1}{\chi_{nl}}\frac{\partial \chi_{nl}}{\partial t}\right| \ll \Omega , \tag{2.13d}$$

$$\chi_{nl} \ll n_0 , \tag{2.13e}$$

we obtain readily the wave equation for the slowly varying amplitude $E(z,\mu)$:

$$\frac{\partial E(z,\mu)}{\partial z}=\left[\frac{2\pi i\Omega(\mu)}{c}\right]\chi_{nl}E(z,\mu) . \tag{2.14}$$

Equations (2.1) and (2.14) compose the coupled Maxwell-Debye equations. Since we neglect absorption in our model of a bistable device, the amplitude of the fields is conserved while propagating through the resonator. Thus introducing

$$E(z,\mu)=|E(0,\mu)|e^{i\psi(z,\mu)} \tag{2.15}$$

yields readily

$$\frac{\partial\psi}{\partial z}=\frac{2\pi n_0\Omega(\mu)}{c}\chi_{nl} , \tag{2.16}$$

which after integration gives

$$\psi(z,\mu)=\psi(0,\mu)+\frac{2\pi n_0\Omega(\mu)}{c}\int_0^z dz'\chi_{nl}(z',\mu) . \tag{2.17}$$

The electric field (2.15) becomes

$$E(z,\mu)=E(0,\mu)\exp[i\phi_{nl}(z,\mu)] , \tag{2.18}$$

where

$$\phi_{nl}=\frac{2\pi n_0\Omega(\mu)}{c}\int_0^z dz'\chi_{nl}(z',\mu) \tag{2.19}$$

is the nonlinear phase shift experienced by the field inside the resonator. The Debye relaxation equation (2.1) for the nonlinear susceptibility χ_{nl} can be used to obtain an equation of motion for ϕ_{nl}. Differentiating (2.19) with respect to μ yields

$$\begin{aligned}\frac{\partial\phi_{nl}}{\partial\mu}&=\frac{2\pi n_0}{c}\frac{\partial\Omega}{\partial\mu}\int_0^z dz'\chi_{nl}(z',\mu)\\&\quad+\frac{2\pi n_0\Omega}{c}\int_0^z dz'\frac{\partial\chi_{nl}(z',\mu)}{\partial\mu}\\&=\frac{1}{\Omega}\frac{\partial\Omega}{\partial\mu}\phi_{nl}\\&\quad+\frac{2\pi n_0\Omega}{c}\int_0^z dz'[\beta|E|^2(\mu)-\Gamma\chi_{nl}(z',\mu)] ,\end{aligned}$$

or finally

$$\frac{\partial\phi_{nl}(z,\mu)}{\partial\mu}+\left[\Gamma-\frac{1}{\Omega}\frac{\partial\Omega(\mu)}{\partial\mu}\right]\phi_{nl}(z,\mu)=\frac{2\pi n_0\Omega}{c}\beta z|E|^2(\mu) . \tag{2.20}$$

This equation has the same structure as the Debye equation (2.1), except for the appearance of a supplementary stochastic relaxation term finding its origin in the field frequency fluctuations. This shows that in general, the dynamics of the nonlinear phase shift is governed by a stochastic equation, even for a deterministic χ_{nl}. In the original (z,t) variables. Equation (2.20) becomes simply

$$\frac{\partial\phi_{nl}(z,t)}{\partial t}+\left[\Gamma-\frac{1}{\Omega}\frac{\partial\Omega}{\partial t}\right]\phi_{nl}(z,t)=\frac{2\pi n_0\Omega}{c}\beta z\,|E\,|^2(z,t)\;. \tag{2.21}$$

A complete description of the system still requires the use of the standard boundary condition

$$\mathbf{E}(0,t)=\sqrt{T}\,\mathbf{E}_i(0,t)+R\mathbf{E}(L,t-\Delta t)\;, \tag{2.22}$$

where R is the reflection coefficient of the mirrors at the input and output ports (the other two mirrors are taken to have unity reflection coefficients), $T=1-R$, and $\Delta t=(2l+L)/c$. With Eqs. (2.2), (2.4), and (2.5), this yields

$$E(0,t)=\sqrt{T}E_i(t)+Re^{i[\Phi(t-\Delta t-n_0L/c)-\Phi(t)]}E(L,t-\Delta t)\;. \tag{2.23}$$

Introducing the cavity round-trip time

$$t_R=\Delta t+n_0L/c \tag{2.24}$$

and (2.18) finally gives

$$E(0,t+t_R)=\sqrt{T}E_i(t+t_R)+Re^{i[\Phi_i(t)-\Phi_i(t+t_R)+\phi_{nl}(t)]}E(0,t)\;, \tag{2.25}$$

where we have also performed the transformation $t\rightarrow t+t_R$, and defined $\phi_{nl}(t)=\phi_{nl}(L,t)$.

