Subharmonic bifurcation and bistability of periodic solutions in a periodically modulated laser

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We consider the effect of periodically modulated cavity losses on the bifurcation diagram of the laser rate equations. The double limit of very slow atomic inversion relaxation and small modulation amplitude is investigated. Our control parameter λ is the ratio of the damped oscillation frequency of the rate equations (at zero modulation amplitude) to the forcing frequency. We concentrate on the case of pure resonance ($\lambda=1$) and the first subharmonic resonance ($\lambda=\frac{1}{2}$) because they are most representative of the effects of the periodic control. We first reformulate the laser problem as a weakly perturbed conservative system and construct harmonic and subharmonic large-amplitude periodic solutions by a regular perturbation analysis. The conditions for the existence of these solutions are analyzed and evaluated numerically. We show that harmonic and subharmonic solutions may coexist. We then determine a small-amplitude periodic solution oscillating at the forcing frequency. We show that perturbations of this basic state lead to slowly decaying quasiperiodic oscillations except in the vicinity of the critical points $\lambda=\frac{1}{2}$ and 1. In the first case, subharmonic bifurcation may occur and the period of the oscillations suddenly doubles. In the second case, bistability of periodic solutions is observed.

I. INTRODUCTION

Homogeneously broadened unidirectional tuned single-mode ring lasers are described by a rather simple set of three nonlinear ordinary differential equations. It was realized by Haken[1] that these equations are equivalent to the Lorenz equations[2] and that this system was therefore a good candidate to study deterministic chaos. The conflicting requirements imposed by the theory for the onset of chaos (bad cavity but nevertheless high gain), however, suggest that alternative possibilities be explored. The guideline for these alternative schemes has been that a necessary condition for the onset of chaos is the presence of three degrees of freedom. Hence, Yamada and Graham[3] suggested to perturb a good-cavity laser with an external detuned field whose amplitude is periodically modulated. Next, Scholz *et al.*[4] suggested to have a good-cavity laser perturbed by a cw external field and a time-periodic modulation of the population inversion. A more accessible setup was studied theoretically by Ivanov *et al.*[5] They proposed to periodically modulate the cavity losses of a Nd^{3+}: YAG laser (where YAG denotes yttrium aluminum garnet). In such a laser the atomic time scales are such that the atomic polarization can be adiabatically eliminated. Hence the laser is described by the usual rate equations for the field and the population inversion. The modulation of the cavity losses then provides the additional degree of freedom required to access the chaotic domain. Simultaneously and independently, Arecchi *et al.*[6] proposed the same scheme for a CO_2 laser and proved experimentally the soundness for these ideas. Alternatively, the same lasers may develop chaos if the cavity length (and therefore the cavity frequency) is periodically modulated as shown by Midavaine *et al.*[7] Finally, the conjecture of Ivanov *et al.*[5] was verified experimentally by Khandokhin and Khanin.[8]

The requirement of three degrees of freedom is not only a necessary condition for the occurrence of chaos. It is also a necessary condition for the existence of a rich variety of periodic states. Because the focus of the work reported in Refs. 3—8 was deterministic chaos, rather little attention was paid to the domain of stable periodic solutions which pave the way to chaos, save for the work of Matorin *et al.*[9] which deals with numerical analyses. This domain is of great interest in its own right. Indeed, as soon as a loss modulation is introduced in the rate equations, a complicated set of periodic solutions is expected to appear even though it need not necessarily lead to a chaotic regime.

The purpose of this paper is to investigate analytically the properties of these periodic solutions. We shall concentrate on the laser rate equations which are two ordinary differential equations for E, the slowly varying real amplitude of the electrical field and N, the population inversion:

$$E_{t'} = -kE + (g^2/\gamma_\perp)EN \; ,$$
$$N_{t'} = -\gamma_{\|}(N - \overline{N}) - (4g^2/\gamma_\perp)\,|E\,|^2 N \; , \tag{1.1}$$

where $\gamma_{\|}\overline{N}$ is the excitation rate and k, $\gamma_\perp$, and $\gamma_{\|}$ are the loss rates for the field, polarization, and population inversion, respectively; g is the atom-field coupling constant.

As in Ref. 6, we shall control the cavity decay rate and study the effects of small periodic modulations of its amplitude. Specifically, we consider

$$k(t') = k_0[1 + m\cos(\omega' t')] \; , \tag{1.2}$$

where $m \ll 1$ and ω' represents the frequency of the periodic forcing.

