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Contractions and expansions of Lie groups and the algebraic approach to scattering

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A recently introduced algebraic approach to scattering is recast in the language of group contractions and expansions. The expansion describes the deformation of the physical states from one kind of symmetry to another and allows the algebraic derivation of an S matrix. The method is illustrated for scattering models in n-dimensional space of the type SO(n,1) and SO(n,m) with $m \geq 2$.

Algebraic methods, which were proven to be useful in a variety of bound-state models in nuclear[1] and molecular[2] physics have recently been extended to the continuum.[3-6]

In a first step,[3] we have shown that certain one-dimensional scattering problems can be described algebraically by using noncompact groups which possess continuous representations, in particular, SU(1,1)≃SO(2,1). The next major step was the development of an algebraic technique to calculate the S matrix. This was first accomplished for a particular potential using a coordinate realization,[4] and was then cast into purely algebraic form by introducing the concept of an "Euclidean connection."[5] Recently, we have been able to generalize these techniques to a higher number of dimensions for problems with orthogonal dynamical symmetries.[6] This led us to consider realistic models related to modified Coulomb problems which can be useful for the study of heavy-ion reactions.[6]

The construction of the S matrix is achieved by connecting the dynamical algebra which describes the scattering problem with a corresponding Euclidean algebra which describes the symmetry of the undistorted waves in the asymptotic regime.[5] Of special importance is the connection formula which expresses the generators of the scattering algebra as a function of those of its Euclidean partner. The formula was constructed to satisfy the appropriate commutation rules of the scattering algebra. However, the relation between the dynamical algebra and the algebra describing the asymptotic symmetry, as well as the general structure of such connection formulas, have not been fully investigated up to now. The purpose of this Rapid Communication is to show that these algebras are closely related by the group-theoretical mechanisms known as contraction and expansion.[7] By means of this procedure, connection formulas can then be studied in a systematic fashion and more general scattering theories can be investigated. As an example, we shall derive an S matrix for scattering problems associated with general orthogonal symmetries.

To illustrate our procedure we shall first analyze problems with SO(n,1) as their symmetry group. These describe Coulomb-type scattering problems in n dimensions.[8] We shall denote the generators of SO(n,1) by $K_{\alpha\beta}(1 \leq \alpha,\ \beta \leq n+1;\ \alpha \neq \beta)$, describing a rotation in the x_α-x_β plane.[9] In the following, we shall use Latin indices $i,j,\ldots$ to denote the spatial range $1,\ldots,n$ and the standard SO(n,1) metric $g_{ij}=\delta_{ij}$, $g_{n+1,n+1}=-1$. In the n-dimensional Coulomb problem this algebra is generated by the angular momentum tensor L_{ij} in the n-dimensional space, and the Runge Lenz vector A_i measured in units of the momentum k. The latter plays the role of the (noncompact) rotation $K_{i,n+1}=A_i$. The Coulomb Hamiltonian $H=\frac{1}{2}p^ip_i+\beta/(x^ix_i)^{1/2}$, where p_i and x_i are the physical momenta and coordinates, is then related to the SO(n,1) Casimir invariant

$$C_2=K^{\alpha\beta}K_{\alpha\beta}=-\sum_i A_i^2+\sum_{i,j} L_{ij}^2 \tag{1}$$

by

$$H=-\beta^2/\{C_2+[\tfrac{1}{2}(n-1)]^2\} \ . \tag{2}$$

The physical states $|\omega,l,(m)\rangle$ are characterized by

$$\begin{aligned} C_2\,|\omega,l,(m)\rangle &= \omega(\omega+n-1)\,|\omega,l,(m)\rangle \ , \\ L^2\,|\omega,l,(m)\rangle &= l(l+n-2)\,|\omega,l,(m)\rangle \ , \end{aligned} \tag{3}$$

where l is the n-dimensional angular momentum ($L^2 \equiv \sum_{i,j} L_{ij}^2$) and (m) is any additional set of angular momentum projection quantum numbers required for a complete labeling of the O(n) states. We are interested here in the scattering eigenstates which belong to a continuous principal series representation $\omega=-(n-1)/2+i\beta/k$ derived from (2). In a general Coulomb-type problem the relation (2) between the Hamiltonian and the