III. SMALL-CAVITY LIMIT

Most potential applications of optical bistable elements use micrometer-size semiconductor devices,[4] for which the round-trip time t_R is on the order of picoseconds, a short time compared to typical medium relaxation times Γ^{-1}. If, furthermore, the typical noise characteristic time t_N (correlation time) is long compared to t_R, then it is safe to perform an adiabatic elimination of the field. This is, for instance, the case for input lasers of bandwidths as large as tens of gigahertz.

The linear part of the phase in the recurrence relation (2.25) has the form

$$\Delta\Phi_i=\Phi_l(t)-\Phi_i(t+t_R)\equiv\omega t_R+\omega[\tau(t+t_R)-\tau(t)]\;. \tag{3.1}$$

Here ω is the nominal pump laser frequency in the absence of noise, and $\tau(t)$ a stochastic function of zero average which is assumed, in accordance with our earlier remarks, to varying little in the time t_R. Hence, we can expand the square bracket in (3.1) as

$$\tau(t+t_R)-\tau(t)\cong t_R\frac{d\tau}{dt}\equiv t_R(\delta\omega/\omega)\;, \tag{3.2}$$

so that

$$\Delta\Phi_i\cong\omega t_R(1+\delta\omega/\omega)\;. \tag{3.3}$$

Similarly, we expand the incident field as

$$E_i(t+t_R)\cong E_i(t)+t_R\frac{dE_i}{dt}+\cdots\cong E_i(t)\;. \tag{3.4}$$

The leading term of Eq. (2.25) becomes then

$$E(t+t_R)=\sqrt{T}E_i(t)+Re^{i[(\omega+\delta\omega)t_R+\phi_{nl}(t)]}E(t)\;, \tag{3.5}$$

where we have dropped the unambiguous $z=0$ argument for clarity. The solution of Eq. (3.5) is

$$E(t+nt_R)=\Lambda^nE(t)+\frac{1-\Lambda^n}{1-\Lambda}\sqrt{T}E_i(t)\;, \tag{3.6}$$

where

$$\Lambda=Re^{i[(\omega+\delta\omega)t_R+\phi_{nl}(t)]}\;. \tag{3.7}$$

Since $|\Lambda|=R=1-T<1$, the adiabatic solution converges to its asymptotic value

$$E=\frac{\sqrt{T}E_i(t)}{1-R\exp\{i[(\omega+\delta\omega)t_R+\phi_{nl}(t)]\}}\;, \tag{3.8}$$

after N round trips, where $N\cong 30$ for $R=0.9$. By our assumptions the total convergence time is small compared to the material response time so that the asymptotic value can be used in the equation of motion (2.21) for ϕ_{nl}, giving the approximate model

$$\frac{d\phi_{nl}}{dt}+(\Gamma+\delta\Gamma)\phi_{nl}=2\pi n_0\beta L\frac{\omega(l+\delta\omega/\omega)}{c}\times\left|\frac{\sqrt{T}E_i(t)}{1-R\exp\{i[(\omega+\delta\omega)t_R+\phi_{nl}(t)]\}}\right|^2\;. \tag{3.9}$$

where

$$\delta\Gamma=-\left[\frac{1}{\Omega}\right]\left[\frac{\partial\Omega(\mu)}{\partial\mu}\right]\;. \tag{3.10}$$

