Bunching and antibunching properties of various coherent states of the radiation field

M. H. Mahran*
Department of Mathematics and Computer Science, Faculty of Science, Suez Canal University, Ismailia, Egypt

M. Venkata Satyanarayana
The Institute of Mathematical Sciences, Madras 600 113, Tamilnadu, India
(Received 25 October 1985; revised manuscript received 8 January 1986)

In contrast to coherent states $|\alpha\rangle$ which have zero Hanbury Brown and Twiss Effect [i.e., $g^{(2)}(0)=1$], it is shown that generalized coherent states $|n,\alpha\rangle$ are antibunched for $|\alpha|^2<\frac{1}{2}$. The range of values for α (real) in terms of the squeezing parameter r (real) for the squeezed coherent state $|\alpha,r\rangle$ in order to exhibit bunching and antibunching are obtained. The conditions and the exact range of values for r and α for a given n for generalized squeezed coherent states $|n,\alpha,r\rangle$ to exhibit bunching and antibunching are also obtained.

The quantity which determines bunching and antibunching of a state[1] of the radiation field is decided by the second-order correlation function $g^{(2)}(0)$ given by

$$g^{(2)}(0)=\frac{\langle a^\dagger a^\dagger aa\rangle}{\langle a^\dagger a\rangle^2} \quad (1)$$

and it could be written as

$$g^{(2)}(0)=\frac{\langle a^\dagger a a^\dagger a\rangle-\langle a^\dagger a\rangle}{\langle a^\dagger a\rangle^2}\ , \quad (2)$$

where $a^\dagger$ and a are the photon creation and destruction operators.

A light field (or the Fock-space state describing the light field) is said to be antibunched if $g^{(2)}(0)<1$, which means that the probability of detecting a coincident pair of photons is less than that from a coherent field described by a coherent state which has Poisson distribution for photon counts. Antibunching is considered to be "a clear demonstration of the quantum nature of light which is not explained by classical theory," since it means "anticorrelation" in the photon detection. The method of generating antibunched states has been described by Stoler[2] and the subject has been attracting a lot of theoretical and experimental activity[3] (see the review of Paul[4]).

There are many states in the Fock space which are antibunched. For example, the number state $|n\rangle$ is one such since $g^{(2)}(0)=1-1/n$, which is a reflection of the fact that the state $|n\rangle$ contains a definite number of photons. The binomial states of the radiation field recently introduced by Stoler *et al.*[5] are antibunched for certain parameter ranges. Also, Simon and Satyanarayana[6] very recently introduced the logarithmic states of the radiation field defined as

$$|q\rangle=c\,|0\rangle+\beta\sum_{n=1}^{\infty}\left[\frac{q^n}{n}\right]^{1/2}|n\rangle \quad \text{for } -1\le q<1\ , \quad (3)$$

where

$$\beta=\left[\frac{-(1-|c|^2)}{\ln(1-q)}\right]^{1/2}, \quad (4)$$

and c is the point inside a unit circle. These states are antibunched for certain ranges of q and c. But the phase state $|\phi\rangle$ (Ref. 7) is bunched. For the coherent state $|\alpha\rangle$, $g^{(2)}(0)=1$ which means that it has null Hanbury Brown and Twiss correlation.

In this paper we discuss the bunching and antibunching properties of various coherent states, since those are the states which are useful for the description of the optical fields. First we consider the generalized coherent state[8] $|n,\alpha\rangle$ as defined as

$$|n,\alpha\rangle\equiv\exp(\alpha a^\dagger-\alpha^* a)\,|n\rangle\ , \quad (5)$$

where $|n\rangle$ is the nth state of the oscillator. These states have been studied in detail.[9] Its $g^{(2)}(0)$ is given by

$$g^{(2)}(0)=1+\frac{n(2|\alpha|^2-1)}{(|\alpha|^2+n)^2}\ , \quad (6)$$

which means that the states $|n,\alpha\rangle$ are bunched only for $2|\alpha|^2-1\rangle 0$, and for $2|\alpha|^2<1$ the states clearly have antibunching. So unlike the coherent states $|\alpha\rangle$ for which $g^{(2)}(0)=1$, the generalized coherent states (GCS) $|n,\alpha\rangle$ has sub-Poissonian statistics for $2|\alpha|^2<1$. This means if α were to lie within the unit phase cell around the origin then the corresponding states are antibunched. Here we have an interesting comment regarding the counting statistics of $|n,\alpha\rangle$. "The appropriate generalizations of the Poisson distribution" as stated by Roy and Virendra Singh also contain sub-Poissonian statistics for $2|\alpha|^2<1$.

