

Fluctuations of dynamical scaling indices in nonlinear systems

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Due to fluctuations around the metric entropy of chaotic dynamical systems there exists a spectrum of invariant dynamical scaling indices, complementary to the range of invariant static scaling indices which have been discovered and applied recently. The basic scaling behavior in both cases is shown to be rooted in the thermodynamic formalism of dynamical systems. An explicit simple example of these ideas is given, stressing their use in characterizing complex behavior.

Despite numerous successes, there is still a gap between experimentation on nonlinear time-dependent physical systems and dynamical systems theory. One of the achievements in bridging this gap has been the definition and measurement of several invariants that describe characteristic aspects of the signals that one faces in the laboratory.¹⁻³ In recent years, it has become commonplace to measure dimensions, entropies, and Liapunov exponents for signals that come from the dynamics on a strange attractor. Obviously, these invariants are insufficient for a complete characterization of a system; very different attractors may have, for example, the same dimension. To gain further insight into the detailed nature of strange sets in general, and of dynamical systems in particular, it has recently been proposed⁴ that one should consider a new set of invariants which consist of a *range* of scaling indices existing on the strange attractor.

Let us illustrate this for the case of the dimension. Picking a point \mathbf{x}_i in the attractor, one asks for the probability that other points of the attractor fall within a small distance l of that point. Denoting that probability by $p_i(l)$, one defines $\alpha_i(l)$ by

$$p_i(l) = l^{\alpha_i(l)}.$$

In terms of the invariant measure $\rho(\mathbf{x})d\mathbf{x}$ of the system on the attractor,

$$p_i(l) = \int_{|y - x_i| < l} \rho(y) dy. \tag{1}$$

As l in Eq. (1) tends to zero, the $\alpha_i(l)$ tend for almost all \mathbf{x}_i (with respect to the measure ρ) to the same limit D , which is called the fractal dimension. However, for small (but nonzero) l , the $\alpha_i(l)$ take, on typical sets of i 's, a range of values between α_{\min} and α_{\max} . It has also been pointed out that if the system is partitioned into boxes of size l , then the number of times one finds the scaling index $\alpha_i(l)$ in the interval $[\alpha, \alpha + d\alpha]$ is proportional to $l^{-f(\alpha)}d\alpha$, so that $f(\alpha)$ can be interpreted as the fractal dimension of the set of boxes that have α as their scaling index. The function $f(\alpha)$ can be therefore used as an additional way to characterize structural properties of strange attractors. Successful applications of these ideas to experimental systems have been reported recently.⁵

In this paper, we show that a range of scaling indices also exists for the *dynamical properties* of chaotic systems.

Furthermore, we give a reinterpretation of these scaling indices and of the underlying scaling hypothesis in terms of the thermodynamic formalism in dynamical systems.⁶

To see this range of dynamical indices it is easiest to begin with generalized entropies.^{7,8} Partitioning phase space into boxes of size l and dividing the time axis into segments of size τ , we define the probability $P(i_1, \dots, i_n)$ to be the joint probability that an orbit visits box i_1 at time τ , is at time 2τ in box i_2 , etc. The generalized entropies K_q are defined by

$$K_q = \frac{1}{q-1} \lim_{l \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{\ln T} \ln \sum_{i_1, \dots, i_n} P(i_1, \dots, i_n)^q, \tag{2}$$

where $T = e^{-n\tau}$. It has been argued before that $\lim_{q \rightarrow 0} K_q$ and $\lim_{q \rightarrow 1} K_q$ are the topological and metric (Kolmogorov) entropies, respectively. K_2 has been used as a convenient lower bound for K_1 in experimental applications.

We first define $\Lambda(i_1, \dots, i_n)$ by

$$P(i_1, \dots, i_n) = T^{\Lambda(i_1, \dots, i_n)}.$$

Furthermore, we make the *scaling hypothesis* (to be justified below) that for l sufficiently small and fixed n , the number of times one finds $\Lambda(i_1, \dots, i_n)$ in the interval $[\Lambda, \Lambda + d\Lambda]$ is

$$\text{number of } \Lambda(i_1, \dots, i_n) \in [\Lambda, \Lambda + d\Lambda] = T^{-g(\Lambda)} d\Lambda. \tag{3}$$

Using this hypothesis and the definition of Λ in Eq. (2) one sees^{4,9} that in the limit as T goes to 0,

$$K_q = \frac{1}{q-1} [\Lambda q - g(\Lambda)],$$

and therefore the knowledge of K_q yields both Λ and $g(\Lambda)$ according to

$$\Lambda = \frac{\partial}{\partial q} \tau(q),$$

and

$$g(\Lambda) = \tau(q) - q \frac{\partial}{\partial q} \tau(q),$$

where $\tau(q)=(q-1)K_q$. These formulas are Legendre transforms of the same type as the formulas that relate the α scaling indices to the generalized dimensions D_q .⁴