Until now, we have considered pump-field frequency fluctuations only, but intensity fluctuations are readily included by the substitution

$$|E_i|^2\rightarrow\langle I_i\rangle(1+a)\;, \tag{3.11}$$

where $\langle I_i\rangle$ is the mean incident intensity and

$$a(t)=\frac{\delta I_i(t)}{\langle I_i\rangle} \tag{3.12}$$

is a stochastic variable of zero mean describing the intensity fluctuations. It is also convenient at this point to introduce explicitly the deterministic part Δ of the linear cavity phase shift

$$\Delta=(\omega_c-\omega)t_R\;, \tag{3.13}$$

where ω_c is the frequency of the cavity mode closest to ω. Note that the phase $(\omega+\delta\omega)t_R$ may be rewritten as

$$(\omega+\delta\omega)t_R=-\Delta+\omega_ct_R+\delta\omega t_R \tag{3.14}$$

and that the second term on the right-hand side of this equality is equal to $2N\pi$ (N integer) by definition of the cavity modes. In order to cast Eq. (3.9), into its final, dimensionless form, we now introduce a dimensionless time variable by $t\rightarrow\Gamma^{-1}t$, i.e., from now in t is understood to be measured in units of Γ^{-1}. Equation (3.9) becomes

$$\frac{d\phi_{nl}}{dt}+(1+\delta\gamma)\phi_{nl} = \frac{G(1+\delta\omega/\omega)(1+a)}{(1-R)^2+4R\sin^2\left[\frac{\phi_{nl}(t)+\delta\omega t_R-\Delta}{2}\right]} . \quad (3.15)$$

Here,

$$G=\frac{2\pi n_0 L\omega}{c}\frac{\beta T\langle I_i\rangle}{\Gamma}=\frac{\langle I_i\rangle}{I_0} , \quad (3.16a)$$

$$I_0^{-1}=\left[\frac{n_0 L\omega}{c}\right]n_0 n_2 T , \quad (3.16b)$$

$$\delta\gamma=\delta\Gamma/\Gamma . \quad (3.17)$$

In defining the scale intensity I_0 in (3.16b), we have used

$$n^2=1+4\pi\chi=n_0^2(1+4\pi\chi_{nl}/n_0^2)=n_0+n_2 I$$

and the steady-state from of Eq. (2.1). Thus the parameter G is the laser intensity measured in units of the characteristic intensity I_0.

IV. SMALL FLUCTUATIONS

To study the influence of noise on the bistable device, we expand the equation of motion (3.15) for the nonlinear phase shift ϕ_{nl} about the noiseless stationary solution ϕ_0 corresponding to the constant input intensity $\langle I_i\rangle$ at the nominal frequency ω. We proceed by introducing the new variable

$$u=\phi_{nl}-\phi_0+\delta\omega t_R . \quad (4.1)$$

In the absence of noise, Eq. (3.15) has a stationary-state solution defined by

$$\phi_0=A(G,\phi_0) , \quad (4.2)$$

where the function

$$A(G,\phi_{nl})=\frac{G}{(1-R)^2+4R\sin^2\left[\frac{\phi_{nl}-\Delta}{2}\right]} \quad (4.3)$$

is the rhs of Eq. (3.15) in the absence of amplitude and frequency fluctuations. To second order in u,

$$A(u)=\phi_o+A_1 u+\tfrac{1}{2}A_2 u^2 , \quad (4.4)$$

where

$$A_1=\left.\frac{\partial A}{\partial\phi_{nl}}\right|_{\phi_{nl}=\phi_0} \quad (4.5)$$

and

$$A_2=\left.\frac{\partial^2 A}{\partial\phi_{nl}^2}\right|_{\phi_{nl}=\phi_0} . \quad (4.6)$$

In order to proceed, we now make the following additional assumptions.

(i) The relative frequency fluctuations

$$f=\frac{\delta\omega}{\omega} \quad (4.7)$$

and amplitude fluctuations

$$a=\frac{\delta I_i}{\langle I_i\rangle} \quad (4.8)$$

can be neglected compared to unity.

(ii) Products of u and f and u and a can be neglected.

(iii) The frequency and amplitude fluctuations $f(t)$ and $a(t)$ are stationary random processes of zero mean, with correlation functions that are smooth enough to make all following manipulations legal.

(iv) The stochastic processes $f(t)$ and $a(t)$ are uncorrelated, and the frequency correlation time τ_f and amplitude correlation time τ_a are both short compared to the deterministic relaxation time of the nonlinear phase shift ϕ_{nl}.