Casimir operator C_2 can be generalized, in which case

$$\omega = -\frac{n-1}{2} + if(k) , \tag{4}$$

where $f(k)$ is some real function of k. In the asymptotic region the waves are undistorted and the space symmetry algebra is the Euclidean algebra[9] $E(n)$. For $n=2$ we have shown[5] that the asymptotic algebra $E(2)$ is obtained from the scattering algebra O(2,1) by contraction. The concept of contractions of a Lie algebra goes back to the work of Inönu and Wigner[10] and of Saletan,[11] who introduced it as a formal way to study the properties of nonsemisimple groups, from those of the better known semisimple groups. Contraction of a Lie algebra arises when we analyze the effect of a linear change of basis which becomes singular in a certain limit. If the transformed structure constants approach a well defined limit as the transformation becomes singular, a new Lie algebra is found. In the $O(n,1)$ scattering problem, we make the transformation

$$P_i^\varepsilon = \varepsilon K_{i,n+1} \tag{5}$$

for a small ε, while the L_{ij} are left unchanged. In the limit $\varepsilon \to 0$ the P_i (we have dropped the superscript ε) become an Abelian invariant subalgebra, namely,

$$[P_i,P_j]=0,\ [L_{ij},P_k]=ig_{ik}P_j - ig_{jk}P_i . \tag{6}$$

In fact, according to (6) P_i transforms as a *vector* under $O(n)$. Thus, a new algebra $\{L_{ij};P_k\}$ is obtained in that limit which is the n-dimensional Euclidean algebra $E(n)$. In the physical application the P_i are just the linear momenta of the scattered particle. While the rotational invariance is left unchanged in the asymptotic region, the system also possesses translational symmetry in that limit, generated by the P_i. In the contraction process the $SO(n,1)$ Casimir operator

$$\varepsilon^2 C_2 \to -P^i P_i \equiv -\bar{C}_2 \tag{7}$$

and becomes the Casimir invariant $\bar{C}_2$ of $E(n)$, whose eigenvalue is the energy k^2. The eigenstates of the $E(n)$ algebra,

$$\begin{aligned} \bar{C}_2 | \pm k,l,(m)\rangle &= k^2 | \pm k,l,(m)\rangle , \\ L^2 | \pm k,l(m)\rangle &= l(l+n-2) | \pm k,l,(m)\rangle , \end{aligned} \tag{8}$$

describe incoming $(-k)$ and outgoing $(+k)$ free spherical waves. The S matrix in the partial-wave basis is obtained from

$$S_l(k) = (-)^{l+(n-1)/2} B_l(k)/A_l(k) ,$$

where

$$|\omega,l,(m)\rangle = A_l(k) | -k,l,(m)\rangle + B_l(k) | +k,l,(m)\rangle \tag{9}$$

and is (m) independent due to $O(n)$ rotational invariance. As explained in Ref. 5, the algebraic calculation of this S matrix is made feasible by the "Euclidean connection," which expresses the generators of $SO(n,1)$ in terms of those of its contraction $E(n)$. In the theory of Lie groups, this process, which is the opposite of contraction, is known as expansion[7] or deformation. In an expansion process, one replaces some of the generators of the nonsemisimple group by nonlinear functions of the generators of that group. If these generators, together with the unaltered ones, close under commutation, we say that we have "expanded" the original group.

The general problem of an expansion is an unsolved one, but several cases have been studied and a good overview can be found in the book by Gilmore. This nonlinear process of expansion is crucial to our scattering theory since it gives rise to the connection formulas.

We shall explain a systematic way to expand which works in the case of $E(n)$. The generators that need to be replaced are the momenta P_i. The Casimir invariant $C_2(SO(n)) = L^{ij}L_{ij}$ of the compact subalgebra $O(n)$ (which was not modified in the contraction process) disappears in the contracted $O(n,1)$ Casimir invariant (7). We should therefore use it if we want to reconstruct the original scattering algebra. Since $C_2(SO(n))$ is a scalar under $O(n)$ rotations and P_i is a vector, we can easily construct a new $SO(n)$ vector which is nonlinear in the $E(n)$ generators by means of the algorithm[7]

$$\bar{K}_{i,n+1} = \frac{1}{2i}[C_2(SO(n)),P_i] + \rho P_i \quad (\rho \text{ real}) . \tag{10}$$

Since $\bar{K}_{i,n+1}$ is an $SO(n)$ vector it automatically satisfies the correct $(O(n,1))$ commutation relations with L_{ij}.