Next, we consider the squeezed coherent states (SCS) defined as[1,10–12]

$$|\alpha,Z\rangle\equiv D(\alpha)S(Z)\,|0\rangle\ , \quad (7)$$

where $D(\alpha)$ is the displacement operator given by

$$D(\alpha)\equiv\exp(\alpha a^\dagger-\alpha^* a)\ , \quad (8)$$

and

$$S(Z)\equiv\exp\left[\frac{Z}{2}a^\dagger a^\dagger-\frac{Z^*}{2}aa\right] \quad (9)$$

is known as the squeeze operator. Also,

$$Sa S^{\dagger}=a \cosh r+e^{i\theta}a^{\dagger}\sinh r\ ,$$
$$Sa^{\dagger}S^{\dagger}=e^{-i\theta}a \sinh r+a^{\dagger}\cosh r\ , \tag{10}$$

where $Z=re^{i\theta}$. SCS were also introduced by Rowe as the "breathing modes" of the radiation field in the context of two photon processes (Ref. 13).

The $g^{(2)}(0)$ of SCS is given by

$$g^{(2)}(0)=\frac{[(\alpha^2-e^{-i\theta}\sinh r \cosh r)(\alpha^{*2}-e^{i\theta}\sinh r \cosh r)+4\,|\alpha|^2\sinh^2 r+2\sinh^4 r]}{(|\alpha|^2+\sinh^2 r)^2}\ . \tag{11}$$

For $\theta=0$ and α (real),

$$g^{(2)}(0)=1+\frac{2\sinh^4 r+\sinh^2 r(2\alpha^2+1)-\alpha^2\sinh 2r}{(\alpha^2+\sinh^2 r)^2}\ . \tag{12}$$

The above form of $g^{(2)}(0)$ could be reduced to Eq. (2.30) of Walls and Milburn.[1] The state $|\alpha,r\rangle$ is bunched only if the numerator of the second term of Eq. (12), which could be written as

$$f(\alpha)=\alpha^2(2\sinh^2 r-\sinh 2r)+(2\sinh^4 r+\sinh^2 r)\ , \tag{13}$$

is positive. $f(\alpha)$ is a quadratic expression and its analysis is simple and given below. The roots of $f(\alpha)$ are

$$\alpha_1,\alpha_2=\pm\left[\frac{\sinh r(1+2\sinh^2 r)}{2(\cosh r-\sinh r)}\right]^{1/2}\ . \tag{14}$$

Case 1: $r>0$. The roots are real and distinct and the coefficient of α^2, namely $2\sinh r(\sinh r-\cosh r)$, is negative. Therefore for a given value of the squeezing parameter r, in order to have an antibunched state, α should be chosen such that $\alpha<\alpha_1$ or $\alpha>\alpha_2$. For $\alpha_1<\alpha<\alpha_2$, we have a bunched state.

Case 2: $r<0$. The roots α_1 and α_2 are purely imaginary quantities and the coefficient of α^2 namely $2\sinh r(\sinh r-\cosh r)$ is positive which means $f(\alpha)$ is positive, and therefore for all values of α we have only bunched states.

The discussion in case 1 and case 2 given above are to be compared with Ref. 1 and Eq. (6.7) of Yuen.[11] Our results are exact for α (real) and r (real) and fix the exact range of values for α in terms of r whereas the discussion of Walls[1] is based on the limit $|\alpha|^2 \gg \sinh^2 r$.

Now for $\alpha=0$, i.e., for the squeezed vacuum state $|0,Z\rangle$,

$$g^{(2)}(0)=2+\coth^2 r\ , \tag{15}$$

which is a rewritten form of Eq. (17) of Walls,[1] which is to be compared with $g^{(2)}(0)=2$ for a chaotic light beam in an optical cavity. This means that the cavity filling due to squeezing is more bunched than chaotic light and the counting statistics are similar.