We substitute Eq. (4.1) into (3.15) and expand the right-hand side to second order in u and to first order in the fluctuations; the result is

$$\frac{\partial u}{\partial t}=F_0(u)+F_1(u) , \quad (4.9)$$

where

$$F_0=-\frac{u}{T_0}+\tfrac{1}{2}A_2 u^2 , \quad (4.10)$$

$$F_1=(\phi_0+\omega t_R)(f+\dot f)+\phi_0 a , \quad (4.11)$$

where the overdot means derivative with respect to the dimensionless time and

$$T_0=\frac{1}{1-A_1} . \quad (4.12)$$

Note that since $A_1\to 1$ as the system approaches the turning points, its effective deterministic response time T_0 becomes very large in these regions. This is a signature of critical slowing down. Equation (4.9) allows the study of the interplay between the rapid noise fluctuations and this sluggish deterministic response of the nonlinear phase ϕ_{nl}.

V. FOKKER-PLANCK EQUATION

Following Van Kampen,[5] the Fokker-Planck equation associated with the Langevin-type equation (4.9) is

$$\frac{\partial}{\partial t}p(u,t)=\frac{\partial}{\partial u}\left[\left[-F_0(u)-\int_0^\infty d\tau\left\langle\frac{\partial F_1(u,t)}{\partial u}F_1(u_{-\tau},t-\tau)\right\rangle\frac{du}{du_{-\tau}} + \frac{\partial}{\partial u}\int_0^\infty d\tau\langle F_1(u,t)F_1(u_{-\tau},t-\tau)\rangle\frac{du}{du_{-\tau}}\right]p(u,t)\right] , \quad (5.1)$$

where

$$u_{-\tau}=u(t-\tau)\ , \tag{5.2}$$

and $u(t)$ is defined as the solution of the deterministic equation

$$\frac{du}{dt}=F_0(u)\ . \tag{5.3}$$

This equation defines a mapping $u(0)\to u(t)$, or more generally, $u(t)\to u(t+\tau)$, and the expression $du/du_{-\tau}$ is the Jacobian of the inverse of this mapping.

Under the assumption (iii) that products of u and f, a and f, respectively, can be neglected, $\partial F_1/\partial u=0$. In other words, we assume that noise corrections to the drift term of Eq. (5.1) are negligible, i.e., that the noise is additive. The diffusion term $B(u)$ is then

$$\begin{aligned}B(u)&=\int_0^\infty d\tau\langle F_1(u,t)F_1(u_{-\tau},t-\tau)\rangle\frac{du}{du_{-\tau}}\\&=\int_0^\infty d\tau(\phi_0+\omega t_R)^2\langle f(t)f(t-\tau)+\dot f(t)\dot f(t-\tau)\rangle\frac{du}{du_{-\tau}}+\int_0^\infty d\tau\,\phi_0^2\langle a(t)a(t-\tau)\rangle\ ,\end{aligned} \tag{5.4}$$

where we have used the explicit form (4.11) of F_1 and assumption (iv), as well as the relation

$$\langle f(t_1)\dot f(t_2)\rangle=-\langle f(t_2)\dot f(t_1)\rangle\ , \tag{5.5}$$

which is valid for stationary random processes, to obtain the second equality.

In the expression (5.4) for B, the term depending on frequency fluctuations is multiplied by the coefficient $(\phi_0+\omega t_R)^2$, while the amplitude fluctuation term is multiplied only by ϕ_0^2. At optical frequencies, and for devices of several tens of micrometers in size,

$$\omega t_R(\cong 10-1000)\gg\phi_0[\cong O(1)]\ ;$$

therefore, frequency fluctuations will dominate over amplitude fluctuations unless the normalized amplitude fluctuations are very much larger than the normalized frequency fluctuations. We assume that this is not the case so that it is permissible to neglect the contribution from amplitude fluctuations in (5.4).

To evaluate the contribution from frequency fluctuations to the diffusion coefficient (5.4), we decompose it as

$$B=B_0+B_1\ , \tag{5.6}$$

where

$$B_0=(\phi_0+\omega t_R)^2\int_0^\infty d\tau\langle f(t)f(t-\tau)\rangle\frac{du}{du_{-\tau}}\ , \tag{5.7}$$

and

$$B_1=(\phi_0+\omega t_R)^2\int_0^\infty d\tau\langle \dot f(t)\dot f(t-\tau)\rangle\frac{du}{du_{-\tau}}\ . \tag{5.8}$$

We will see later that B_1, which is given in terms of the correlation function for the frequency derivative, $(\phi_0+\omega t_R)\dot f$, is normally much larger than B_0. The ϕ_0 part of B_1 can be traced to the $\delta\gamma$ term in (3.15), i.e., to the stochastic correction to the material-response time. The ωt_R part comes from the frequency fluctuation term $t_R\delta\omega$ in the denominator on the right-hand side of (3.15). Since $\omega t_R\gg\phi_0$, it is clear that the main physical effect of the noise enters through the fluctuations of the frequency in the Airy denominator. The correlation function in B_1 can be evaluated by using the identity

$$\langle\dot f(t)\dot f(t-\tau)\rangle=-\frac{\partial^2}{\partial\tau^2}\langle f(t)f(t-\tau)\rangle\ , \tag{5.9}$$

which is valid for stationary processes.