The periodically perturbed laser problem (1.1) and (1.2) is mathematically difficult because Eqs. (1.1) do not admit any bifurcation other than the laser first-threshold steady bifurcation. This contrasts with most analytic studies of resonance phenomena which investigate the effect of a small-amplitude periodic forcing near a Hopf bifurcation point (see, for example, Refs. 10—12). To determine the response of the laser equations (1.1) subject to the periodic perturbation (1.2), we shall develop a new asymptotic method based on the fact that the ratio $\gamma = \gamma_{\|}/\gamma_\perp$ is small. (For example,[6] for a CO_2 laser, $\gamma_\perp \simeq 10^8$ s^{-1}, $\gamma_{\|} \simeq 10^3$ s^{-1}, and thus $\gamma \simeq 10^{-5}$.) If $m = 0$ and $\gamma \to 0$, we shall note from Eqs. (1.1) that the non-zero steady-state solutions correspond to double zero eigenvalues of the linearized theory. Thus, each point of the branch of the steady state becomes a degenerate Hopf bifuraction point with zero frequency. This suggests that we analyze the simultaneous limit $m \to 0$ and $\gamma \to 0$ of Eqs. (1.1) and (1.2).

II. FORMULATION

We first introduce the usual nondimensional variables and parameters defined by

$$I \equiv (4g^2/\gamma_\perp\gamma_{\|})\,|E\,|^2 \; , \quad F \equiv N/\overline{N} \; , \quad t \equiv \gamma_\perp t' \; ,$$
$$A \equiv g^2\overline{N}/k_0\gamma_\perp \; , \quad \gamma \equiv \gamma_{\|}/\gamma_\perp \; , \tag{2.1}$$
$$K_0 \equiv k_0/\gamma_\perp \; , \quad \Omega \equiv \omega'/\gamma_\perp$$

and rewrite Eqs. (1.1) and (1.2) in terms of I, F, t, A, γ, K_0, and Ω:

$$I_t = 2K_0 I(-1 + AF) - 2IK_0 m\cos(\Omega t) \; ,$$
$$F_t = \gamma[1 - F(1 + I)] \; . \tag{2.2}$$

When $m = 0$, Eqs. (2.2) admit the following steady-state solutions.

(i) The zero-intensity solutions for all values of A

$$I = 0 \; , \quad F = 1 \; . \tag{2.3}$$

(ii) The nonzero-intensity solutions for $A \ge 1$

$$I = I_0 \equiv A - 1 > 0 \; , \quad F = F_0 \equiv 1/A \; . \tag{2.4}$$

From the linearized theory, we note that (2.3) is stable (unstable) if $A < 1$ $(A > 1)$. On the other hand, the characteristic equation describing the stability of (2.4) is given by

$$\sigma^2 + \gamma A\sigma + 2K_0\gamma(A - 1) = 0 \; , \tag{2.5}$$

where σ is the growth rate. If $\gamma \neq 0$, we find from (2.5) that $\mathrm{Re}(\sigma_1) < 0$ and $\mathrm{Re}(\sigma_2) < 0$ for all $A > 1$ and therefore (2.4) represents a stable steady-state solution. However, in the limit $\gamma \to 0$ we have, from (2.5),

$$\sigma = R + i\omega \; ,$$
$$\omega = [2K_0\gamma(A - 1)]^{1/2} + O(\gamma^{3/2}) \; , \tag{2.6}$$
$$R = O(\gamma) \; .$$

Hence, if $\gamma = 0$, $\sigma_1 = \sigma_2 = 0$ and the steady-state solutions are marginally stable. The points $(I, F, m, \gamma) = (A - 1, 1/A, 0, 0)$ are therefore degenerate bifurcation points because they correspond to double zero eigenvalues of the linearized theory. To analyze the perturbation produced by small values of m and γ, it will be convenient to introduce the deviations from the steady-state solutions (2.4) defined by

$$x \equiv (F - F_0)/\alpha \; ,$$
$$y \equiv (I - I_0)/I_0 \; , \tag{2.7}$$

where α is given by

$$\alpha \equiv (F_0 I_0 \gamma / 2K_0 A)^{1/2} \; . \tag{2.8}$$

After inserting (2.7) and (2.8) into (2.2) and redefining our reference time as

$$T = \Omega t = \omega' t' \; , \tag{2.9}$$

we obtain

$$x_T = -\lambda y - \epsilon b x[1 + \lambda^2(1 + y)/a^2] \; ,$$
$$y_T = \lambda x(1 + y) - \epsilon c(1 + y)\cos(T) \; . \tag{2.10}$$

The parameters a, b, and c are fixed $O(1)$ quantities defined by

$$a = (2K_0)^{1/2}/(\Omega\gamma^{-1/2}) \; , b = 1/(\Omega\gamma^{-1/2}) \; ,$$
$$c = (2K_0 m\gamma^{-1})/(\Omega\gamma^{-1/2}) \; . \tag{2.11}$$

The small parameter $\epsilon \ll 1$ and the control (or bifurcation) parameter $\lambda = 0(1)$ are given by

$$\epsilon = \lambda^{1/2} \; , \quad \lambda = aI_0^{1/2} = a(A - 1)^{1/2} = \omega/\Omega \; . \tag{2.12}$$

This shows that the control parameter λ has a very simple physical meaning: it is the ratio of the damped-oscillation frequency ω given by (2.6) to the forcing frequency.