To see the physical meaning of the numbers Λ we consider a one-dimensional dynamical system $x_{n+1}=f(x_n)$. (The consequences are immediately generalizable to higher-dimensional systems.) Starting at a box $\Delta(i_1)$ of size l (small enough) the system can go to $|1/f'(x)|$ boxes in the next step, where $x \in \Delta(i_1)$. Generalizing, we see that

$$P(i_1, \dots, i_n) = \frac{\rho(\Delta(i_1))}{|f'(x_1)f'(x_2) \cdots f'(x_{n-1})|},$$

where $x_j \in \Delta(i_j)$, and $\rho(\Delta(i_1))$ is the invariant measure of the first box. Evidently,

$$\ln P(i_1, \dots, i_n) = \ln \rho(\Delta(i_1)) - \sum_j \ln |f'(x_j)|,$$

or

$$\frac{1}{n} \ln P(i_1, \dots, i_n) \approx -\frac{1}{n} \sum_j \ln |f'(x_j)|,$$

for large n . Writing this as

$$P(i_1, \dots, i_n) \approx T^{(1/n) \sum_j \ln |f'(x_j)|}$$

and realizing that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum \ln |f'(x_i)| = \lambda,$$

where λ is the Liapunov exponent, we see that for large but finite n the range of possible $\lambda(i_1, \dots, i_n)$ is the range of possible fluctuations around the Liapunov exponent. Since we assumed that the number of times Λ is in $[\Lambda, \Lambda + d\Lambda]$ is $T^{-g(\Lambda)} d\Lambda$, the probability to see Λ is

$$T^{-[g(\Lambda) - g(\Lambda^*)]},$$

where $g(\Lambda^*)$ is the largest g (i.e., the most probable). This result can be derived by the same saddle-point approximation that has been used to get the Legendre transforms. Since $T = e^{-n\tau}$, we see that the probability to observe a deviation from the most probable Λ (namely Λ^*) decays exponentially with n . This result generalizes in the case of multidimensional systems to the notion of fluctuations around the sum of positive Liapunov exponents.

To make these ideas concrete, and to obtain an analytic example of a $g(\Lambda)$ curve, we consider the simple two-dimensional "skinny" baker's transformation. Given $0 < \xi_1, \xi_2 < \frac{1}{2}$, and an η in the interval $0 < \eta < \frac{1}{2}$, this baker's transformation is a map of the unit square to itself defined by

$$(x', y') = \begin{cases} (\xi_1 x, y/\eta) & \text{if } y < \eta \\ (\frac{1}{2} + \xi_2 x, (y - \eta)/(1 - \eta)) & \text{if } y > \eta \end{cases} \quad (4)$$

We notice that the geometric structure of the strange attractor of this map is independent of η . We shall show,

however, that systems with different η can be distinguished via their $g(\Lambda)$ function. One could calculate K_q analytically and perform the Legendre transform to obtain $g(\Lambda)$. However, one can find $g(\Lambda)$ directly, and this calculation will shed some additional light into the connection of the fluctuations with the thermodynamic formalism for dynamical systems.

Observe that the map in Eq. (4) is expanding in the y direction with rates η^{-1} and $(1-\eta)^{-1}$ for $y < \eta$ and $y > \eta$, respectively. These expanding rates are encountered with probabilities η and $(1-\eta)$, respectively. When we compute the Liapunov exponent, which is the logarithm of the product of these expansion rates (since the Jacobian is diagonal), we are led to ask for the probability of seeing, in n steps, a product

$$\eta^{-p}(1-\eta)^{-(n-p)}.$$

This probability is

$$\binom{n}{p} \eta^p (1-\eta)^{n-p}.$$

As $n \rightarrow \infty$, a use of Stirling's formula shows that the logarithm of this probability is

$$-x \ln(x/\eta) - (1-x) \ln[(1-x)/(1-\eta)] \equiv s(x),$$

where $x = p/n$. Reexpressing this result in terms of the original question of large deviations, we find that the probability of seeing an expansion rate $\Lambda = \ln[\eta^{-x}(1-\eta)^{-(1-x)}]$, when looking at a segment of the orbit which is of length n , is proportional to

$$e^{n[s(x) - s(\eta)]}.$$

Expressing x as a function of Λ , we obtain g by writing

$$e^{n[s(x) - s(\eta)]} = T^{-[g(\Lambda) - g(\Lambda^*)]},$$

where $\Lambda^* = \ln[\eta^{-\eta}(1-\eta)^{-(1-\eta)}]$. Figure 1 shows $g(\Lambda)$ for a few values of η , and it is clear that the fluctuations of the Liapunov exponent will be different for different η .

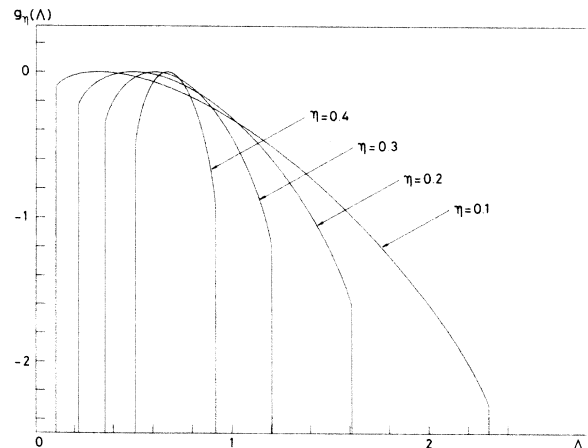


FIG. 1. $g_\eta(\Lambda)$ for different values of η . The range of Λ extends, for fixed η , from $-\ln(1-\eta)$ to $-\ln(\eta)$.