For any function $W(\tau)$, one has then

$$\begin{aligned}\int_0^\infty d\tau\,W(\tau)\langle\dot f(t)\dot f(t-\tau)\rangle&=-\int_0^\infty d\tau\,W(\tau)\frac{\partial^2}{\partial\tau^2}\langle f(0)f(-\tau)\rangle\\&=-W(\tau)\frac{\partial}{\partial\tau}\langle f(0)f(-\tau)\rangle\Big|_0^\infty+\int_0^\infty d\tau\frac{\partial W(\tau)}{\partial\tau}\frac{\partial}{\partial\tau}\langle f(0)f(-\tau)\rangle\\&=-W(0)\langle f(0)\dot f(0)\rangle+\frac{\partial W(\tau)}{\partial\tau}\langle f(0)f(-\tau)\rangle\Big|_0^\infty-\int_0^\infty d\tau\frac{\partial^2W(\tau)}{\partial\tau^2}\langle f(0)f(-\tau)\rangle\\&=-W(0)\langle f(0)\dot f(0)\rangle-\frac{\partial W(0)}{\partial\tau}\langle f(0)^2\rangle-\int_0^\infty d\tau\frac{\partial^2W(\tau)}{\partial\tau^2}\langle f(0)f(-\tau)\rangle\ .\end{aligned} \tag{5.10}$$

Here we have used the fact that $\langle f(0)f(-\tau)\rangle\to 0$ for $\tau\to\infty$. The first term in the last equality is equal to $W(0)(d/d\tau)\langle f(0)f(-\tau)\rangle\,|_{\tau=0}$, and is identically zero for a stationary process. Furthermore, the integral can be calculated by evaluating $\partial^2W(\tau)\partial\tau^2$ at $\tau=0$, a procedure valid provided that $W(\tau)$ is slowly varying over the correlation time of the noise. This yields, identifying $W(\tau)$ with $du/du_{-\tau}$,

$$B_1 = -\frac{\partial}{\partial \tau}\left[\frac{du}{du_{-\tau}}\right]\Bigg|_{\tau=0}\langle f(0)^2\rangle - \frac{\partial^2}{\partial \tau^2}\left[\frac{du}{du_{-\tau}}\right]\Bigg|_{\tau=0}\int_0^\infty d\tau\langle f(0)f(-\tau)\rangle \ . \tag{5.11}$$

All that remains to do in evaluating the diffusion coefficient B is to determine the Jacobian $du/du_{-\tau}$. This is readily done by expanding $u(t-\tau)$ as a Taylor series about $u(t)$ and making use of Eq. (5.3). After dropping terms of third order in τ, this gives

$$\frac{du}{du_{-\tau}} = 1 - \frac{\tau}{T_0} + \frac{\tau^2}{2T_0^2} + O(\tau^3) + O(u) \ , \tag{5.12}$$

where the small size of u has also been used. Keeping only the lowest-order terms in Eqs. (5.7) and (5.12) yields finally

$$B_0 = (\phi_0 + \omega t_R)^2 D_{ff} \ , \tag{5.13}$$

and

$$B_1 = (\phi_0 + \omega t_R)^2\left[\frac{\langle f(0)^2\rangle}{T_0} - \frac{D_{ff}}{T_0^2}\right] , \tag{5.14}$$

where

$$D_{ff} = \int_0^\infty d\tau\langle f(t)f(t-\tau)\rangle \ . \tag{5.15}$$

VI. NOISE MODEL

So far the noise model has been limited only by the reasonable assumptions (i)—(iv), but now we must be more specific. The familiar white-noise model for frequency fluctuations does not satisfy the smoothness part of assumption (iii). This follows from the fact that B_1 depends on derivatives of the frequency correlation function which do not exist for white noise. The same objection applies to an Orenstein-Uhlenbeck process, therefore in order to get a finite diffusion constant it is necessary to use a more elaborate colored-noise model. (For the use of colored-noise models in optical bistability, see, e.g., Ref. 6.) We have chosen to represent the frequency fluctuations $f(t)$ by the following process:

$$\dot{f}(t) = -f(t)/\tau_f + \eta/\tau_f \ , \tag{6.1}$$

$$\dot{\eta}(t) = -\eta(t)/\tau_f + \frac{2}{\omega\tau_f}\sqrt{\Delta_l}\,\xi(t) \ , \tag{6.2}$$

where $\xi(t)$ is a Gaussian stochastic process

$$\langle \xi(t)\xi(t+\tau)\rangle = \delta(\tau) \ . \tag{6.3}$$

The field correlation function for the laser can be evaluated by standard techniques[7] which lead to the conclusion that Δ_l can be thought of as the laser linewidth.

From Eqs (6.2) and (6.3), we get readily

$$\eta(t) = \int_{t_0}^t dt'\xi(t')e^{-(t-t')/\tau_f} + \eta(t_0)e^{-(t-t_0)/\tau_f} \ , \tag{6.4}$$

and

$$\langle \eta(t)\eta(t+\tau)\rangle = \frac{\Delta_l}{\omega}\frac{1}{\omega\tau_f}\exp(-|\tau|/\tau_f) \ , \tag{6.5}$$

this last relation being valid in the limit $t_0 \to -\infty$. Similarly,

$$\langle f(t)f(t+\tau)\rangle = \frac{\Delta_l}{\omega}\frac{1}{2\omega\tau_f}(1+|\tau|/\tau_f) \times \exp(-|\tau|/\tau_f) \ . \tag{6.6}$$

With the definition (5.13) of D_{ff}, (6.6) gives

$$D_{ff} = \frac{1}{\omega}\frac{\Delta_l}{\omega} = 2\tau_f\langle f(0)^2\rangle \ . \tag{6.7}$$

The two contributions B_0 and B_1 to the diffusion coefficient become then

$$B_0 = \frac{1}{\omega}\frac{\Delta_l}{\omega}(\phi_0 + \omega t_R)^2 \tag{6.8}$$

and

$$B_1 = B_0\left[\frac{1}{2\tau_f T_0} - \frac{1}{T_0^2}\right] . \tag{6.9}$$

In the next section we use these results to estimate the average time between noise-induced switching events.

VII. AVERAGE TIME FOR NOISE-INDUCED SWITCHING

By combining Eqs. (4.10), (5.13), and (5.14) the Fokker-Planck equation (5.1) can be written in the form

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial u}\left[\frac{\partial V}{\partial u}p\right] + B\frac{\partial^2}{\partial u^2}p \ , \tag{7.1a}$$

where

$$V = \frac{u^2}{2T_0} - \tfrac{1}{6}A_2 u^3 \ . \tag{7.1b}$$

Thus the phase diffusion problem is equivalent to the diffusion of a particle trapped in the potential well $V(u)$ and subjected to stochastic forces characterized by the dif-

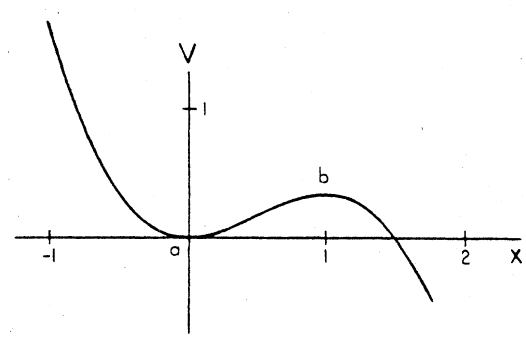

FIG. 2. Normalized potential $V(x) = V(u)(2/A_2^3T_0^3)$, where $x = A_2T_0u/2$. The minimum of $V(x)$ is at the holding point $a = u = 0$ and the maximum occurs at $u = b = 2/(A_2T_0)$.