Consider now the commutators $[\bar{K}_{i,n+1},\bar{K}_{j,n+1}]$. They form an antisymmetric tensor of $SO(n)$ (with respect to their indices i,j). Since they are at most quadratic in P and linear in L, the only possible antisymmetric tensor to this power is $L_{ij}P^mP_m$. Since $P^mP_m = k^2$, it follows that $K_{i,n+1} = \bar{K}_{i,n+1}/k$ must close under commutation with L_{ij}. In fact, we get precisely the $O(n,1)$ algebra. By calculating its Casimir invariant we find that

$$\rho \equiv \pm f(k) . \tag{11}$$

The expansion (10) coincides with the connection formula derived in Refs. 5 and 6 for $n=2,3$. The sign in (11) depends on whether we work in the $+k$ or $-k$ representation of $E(n)$. It leads (by the same method explained in Ref. 5) to an S matrix of the form

$$S_l(k) = \frac{\Gamma[l+\frac{1}{2}(n-1)+if(k)]}{\Gamma[l+\frac{1}{2}(n-1)-if(k)]} e^{i\phi(k)} , \tag{12}$$

where $\phi(k)$ is an arbitrary phase. Equation (12) describes Coulomb-type S matrices in n dimensions.

To illustrate further the interplay between contractions and expansions within the algebraic theory of scattering, we now analyze a scattering problem of the type $SO(n,m) \supset SO(n)\times SO(m)$ with $m \geq 2$. Such scattering models with $m=2$ and $n=1,2,3$ were discussed in Ref. 6. Here $SO(n,m)$ plays the role of a potential algebra[4,5,12] (rather than just a symmetry algebra). The subalgebra with $SO(m)$ with generators $M_{\alpha\beta}$ is used to label the interaction parameters of the model, and they can be raised or lowered (together with the angular momentum l) by the noncompact generators $K_{i\alpha}$ of $SO(n,m)$. Here and in the following we shall use Latin letters $i,j,\ldots$ to denote the range $1,\ldots,n$ and Greek letters $\alpha,\beta,\ldots,$ to denote the range $n+1,\ldots,n+m$. The subgroup $SO(n)$ is still the rotational algebra L_{ij} in the n-dimensional physical space.

The scattering eigenstates $|\omega,l,(m);v,(v)\rangle$ now contain additional labels $v,(v)$ which denote the interaction strength (see Ref. 6) such that $v(v+m-2)$ is the eigenvalue of $M^2 \equiv M^{\alpha\beta}M_{\alpha\beta}$ and (v) labels a complete set of projections in SO(m). The angular momentum labels $l,(m)$ are as before, but now the eigenvalue of the SO(n,m) Casimir invariant

$$C_2 = -\sum_{i,\alpha} K_{i\alpha}^2 + L^2 + M^2 \tag{13}$$

is $\omega(\omega+n+m-2)$.

In the contraction process we leave L and M unaltered while $\varepsilon K_{i\alpha} \to P_{i\alpha}$ and $\varepsilon^2 C_2 \to P^{i\alpha}P_{i\alpha} \equiv \bar{C}_2$. Now C_2 is the Casimir operator of the contracted algebra $\{L_{ij}, M_{\alpha\beta}, P_{\alpha i}\}$ known in the literature as ISO(n,m) (the inhomogeneous special orthogonal group[13]). It is obtained by complementing the algebra SO$(n)\times$SO(m) with an Abelian tensor (double vector) $P_{i\alpha}$ of SO$(n)\times$SO(m), where i is the SO(n) index and α is the SO(m) index

$$\begin{aligned} [L_{ij}, P_{m\gamma}] &= i g_{im} P_{j\gamma} - i g_{jm} P_{i\gamma} , \\ [M_{\alpha\beta}, P_{m\gamma}] &= i g_{\alpha\gamma} P_{m\beta} - i g_{\beta\gamma} P_{m\alpha} . \end{aligned} \tag{14}$$

Note that when $m=1$ (i.e., we "freeze" the potential degree of freedom) there is no index α, and we are back to the Coulomb-type problems where ISO$(n,1) = E(n)$.