All the above discussed results could be obtained as various special cases of the $g^{(2)}(0)$ of the generalized squeezed coherent states[9] (GSCS) introduced by one of the authors, earlier defined as

$$|n,Z,\alpha\rangle \equiv D(\alpha)S(Z)\,|n\rangle\ , \tag{16}$$

and its $g^{(2)}(0)$ is given by

$$g^{(2)}(0)=\{[\alpha^2-(2n+1)\sinh r \cosh r e^{-i\theta}][\alpha^{*2}-(2n+1)\sinh r \cosh r e^{i\theta}]+4\,|\alpha|^2\sinh^2 r(n+1)$$
$$+\sinh^4 r(n+1)(n+2)+4\,|\alpha|^2 n\cosh^2 r+\cosh^4 r\,n(n-1)\}/(|\alpha|^2+\sinh^2 r+n\cosh^2 r)^2\ . \tag{17}$$

Now, for case 1, $Z=0$, Eq. (17) becomes Eq. (6); for case 2, $n=0$, Eq. (17) becomes Eq. (11); for case 3, $n=0$ and $\alpha=0$, Eq. (17) becomes Eq. (15); for case 4, $n=0$; $\theta=0$, and α (real), Eq. (17) becomes Eq. (12).

Now we proceed to get the conditions for bunching and antibunching of GSCS

$$g^{(2)}(0)=1+[\sinh^4 r(n^2+3n+1)+2\alpha^2\sinh^2 r+(4n^2+2n+1)\sinh^2 r\cosh^2 r+2n\alpha^2\cosh^2 r$$
$$-\cosh^4 r-2(2n+1)\alpha^2\sinh r\cosh r]/(\alpha^2+\sinh^2 r+n\cosh^2 r)^2\ . \tag{18}$$

TABLE I. Range of values for r and α.

r		α	
r	$x=\sinh^2 r$	Bunching	Antibunching
$-\infty<r<r_2$	$x>x_2$	$-\infty<\alpha<\infty$	No
	$0<x<x_2$	$\alpha<\alpha_1$ and $\alpha>\alpha_2$	No
		No	$\alpha_1<\alpha<\alpha_2$
$r>r_2$	$x>x_2$	No	$\alpha<\alpha_1$ and $\alpha>\alpha_2$
		$\alpha_1<\alpha<\alpha_2$	

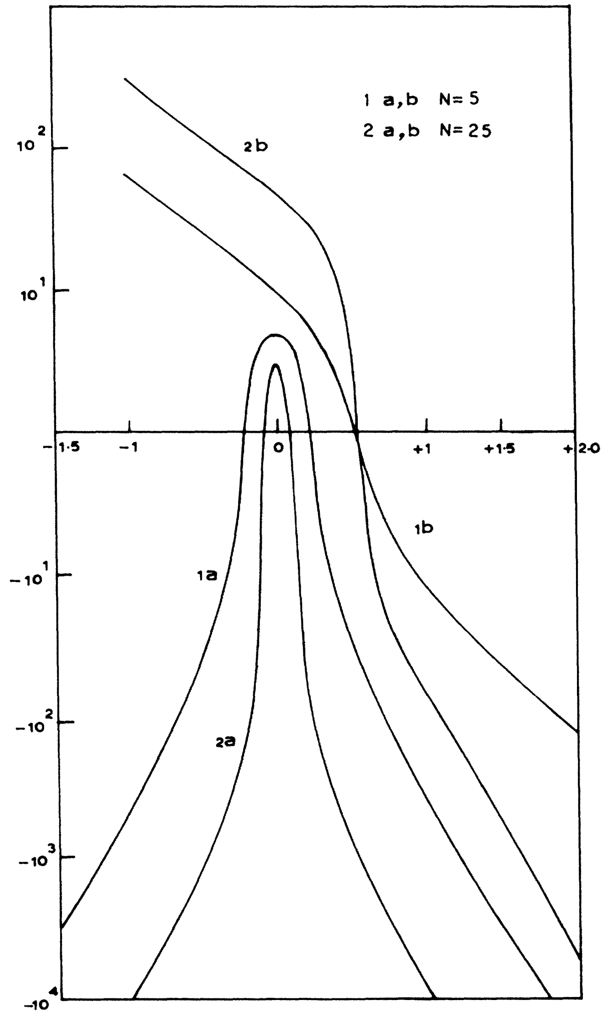

FIG. 1. Behavior of $f(r)$ (1a and 2a) and $G(r)$ (1b and 2b) for $n=5$ and 25, respectively.