We shall investigate the laser equations (2.10) by fixing the amplitude $m = O(\gamma)$ and the frequency $\Omega = O(\gamma^{1/2})$ of the periodic variations of $k(t)$ as suggested by the experimental work.

To analyze the perturbation of the double zero eigenvalue, we propose an asymptotic study of Eqs. (2.10) as $\epsilon \to 0$. This paper is divided into four parts. In Sec. III, we seek $O(1)$ periodic solutions of (2.10) of the form

$$x(T,\epsilon) = x_0(T) + \epsilon x_1(T) + \cdots \; ,$$
$$y(T,\epsilon) = y_0(T) + \epsilon y_1(T) + \cdots \tag{2.13}$$

which are defined for all values of $\lambda > 0$. The expansion

(2.13) may, however, become singular if (x_0,y_0) is too small in amplitude and is in the vicinity of particular values of λ. To understand this problem in detail, we determine in Sec. IV the small-amplitude solutions of (2.10) of the form

$$x(T,\lambda,\epsilon)=\epsilon x_1(T,\lambda)+\epsilon^2 x_2(T,\lambda)+\cdots ,$$
$$y(T,\lambda,\epsilon)=\epsilon y_1(T,\lambda)+\epsilon^2 y_2(T,\lambda)+\cdots . \quad (2.14)$$

We discuss the validity of this expansion and show that it may become singular in the vicinity of critical values of λ. Two singularities are investigated to complete our bifurcation analysis.

III. HARMONIC AND SUBHARMONIC $O(1)$ PERIODIC SOLUTIONS

In this section, we will treat ϵ as a small parameter and use the expansion (2.13) to determine the possible periodic solution of (2.10). Setting $\epsilon=0$ in (2.10), we obtain the reduced equations

$$x_T=-\lambda y ,$$
$$y_T=\lambda x(1+y) . \quad (3.1)$$

The system (3.1) is conservative with a first integral given by

$$J=x^2+2y-2\ln|1+y| . \quad (3.2)$$

The essential feature of these equations is described in the phase plane (x,y) (see Fig. 1 of Ref. 13). The origin is surrounded by periodic orbits with periods ranging continuously from $2\pi/\lambda$ near the origin to ∞ near the line $y=-1$. Thus for a fixed value of λ, there exists among the solutions of (3.1) a set of periodic solutions $(x,y)=(x_n(T),y_n(T))$ characterized by their period $P=2\pi n$ $(n=\ldots,\frac{1}{3},\frac{1}{2},1,2,3,\ldots)$ with $n>1/\lambda$. As a consequence, if we now consider $\epsilon\neq0$ but sufficiently small, only these periodic solutions may be excited by the 2π-periodic forcing. Our goal is precisely to study the possible resonances.

The two-dimensional system (3.1) has appeared in some earlier work on the laser rate equations. If we define $s=\ln|1+y|$ and use (3.1) to write a second-order differential equation for s, we obtain $s_{TT}+\lambda^2(e^s-1)=0$. This equation was derived by Oppo and Politi[14] to find the period of the damped laser oscillations. It is also interesting to note that Eqs. (3.1) have been obtained recently in different singular Hopf bifurcation problems.[13,15]

The mathematical approach we consider follows closely the work by Schwartz and Smith[13] which is based on the ideas developed by Chow, Hale, and Mallet-Paret.[16] Specifically, we shall seek periodic solutions of Eq. (2.10) of the form

$$x(T,\epsilon)=x_n(T+\varphi)+\epsilon X_1(T)+\epsilon^2 X_2(T)+\cdots ,$$
$$y(T,\epsilon)=y_n(T+\varphi)+\epsilon Y_1(T)+\epsilon^2 Y_2(T)+\cdots , \quad (3.3)$$

where $(x_n(T),y_n(T))$ represents a periodic solution of (3.1) of period $P=2\pi n$ $(n=\cdots,\frac{1}{3},\frac{1}{2},1,2,\ldots>1/\lambda)$ and φ is an arbitrary constant phase to be determined by the perturbation procedure. Introducing (3.3) into (2.10) and equating to zero the coefficients of each power of ϵ leads to the following equations for $X_1(T)$ and $Y_1(T)$:

$$X_{1T}+\lambda Y_1=-bx_n[1+\lambda^2(1+y_n)/a^2] ,$$
$$Y_{1T}-\lambda X_1(1+y_n)-\lambda x_n Y_1=-c(1+y_n)\cos(T) . \quad (3.4)$$

Since the homogeneous linear problem for X_1 and Y_1 admits a single-bounded periodic solution, namely $(X_1,Y_1)=(x_{nT},y_{nT})$, the inhomogeneous equation must satisfy a solvability condition. The procedure to obtain this condition is described in detail in the Appendix of Ref. 13, so that we only discuss the main results.