The contracted algebra ISO(n,m) describes the undistorted waves in the asymptotic region of the scattering problem. To make contact with the methods of Ref. 6 we note that one can add to the space group O(n) an Abelian vector P_i (the linear momentum) of length $P^iP_i = k^2$, and to the potential O(m) an Abelian vector v_α of length $v^\alpha v_\alpha = 1$, to obtain an asymptotic symmetry group $E(n)\times E(m)$. The generators $P_{i\alpha}$ of the above ISO(n,m) are given by

$$P_{i\alpha} = P_i v_\alpha . \tag{15}$$

It then follows that $\bar{C}_2 = v^\alpha v_\alpha P^i P_i = k^2$ and the incoming $(-k)$ and outgoing $(+k)$ free waves are $|\pm k,l,(m),v,(v)\rangle$, where all other quantum numbers are as before.

To calculate an S matrix[6] we must be able to expand ISO(n,m) back to SO(n,m). In particular, we try to recover the "lost" operators $K_{i\alpha}$ by using the Casimir invariant of the unmodified part SO$(n)\times$SO(m). In analogy with (11) we define[13]

$$\begin{aligned} K_{i\alpha} &= \left\{ \frac{1}{2i} [C_2(\mathrm{SO}(n)) + C_2(\mathrm{SO}(m)), P_{i\alpha}] + f(k) P_{i\alpha} \right\} / k \\ &= (L_{ij} P_j v_\alpha - M_{\alpha\beta} v_\beta P_i - i\omega v_\alpha P_i)/k , \end{aligned} \tag{16}$$

where $\omega = -(n+m-2)/2 + i f(k)$ denotes the (most degenerate) principal series unitary representation of SO(n,m). In analogy with the case $m=1$ it can be shown that relation (16) defines an expansion into SO(n,m).

The connection formula (16) leads to an S matrix which is independent of (m) and (v) [due to the SO$(n)\times$SO(m) symmetry]:

$$S_l(k) = \frac{\Gamma(\frac{1}{2}[l+v+(n+m-2)/2+if(k)])\Gamma(\frac{1}{2}[l-v+(n+m-2)/2+if(k)])}{\Gamma(\frac{1}{2}[l+v+(n+m-2)/2-if(k)])\Gamma(\frac{1}{2}[l-v+(n+m-2)/2-if(k)])} e^{i\phi(k)} . \tag{17}$$

It can be shown that for angular momentum l large compared with v and $f(k)$, the phase shifts in (17) approach the Coulomb-type ones in (12). Thus, (17) corresponds to a second class of modified Coulomb-type potentials.

To conclude, the algebriac theory of scattering [in particular that of the SO(n,m) scattering models] has been recast in a general mathematical framework by means of the group-theoretical concepts of contraction and expansion. This formulation can be applied to the systematic study of other types of dynamical symmetries in scattering problems and their associated S matrices. Modification in the dynamical groups is necessary when one includes spin (and isospin) degrees of freedom and more complicated interactions which can depend on these new dynamical variables. Other interesting generalizations occur when the asymptotic symmetry is not necessarily that of a free particle, but describes distorted waves in some potential for which the problem is solvable. The calculation of the additional phase shift caused by adding another potential can then be solved (in principle) by a deformation (contraction or expansion) of a group G into another G'. We note that expansions analogous to the ones studied here, have been analyzed in a different context for the $U(n,m) \supset U(n)\times U(m)$ and Sp$(n,m)\supset$Sp$(n)\times$Sp(m),[13] as well as for the contraction Sp$(6,R)\to$ W$(6)\wedge$U(3),[14] where $\wedge$ indicates a semidirect product. These may prove useful for the derivation of S matrices for other classes of dynamic symmetry.

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