fusion constant B. As shown in Fig. 2 the potential minimum lies at $u=0$ and the maximum occurs at $u=b=2/(A_2T_0)$. At some finite time a particle initially at $u=0$ will arrive at the potential maximum where it can either fall back or escape. The average time required for the particle to traverse the distance $\Delta u=2/(A_2T_0)$ between the maximum and minimum points of the potential is a measure of the time required for escape from the potential well. In order to relate this description to the original problem we consider the solution of the deterministic steady-state equation (4.2) in the vicinity of an up-switching point, similar considerations hold for down-switching. The conditions for up-switching are [see Eq. (4.2)]

$$\phi=A(G,\phi)\ , \tag{7.2a}$$

$$\frac{\partial A}{\partial\phi}=1\ , \tag{7.2b}$$

$$\frac{\partial^2 A}{\partial\phi^2}>0\ , \tag{7.2c}$$

and they define the switching values (G_s,ϕ_s), as shown in Fig. 3. The solutions of (4.2) in the vicinity of the switching point can be obtained by expanding $A(\phi,G)$ in a Taylor series through second order in $\phi-\phi_s$ and $G-G_s$. This yields the two roots

$$\phi_0=\phi_s-\left[\frac{2\phi_s\delta G}{A_2}\right]^{1/2}\ . \tag{7.3a}$$

$$\phi_1=\phi_s+\left[\frac{2\phi_s\delta G}{A_2}\right]^{1/2}\ , \tag{7.3b}$$

where $\delta G=|G-G_s|$ is the distance, in intensity, to the switching point in units of the scale intensity I_0. To the present order of accuracy, the constant A_2, defined by (4.6), can be evaluated at either of the points ϕ_0 or ϕ_s, but the calculation of A_1, defined by (4.5), involves first-order corrections so it must be evaluated at ϕ_0. This yields

$$A_1=1-\sqrt{2\phi_sA_2\delta G}\ , \tag{7.4}$$

and by (4.12), the deterministic response time T_0 is given by

$$T_0=\frac{1}{\sqrt{2\phi_sA_2\delta G}}\ . \tag{7.5}$$

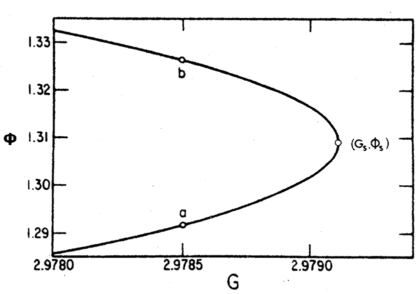

FIG. 3. Bistability curve in the vicinity of the up-switching point (G_s,ϕ_s): a is the operating point and b the corresponding point on the unstable branch. They correspond to the points a and b in Fig. 2.

Substitution of these results into the expression given above for Δu, the distance between the minimum and maximum points of the potential, shows that

$$\Delta u=2\left[\frac{2\phi_s\delta G}{A_2}\right]^{1/2}\ , \tag{7.6}$$

and comparison with (7.3) shows that this is exactly the distance between stable and unstable branches in Fig. 3.

The calculation of the average escape time from the potential well is an example of a "first-passage-time" calculation for which the general result is well known.[7] In the present case the average escape time T^*, for a particle initially at $u=0$, is given by

$$T^*=\frac{1}{B}\int_0^b du\,\exp\left[\frac{V(u)}{B}\right]\int_{-\infty}^{u} dv\,\exp\left[-\frac{V(v)}{B}\right]\ , \tag{7.7}$$

where the boundary conditions include an absorbing barrier at $u=b$, the location of the potential maximum, to represent the escape of the particle. In terms of the original problem, T^* is the time required for a bistable system originally operating at the stable point ϕ_0 to migrate to the unstable point ϕ_1. Since the system will then very rapidly either return to ϕ_0 or switch up to the next stable branch, T^* is a suitable measure of the time for noise-induced up-switching. Thus the particle escape time serves as an estimate of the average time between noise-induced switching events.

If the central maximum in the potential is large compared to the diffusion constant, i.e., $V(b)\gg B$, then the function $\exp[V(u)/B]$ is sharply peaked at $u=b$, and (7.7) can be evaluated by the method of steepest descents to yield[8] the well-known Arrhenius formula from chemical reaction theory:

$$T^*=\frac{2\pi\exp\left[\dfrac{V(b)-V(0)}{B}\right]}{\left[-\left[\dfrac{\partial^2V}{\partial u^2}\right]_b\left[\dfrac{\partial^2V}{\partial u^2}\right]_0\right]^{1/2}}\ , \tag{7.8}$$