We have found the following solvability condition

$$(\lambda^2/a^2+1)bI_1+c\cos(\varphi)I_2+c\sin(\varphi)I_3=0 , \quad (3.5)$$

where $I_1(\lambda,n)$, $I_2(\lambda,n)$, and $I_3(\lambda,n)$ are the integrals defined by

$$I_1=\int_0^{n2\pi} x_n^2(\xi)d\xi ,$$
$$I_2=\int_0^{n2\pi}\cos(\xi)y_n(\xi)d\xi ,$$
$$I_3=\int_0^{n2\pi}\sin(\xi)y_n(\xi)d\xi . \quad (3.6)$$

Equation (3.5) is the bifurcation equation for the undetermined phase φ: If Eq. (3.5) admits a solution for φ then Eq. (2.10) has a $2\pi n$-periodic solution given by

$$x=x_n(T+\varphi)+O(\epsilon) ,$$
$$y=y_n(T+\varphi)+O(\epsilon) . \quad (3.7)$$

In general, the integrals $I_j(\lambda,n)$ must be evaluated numerically. As an illustration, we have considered $n=1$ and 2. In Fig. 1, we present the domains in the (λ,c) parameter plane where a solution of the form (3.7) may exist. Depending on the value of n and the initital value of $y_n(T)$ used to evaluate the integrals I_2 and I_3, we have found that either $I_2=0$ or $I_3=0$. If $I_2=0$, the boundaries of the domain of possible solutions correspond to $|\varphi|=\pi/2$

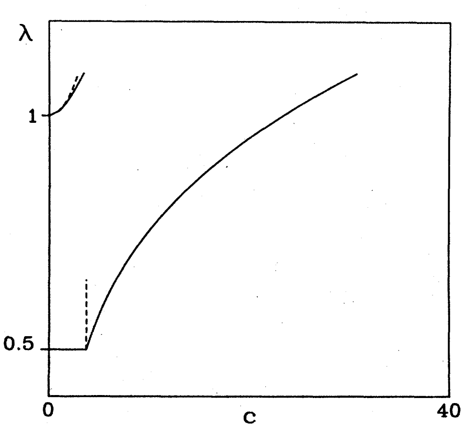

FIG. 1. Domains in the (λ,c) parameter plane of the subharmonic 4π-periodic solutions $(\lambda>\frac{1}{2})$ and the harmonic 2π-periodic solutions $(\lambda>1)$.

and are defined by the two functions

$$(\lambda^2/a^2+1)bI_1 \pm cI_2 = 0 \ . \tag{3.8}$$

On the other hand, if $I_3=0$, the boundaries of the domain of possible solutions correspond to $\varphi=0$ or π and are defined by the two functions

$$(\lambda^2/a^2+1)bI_1 \pm cI_3 = 0 \ . \tag{3.9}$$

The solid lines in Fig. 1 represent the functions defined either by (3.8) or (3.9). Note, however, that this does not imply the stability of the periodic solutions.

In order to obtain an analytic expression for the boundaries shown by the solid lines on Fig. 1, we analyze the periodic solutions with period $2\pi n$ ($n=1$ or 2) in the vicinity of $\lambda=1/n$. We first determine a small-amplitude $2\pi n$-periodic solution of Eq. (3.1) by the Lindstedt-Poincaré method.[10] We seek a solution of the form

$$\begin{aligned} x(T/n,\delta) &= \delta u_1(T/n)+\delta^2 u_2(T/n)+\cdots \ , \\ y(T/n,\delta) &= \delta v_1(T/n)+\delta^2 v_2(T/n)+\cdots \ , \end{aligned} \tag{3.10}$$

where δ is a small parameter defined by

$$\delta = \frac{1}{2\pi n}\int_0^{n2\pi} x(T/n,\delta)e^{-iT/n}dT \ . \tag{3.11}$$

From the expression for x in (3.10), we note that the definition (3.11) implies that $u_1=e^{iT/n}+\text{c.c.}$, while $u_2, u_3, \ldots$ may only involve harmonics of $e^{\pm iT/n}$ (i.e., $e^{\pm imT/n}$ with $m=2,3,\ldots$). Thus δ is simply defined as the amplitude of the basic mode $e^{iT/n}$. We also assume the following expansion of λ:

$$\lambda(\delta) = \frac{1}{n}(1+\delta\lambda_1+\delta^2\lambda_2+\cdots) \ . \tag{3.12}$$