It remains to justify the basic scaling assumption of Eq. (3). The justification is based on the thermodynamic formalism for dynamical systems.¹⁰⁻¹³ The basic idea is to relate the probability of seeing a section of a trajectory (i_1, \dots, i_n) to observing a configuration of spins in an Ising model (where the Ising model is more or less exotic depending on the dynamical system and the partition of phase space). The baker transformation (4) is a case in point. There, the probability to see a trajectory (i_1, \dots, i_n) is simply $\eta^p(1-\eta)^{(n-p)}$ where p is the number of i_j 's corresponding to $y < \eta$. This is also the probability of seeing a configuration of spins in an Ising model without exchange interaction, but in a magnetic field. Obviously, n simply becomes the length of the system, and the thermodynamic limit is $n \rightarrow \infty$. The same can be done for any expanding system f (sometimes even for more general systems) using the ideas of Refs. 10-12. Consider a partition of phase space into disjoint boxes Δ_i , $i=1, \dots, k$. A k state Ising-like one-dimensional lattice model is defined by allowing only those nearest-neighbor configurations i, i' for which a transition is possible, i.e., $f(\Delta_i) \cap \Delta_{i'} \neq \emptyset$. For each allowed configuration i_0, i_1, \dots, i_n , there is a subset $\Delta(i_0, i_1, \dots, i_n)$ of phase space defined by

$$\Delta(i_0, i_1, \dots, i_n) = \{x: x \in \Delta_{i_0}, f(x) \in \Delta_{i_1}, \dots, f^n(x) \in \Delta_{i_n}\}.$$

The size of these subsets goes to zero exponentially as $n \rightarrow \infty$, and we denote $\Delta(i)$ the limit point defined by $\Delta(i) = \Delta(i_0, i_1, \dots, i_n, \dots)$. Therefore, as $n \rightarrow \infty$, we have the important identity

$$|\Delta(i_0, i_1, \dots, i_n)| = |\Delta(i_1, \dots, i_n)| |f'(\Delta(i))|,$$

where $|\Delta|$ is the Lebesgue measure of Δ . Define now

$$h(i) = \ln |f'(\Delta(i))|.$$

The thermodynamic formalism comes about by asking for the conditional probability for x being in Δ_{i_0} , given that $f^k(x) \in \Delta_{i_k}$ for $k=1, 2, \dots$. It is

$$\frac{e^{-h(i_0, i_1, \dots)}}{\sum_{i'_0 \in \{1, \dots, k\}} e^{-h(i'_0, i_1, \dots)}}.$$

This expression should be compared to the identity satisfied

by the invariant measure ρ ,

$$\rho(x) = \sum_{y: f(y)=x} \frac{\rho(y)}{|f'(y)|}.$$

The function h is a sort of total Hamiltonian and the conditional probability looks like an expectation in the canonical ensemble. This Gibbs state for this system turns out to be an invariant measure for f . We now view h as the energy contribution from the lattice site 0, and we want to define a potential function Φ from which this energy function derives. The customary definition for the m -site (m -particle) interaction is

$$\Phi_m(i_0, i_1, \dots, i_{m-1}) = H_m(i_0, i_1, \dots, i_{m-1}) - H_{m-1}(i_0, i_1, \dots, i_{m-2}),$$

where

$$H_s(i_0, i_1, \dots, i_{s-1}) = \inf_{i'_s, i'_{s+1}, \dots} h(i_0, i_1, \dots, i_{s-1}, i'_s, \dots),$$

$$H_{-1} = 0.$$

The potential Φ is naturally extended to all subsets of the lattice by translation invariance. From this we easily obtain the total energy for a configuration i on a finite sublattice Ω , which we denote by $U_\Omega(i)$. If we also define by W the interaction between this sublattice and the remainder of the lattice, then we have the identification

$$U_{\{0, \dots, s\}}(i_0, i_1, \dots, i_s) + W(i_0, i_1, \dots, i_s | i_{s+1}, \dots) = h(i_0, i_1, \dots) + \dots + h(i_s, i_{s+1}, \dots).$$

Thus the identification between the dynamical system and the Gibbs ensemble is complete.

It is now very important to note that Φ_m is bounded by $\max \ln |f'(x)/f'(y)|$, with x, y varying in $\Delta(i_0, i_1, \dots, i_{m-1})$. From the expansivity assumption and a certain regularity of f it follows that Φ_m decays exponentially with m . The invariant measure ρ defined in this way is thus seen to be equivalent to the Gibbs measure of a one-dimensional Ising-like mode with *short range forces*. It is well-known that in such systems, the *probability to see a finite deviation of any observable in volume m is always decaying exponentially with m* . This is the underlying mechanism for the applicability of the scaling hypothesis in dynamical systems, as described above.

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