which becomes, after using (7.1b) for $V(u)$,

$$T^*=2\pi T_0\exp\left[\frac{V(b)}{B}\right]\ . \tag{7.9}$$

In order to check the applicability of this formula, we take as a typical case a bistable device with dimensions $L\cong l\cong 0.01$ cm, and mirrors with $R\cong 0.9$. The nonlinear material is taken to be InSb with $n_0=4$, $n_2=3\times10^{-4}$ cm^2/W, and $\Gamma=10^9$ s^{-1}. This gives a round-trip time $t_R\cong 2$ ps. For the laser we take $\omega=10^{15}$ s^{-1}, noise correlation time $\tau_f=1$ ps, and linewidth $\Delta_l=1$ GHz.[9] The scale intensity $I_0=6.25$ W/cm^2. Assuming a linear cavity phase shift $\Delta=-\pi$, the first up-switching point occurs at $\phi_s=1.31$ and $G_s=2.98$ (switching intensity $I_s=18.6$ W/cm^2). The curvature coefficient is given by

$$T_0^{-1}=2.1\sqrt{\delta G}\ . \tag{7.10}$$

In all of these calculations it should be remembered that times are measured in units of Γ^{-1}. Evaluation of (6.8) and (6.9) gives $B_0=4\times10^{-6}$ and $B_1=4.2\times10^{-3}\sqrt{\delta G}$; therefore if $\delta G \gg 10^{-6}$, $B_1 \gg B_0$ as stated in Sec. V. The potential maximum is given by

$$V(b)=2.1(\delta G)^{3/2}\ , \tag{7.11}$$

so the condition $V(b)\gg B$ becomes $\delta G \gg 2\times10^{-3}$. Thus when the operating intensity is below the upswitching point by a percent or so, the use of (7.9) is justified. For the representative numbers used above (7.9) is

$$T^*=\frac{3}{\sqrt{\delta G}}\exp(500\delta G)\ (\text{ns})\ , \tag{7.12}$$

where we have restored conventional units. For permissible values of δG, this result shows that the average time for noise-induced switching increases very rapidly as the operating intensity I_i departs from the critical switching intensity I_s. This behavior is illustrated in Table I for the representative parameter values used in the numerical estimates.

TABLE I. Average time for noise-induced switching for representative values of $\delta G = |I_i - I_s|/I_0$.

δG	T^* (s)
0.01	0.000 004
0.02	0.000 467
0.03	0.0566
0.04	7.28
0.05	966.0

VIII. SUMMARY AND CONCLUSIONS

We have studied the stability of OBE's against holding field noise by considering the specific case of a purely dispersive Kerr medium. For small devices, i.e., those for which the round-trip time is small compared to the material-response time, the intracavity field can be adiabatically eliminated and this leads to a Langevin-type equation for the nonlinear phase shift. This equation involves the amplitude fluctuation δI, the frequency fluctuation $\delta\omega$, and the time derivative of the frequency fluctuation, $d(\delta\omega)/dt$. This last feature makes the theory quite sensitive to the details of the frequency-noise model. This sensitivity became apparent after the exact Langevin equation was expanded about a stable deterministic solution and the corresponding Fokker-Planck equation was derived. The diffusion constant B in the Fokker-Planck equation is dominated by the frequency-noise contribution and furthermore, B becomes infinite if the frequency fluctuations are described by either a white noise or an Ornstein, Uhlenbeck process. We were therefore compelled to use a suitable colored-noise model to describe the frequency fluctuations.

The approximate Fokker-Planck treatment is valid in the vicinity of any stable solution but the interesting case from the standpoint of applications is when the solution is near the switching point. In this region the phenomenon of critical slowing down is the dominant feature. The deterministic response time T_0 diverges like $1/\sqrt{\delta G}$ near the switching point, and an uncritical use of (7.12) would lead one to conclude that the noise-induced switching time T^* becomes infinite at the switching point ($\delta G=0$). This is not the case, since the validity of (7.12) was seen to impose a lower bound on δG. In fact, for solutions too close to the switching point, it would be necessary to include terms which were neglected in the derivation of the Fokker-Planck equation. This would lead to a qualitative change in the effective potential shown in Fig. 2, and the solution ϕ_0 would no longer be stable.[6,10] In other words, our treatment is not valid when the intensity is too close to the switching point. In practice this makes no difference since the laser intensity cannot be controlled with sufficient precision to violate the lower bound on δG. For permissable values of δG, Table I shows a spectacular increase in T^* as the operating point is moved away from the switching point. We therefore conclude that operation of an optical switch with a holding-field intensity a few percent away from the switching value will not be subject to problems due to the noise in the holding laser.

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