Introducing (3.10) and (3.12) into (3.1) and equating to zero the coefficients of each power of δ, we obtain a sequence of problems for $(u_1,v_1), (u_2,v_2), \ldots$. Then solving each problem sequentially and using (3.11), we obtain the following results:

$$\begin{aligned} x(T/n,\delta) &= \delta(e^{iT/n}+\text{c.c.})+\delta^2\left[-\frac{i}{3}e^{2iT/n}+\text{c.c.}\right] \\ &\quad +\delta^3(-\tfrac{1}{8}e^{3iT/n}+\text{c.c.})+O(\delta^4) \ , \\ y(T/n,\delta) &= -\frac{1}{\lambda}x_T \ , \end{aligned} \tag{3.13}$$

where the $O(\delta^4)$ term involves higher-order harmonics of the form $e^{\pm imT/n}$ ($m=2,3,\ldots$). The unknown coefficients $\lambda_1, \lambda_2, \ldots$ appearing in (3.12) are determined from the solvability conditions. We find that

$$\lambda(\delta) = \frac{1}{n}\left[1+\frac{\delta^2}{6}+O(\delta^4)\right] \ . \tag{3.14}$$

Thus from (3.14), we can express the amplitude δ as a function of the deviation $\lambda-1/n$:

$$\delta = [6n(\lambda-1/n)]^{1/2}+O(\lambda-1/n) \ . \tag{3.15}$$

We now consider the cases $n=1$ and 2 evaluate the integrals in (3.5), (3.8), and (3.9) using the expansion (3.13). The corresponding boundaries of existence are drawn as dotted lines on Fig. 1.

A. $n=1$

As $\delta\to 0$ (i.e., as $\lambda-1\to 0$ or $\omega\simeq\Omega$) we obtain from (3.9)

$$c \simeq \pm 2b(1/a^2+1)\sqrt{6}(\lambda-1)^{1/2}, \quad \lambda>1 \tag{3.16}$$

which represents a parabola in the (λ,c) parameter plane. For the physically relevant values of $\lambda\geq 0$ and $c\geq 0$ located inside this parabola, there exist periodic solutions of the form

$$x(T,\delta) = \delta(e^{i(T+\varphi)}+\text{c.c.})+O(\delta^2) \ . \tag{3.17}$$

Thus these values of (λ,c) correspond to the case of pure resonance.

B. $n=2$

As $\delta\to 0$ (i.e., as $\lambda-\frac{1}{2}\to 0$ or $\Omega\simeq 2\omega$), we now find from (3.8)

$$c \simeq \pm 3b[1/(2a)^2+1], \quad \lambda>\tfrac{1}{2} \tag{3.18}$$

which are vertical lines in the (λ,c) plane. Inside the domain bounded by these lines the parameter values are associated with periodic solutions of the form

$$x(T,\delta) = \delta(e^{i(T/2+\varphi)}+\text{c.c.})+O(\delta^2) \ . \tag{3.19}$$

Thus this region corresponds to the subharmonic case of order 2. From Fig. 1 we see that the vertical line (3.18) is tangent to the numerically determined solution of (3.8).

A similar analysis of (3.5) or (3.8) and (3.9) is possible for the next subharmonic cases ($n=3,4,5\ldots$) or harmonic cases ($n=1/m$, $m=2,3,\ldots$) but will not be given here because the two cases $n=1$ and 2 already capture the essential properties of the next cases.

In this way one can generate in the (λ,c) parameter plane a succession of boundaries corresponding to the existence of subharmonic solutions ($P=2\pi n$, $n=1,2\ldots$) and of harmonic solutions ($P=2\pi/n$, $n=1,2,\ldots$). For a fixed value of c and gradually increasing λ from zero, successive transitions to distinct subharmonic periodic solutions are possible. This analysis, however, does not determine if the transitions between these solutions occur by successive bifurcations or if the different periodic regimes belong to coexisting branches of solutions. We emphasize the fact that the periodic solutions constructed using (3.3) represent $O(1)$ solutions. In general, the small-amplitude limit of this expansion [see (3.10)—(3.19)] valid near particular values of λ ($\lambda\simeq 1/n$) is singular, i.e., the expansion (3.3) becomes nonuniform near and at these values of λ. Thus the small-amplitude analysis presented at the end of this section can only be considered as indicative and inner solutions valid near these critical values of λ have to be constructed. Uniform solutions may then be proposed connecting these inner solutions to the general (outer) solution described by (3.3). In order to clearly understand the role of these singularities, we shall analyze in the next section the expansion (2.14) for $O(\epsilon)$ periodic solutions.

IV. HARMONIC AND SUBHARMONIC $O(\epsilon^p)$ PERIODIC SOLUTIONS

In this section, we determine small-amplitude periodic and quasiperiodic solutions of Eq. (2.10). To this end, we first seek a solution of (2.10) of the form

$$x(T,S,\epsilon)=\epsilon X_1(T,S)+\epsilon^2 X_2(T,S)+\cdots ,$$
$$y(T,S,\epsilon)=\epsilon Y_1(T,S)+\epsilon^2 Y_2(T,S)+\cdots , \qquad (4.1)$$

which depends on two independent time variables: the fast time T of the periodic forcing and a slow time given by

$$S=\epsilon T . \qquad (4.2)$$

Introducing (4.1) and (4.2) into (2.10) and equating to zero the coefficients of each power of ϵ, we obtain a sequence of linear problems for (X_1,Y_1), (X_2,Y_2), ... After applying the solvability conditions, we find the following solution:

$$x=\epsilon\{[\alpha(S)e^{i\lambda T}+\text{c.c.}]+(p_1e^{iT}+\text{c.c.})\}+0(\epsilon^2) ,$$
$$y=\epsilon\{[-i\alpha(S)e^{iT}+\text{c.c.}]\}+(q_1e^{iT}+\text{c.c.})+O(\epsilon^2) , \qquad (4.3)$$

where p_1 and q_1 are two constants defined by

$$p_1=\frac{c}{2}\frac{\lambda}{\lambda^2-1} ,$$
$$q_1=-i\frac{c}{2}\frac{1}{\lambda^2-1} , \qquad (4.4)$$

and the amplitude $\alpha(S)$ is given by

$$\alpha(S)=\alpha(0)\exp\left[-\frac{b}{2}(\lambda^2/a^2+1)S\right]\rightarrow 0 \text{ as } S\rightarrow\infty , \qquad (4.5)$$

where $\alpha(0)$ can be related to the initial conditions for x and y. Thus, the solution (4.3) represents a quasiperiodic solution involving two distinct frequencies $\sigma=1$ and $\sigma=\lambda$. However, as $T\rightarrow\infty$, the solution approaches a 2π-periodic regime. From (4.3) and (4.4), we note that the solution becomes singular as $|\lambda-1|\rightarrow 0$ (the pure resonance case). Moreover, the analysis of the $O(\epsilon^2)$ corrections which involve harmonics of $e^{\pm iT}$ and $e^{\pm i\lambda T}$ and their combinations show that other singularities may develop at $\lambda=1/n$ (the subharmonic cases) or $\lambda=n$ (the resonance cases). This is a general observation for periodically perturbed Hopf bifurcations.[17] To resolve these singularities, new (inner) expansions of the solutions valid near the different singular points must be proposed. To illustrate this phenomenon, we shall analyze one example of resonance case ($n=1$) and one example ($n=2$) of subharmonic case. Since the successive perturbation studies are discussed in detail by Rosenblat and Cohen,[17] we only summarize the main results.

A. $\lambda\simeq 1$

Rosenblat and Cohen[17] have shown that the appropriate expansion of the bifurcation parameter and the dependent variables are given by

$$\lambda-1=\epsilon^{2/3}\lambda_2+\epsilon\lambda_3+\cdots \qquad (4.6)$$

and

$$x=\epsilon^{1/3}X_1(T,\tau)+\epsilon^{2/3}X_2(T,\tau)+\cdots ,$$
$$y=\epsilon^{1/3}Y_1(T,\tau)+\epsilon^{2/3}Y_2(T,\tau)+\cdots , \qquad (4.7)$$

where τ is a new slow time defined by

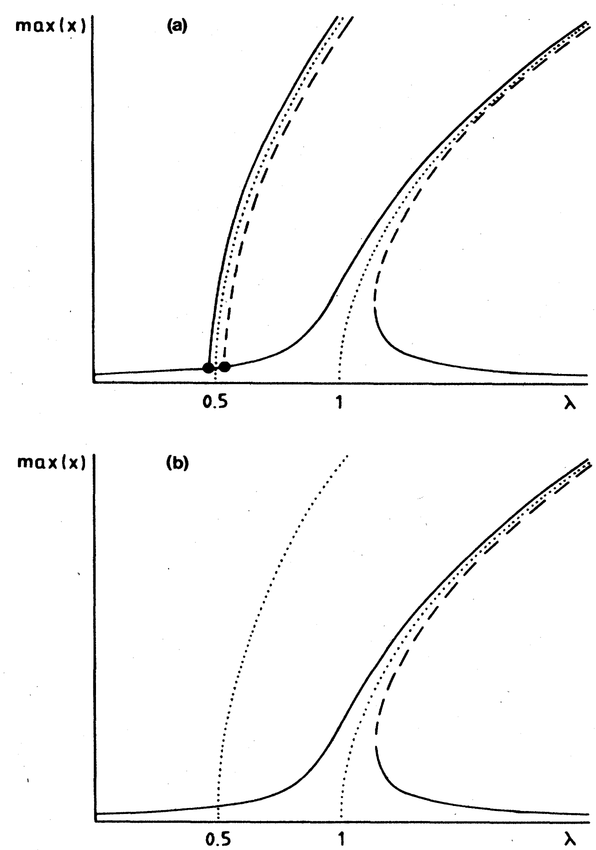

FIG. 2. Global bifurcation diagram of the periodic solutions representing the maximum value of x as a function λ. The figure has been obtained by matching the expansions of the harmonic and subharmonic solutions valid near $\lambda=1$ and $\lambda=\frac{1}{2}$, respectively. In (a) bifurcation to subharmonic solutions (near $\lambda=\frac{1}{2}$) is possible because condition (4.23) is satisfied. If this condition is not verified, the basic 2π-periodic solution remains stable and bifurcation to subharmonic solutions does not occur [see (b)]. Isolated branches of $O(1)$ subharmonic periodic solutions may, however, exist but are not described by the local analysis. Note that the separation between the two bifurcation points near $\lambda=\frac{1}{2}$ is $O(\epsilon)$ in (a) while the deviation of the limit point from $\lambda=1$ is $O(\epsilon^{2/3})$.

$$\tau=\epsilon^{2/3}T \ . \tag{4.8}$$

We substitute the series (4.6) and (4.7) into (2.10) and equate to zero the coefficients of each power of $\epsilon^{1/3}$. Using the solvability conditions, we find that

$$x=\epsilon^{1/3}[\alpha(\tau)e^{iT}+\text{c.c.}]+O(\epsilon^{2/3}) \ ,$$
$$y=\epsilon^{1/3}[-i\alpha(\tau)e^{iT}+\text{c.c.}]+0(\epsilon^{2/3}) \ , \tag{4.9}$$

where the amplitude α satisfies

$$\alpha_\tau=i\lambda_2\alpha-\frac{i}{6}\alpha^2\alpha^*-i\frac{c}{4} \ . \tag{4.10}$$

To analyze the solutions of (4.10), it will be convenient to define $\alpha=r\exp(i\theta)$ and consider the evolution equations for r and θ:

$$r_\tau=-\frac{c}{4}\sin(\theta) \ ,$$
$$r\theta_\tau=\lambda_2 r-\frac{r^3}{6}-\frac{c}{4}\cos(\theta) \ . \tag{4.11}$$

The steady-state solutions of (4.11) are given by solution (i),

$$\theta=0 \ ,$$
$$\lambda_2=\frac{r^2}{6}+\frac{c}{4r} \ ; \tag{4.12}$$

and solution (ii),

$$\theta=\pi \ ,$$
$$\lambda_2=\frac{r^2}{6}-\frac{c}{4r} \ . \tag{4.13}$$

They are represented in Fig. 2. The linear stability of (4.12) and (4.13) can be studied using (4.11). This leads to positive eigenvalues for the middle branch which is therefore always unstable. For the two other branches, the trace of the Jacobian matrix identically vanishes and therefore the eigenvalues are purely imaginary. The first contribution to the real part of the eigenvalues will therefore come from an analysis of the next order in the perturbation expansion. When this is done, it is found that $\text{Re}\lambda=-\epsilon(1+1/a^2)/2<0$ (on the time scale T) and therefore the upper and lower branches are stable.

B. $\lambda\simeq\frac{1}{2}$

We now analyze the second subharmonic. The appropriate expansion for the bifurcation parameter in this case turns out to be

$$\lambda=\tfrac{1}{2}+\epsilon\lambda_2+\cdots \ . \tag{4.14}$$

We substitute (4.14) into (2.10) together with the series

$$x=\epsilon^{1/2}X_1(T,S)+\epsilon X_2(T,S)+\cdots \ ,$$
$$y=\epsilon^{1/2}Y_1(T,S)+\epsilon Y_2(T,S)+\cdots \ , \tag{4.15}$$

where the slow time S is defined by (4.2). Equating coefficients of like powers of $\epsilon^{1/2}$, we obtain a sequence of linear problems for $(X_1,\ Y_1)$, $(X_2,\ Y_2),\ldots$. Analyzing the three first orders, we find that x and y are in first approximation 4π-periodic functions of T and are given by

$$x=\epsilon^{1/2}[\alpha(S)e^{iT/2}+\text{c.c.}]+O(\epsilon) \ ,$$
$$y=\epsilon^{1/2}[-i\alpha(S)e^{iT/2}+\text{c.c.}]+O(\epsilon) \ , \tag{4.16}$$

where the $O(\epsilon)$ correction terms involve contributions from the forcing function of the form $e^{\pm iT}$. From the solvability condition of the $O(\epsilon^{3/2})$ problem, we find that α satisfies

$$\alpha_S=i\lambda_2\alpha-\alpha\frac{b}{2}[1/(2a)^2+1]-\frac{i}{12}\alpha^2\alpha^*+\frac{c}{6}\alpha^* \ . \tag{4.17}$$

By defining $\alpha=\text{r}e^{i\theta}$, we obtain from (4.17)

$$r_S=-\frac{b}{2}[1/(2a)^2+1]r+\frac{c}{6}r\cos(2\theta) \ ,$$
$$r\theta_S=r\lambda_2-\frac{1}{12}r^3-\frac{c}{6}r\sin(2\theta) \ . \tag{4.18}$$

Eliminating θ from the steady-state equations, we find two different branches of steady states: branch (i),

$$r=0 \ , \tag{4.19}$$

and branch (ii),

$$r_\pm=6(2\lambda_2\pm\Gamma^{1/2})^{1/2} \ , \tag{4.20}$$

where

$$\Gamma=\frac{c^2}{9}-b^2[1/(2a)^2+1]^2>0 \ . \tag{4.21}$$

The solution (4.19) corresponds to the basic solution and is given by

$$x=\epsilon\left[-\frac{c}{3}e^{iT}+\text{c.c.}\right]+O(\epsilon^{3/2}) \ ,$$
$$y=\epsilon\left[2i\frac{c}{3}e^{iT}+\text{c.c.}\right]+O(\epsilon^{3/2}) \ . \tag{4.22}$$

It is stable when $\Gamma<0$. From the solution (4.20) and provided that $\Gamma>0$ or equivalently

$$\left|\frac{c}{3}\right|>b[1/(2a)^2+1] \ , \tag{4.23}$$

there exist two distinct branches of 4π-periodic solutions of the form

$$x_\pm=\epsilon^{1/2}(r_\pm e^{iT/2}+\text{c.c.})+O(\epsilon) \ ,$$
$$y_\pm=\epsilon^{1/2}(-ir_\pm e^{iT/2}+\text{c.c.})+O(\epsilon) \ . \tag{4.24}$$

From a linear stability analysis of the steady states (4.20), we have found that r_+ (r_-) corresponds to stable (unstable) solutions. In this case the solution (4.22) is stable except in the domain $-\Gamma^{1/2}<2\lambda_2<\Gamma^{1/2}$. In Fig. 2(a), we have represented the two bifurcations. In Fig. 2(b), there exists no period-doubling bifurcation because condition (4.23) is not satisfied. Figures 2 gives global bifurcation diagrams connecting the $O(1)$ periodic solutions found in Sec. III and the small-amplitude periodic solutions analyzed in this section. In all cases, we have verified analytically the matching of the different asymptotic expansions.

V. DISCUSSION

The problem we have studied in this paper is of an unusual nature because in the absence of external modulation the laser rate equations do not have any Hopf bifurcation (nor any other bifurcation) on the finite-intensity branch. The clue to understanding the rich variety of behaviors displayed by the solutions is the smallness of the relaxation time and frequency. In the limit $\gamma=\gamma_{\parallel}/\gamma_{\perp}\rightarrow 0$ (which corresponds to the domain investigated experimentally) the laser will relax towards its stable steady state through damped oscillations. The oscillation frequency vanishes like $\gamma^{1/2}$ whereas the damping vanishes ever more rapidly, being proportional to γ. In a way, a laser operating in this regime displays critical slowing down for any value of the pump parameter consistent with a finite intensity. We may therefore describe the resulting evolution as the small disipative perturbation of a conservative system. This conservative system has bounded periodic solutions. As a result of the smallness of the damping rate, a small-amplitude external modulation may disrupt the decay sufficiently and stabilize periodic solutions.

With this picture in mind, we expect in general a periodic response at the external modulation frequency. However, anomalous behavior of the solutions may happen at resonances, i.e., when the oscillation frequency (ω) of the rate equations and the external modulation frequency (Ω) are commensurate. We have analyzed the two cases $\lambda=\omega/\Omega=1$ and $\frac{1}{2}$.

For the resonant forcing ($\omega=\Omega$) we have shown that a domain of bistability may occur. The two stable solutions (as well as the intermediate unstable solution) have the same frequency of oscillation but different amplitudes.

A different situation occurs near the first subharmonic resonance ($\Omega\simeq 2\omega$). On both sides of $\lambda=\frac{1}{2}$, bifurcation points can appear whose existence and location depend on the external modulation. Only the solution emerging from the bifurcation point located below $\lambda=\frac{1}{2}$ is stable. This again leads to a domain of bistability beyond the bifurcation point located above $\lambda=\frac{1}{2}$. At the lower bifurcation point, we are dealing with a steady bifurcation of the amplitudes. In terms of the complete solution (which is an expansion in powers of $\epsilon^{1/2}$), the situation is quite different. The vanishing of r means that the leading contribution to the solution is proportional to ϵ whereas when r is finite, the leading contribution is proportional to $\epsilon^{1/2}$. Therefore, the solution corresponding to $r=0$ has the frequency Ω whereas the bifurcating solution with $r\neq 0$ has the frequency $\Omega/2$. As a result the lower bifurcation point is a period-doubling bifurcation. Needless to say, the occurrence of bistability domains implies that different responses will be displayed when λ is progressively increased or decreased. As a consequence, a complete experimental characterization of the bifurcation diagrams requires that both variations of λ be studied.

The choice of scaling for the modulation amplitude which was made throughout this paper imply that the external modulation is a *small* perturbation. This is essential for the derivation of the various solvability conditions which have been discussed. Therefore we cannot infer from the present analysis the behavior of the solutions when the modulation amplitude is large (compared to ϵ). This problem would require a totally different analysis which is beyond the scope of the present paper.

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