The numerator of the second term of $g^{(2)}(0)$ in the above expression could be written as (for α and r real)

$$F(\alpha)=\alpha^2[2(n+1)\sinh^2 r+2n-(2n+1)\sinh 2r]+\sinh^4 r(5n^2+4n+2)+(4n^2+1)\sinh^2 r-n \quad (19)$$

$$=\alpha^2 f(r)-G(r)\ . \quad (20)$$

The roots of $F(\alpha)$ are given by

$$\alpha_1,\alpha_2=\pm\left[\frac{G(r)}{f(r)}\right]^{1/2}\ . \quad (21)$$

Case 1: $r<0$. In this case $f(r)>0$. Taking $x=\sinh^2 r$, $G(r)$ could be rewritten as

$$g(x)=n-x(1+4n^2)-x^2(5n^2+4n+2)\ . \quad (22)$$

Since $x>0$, the positive root of $g(x)$ is

$$x_2=\frac{[(1+4n^2)^2+4n(5n^2+4n+2)]^{1/2}-(1+4n^2)}{2(5n^2+4n+2)}\ . \quad (23)$$

(a) For $r<0$, such that $x>x_2$, $g(x)<0$ and α_1 and α_2 are purely imaginary. Since $f(r)>0$, for all α, $F(\alpha)>0$, i.e., the states are bunched.

(b) For $r<0$, such that $0<x<x_2$, the sign of $g(x)$ is opposite to that of the coefficient of x^2, i.e., positive. Therefore, α_1 and α_2 are real and distinct. For $\alpha<\alpha_1$ and $\alpha>\alpha_2$, $F(\alpha)>0$, i.e., the states are bunched and for $\alpha_1<\alpha<\alpha_2$, $F(\alpha)<0$, i.e., the states are antibunched.

Case 2: $r>0$. The positive root of $f(r)$ is given by

$$r_2=\tfrac{1}{2}\ln\left[\frac{(n-1)+[(n-1)^2+n(3n+2)]^{1/2}}{n}\right]\ . \quad (24)$$

A similar analysis can be done as above in case 1 and the range of values for both the cases are given in Table I. To know whether a given $|n,z,\alpha\rangle$ is bunched or antibunched, one should just calculate x_2 [Eq. (23)] and r_2 [Eq. (24)] and then look at Table I.

To have a feeling for $F(\alpha)$ [Eq. (20)] we have Fig. 1 which gives the behavior of the functions $f(r)$ (1a and 2a) and $G(r)$ (1b and 2b) for $n=5$ and $n=25$, respectively. At a chosen r, the ratio of $G(r)$ to $f(r)$ determines α_1 and α_2. We note that $f(r)$ is a monotonically decreasing function and the root of $f(r)$, namely r_2, tends to $\frac{1}{2}\ln 3$ as n tends to infinity. The positive root of $G(r)$ given by Eq. (23) could also be obtained from the positive zeroes of $G(r)$ from Fig. 1.

ACKNOWLEDGMENTS

One of the authors (M.V.S.) is grateful to Professor T. S. Santhanam, Dr. R. Balasubramanian, and Professor E. C. G. Sudarshan for useful discussions. Both the authors are thankful to Professor Abdus Salam, International Centre for Theoretical Physics (ICTP), International Atomic Energy Agency (IAEA), United Nations Educational, Social and Cultural Organization (UNESCO), and the organizers of the Winter College on LAMP (Lasers, Atomic, and Molecular Physics) for the hospitality at the ICTP, Trieste, where this work was started. One of the authors (M.V.S.) is also thankful to the Council for Scientific and Industrial Research (CSIR), India for partial financial support.

---

*Present address: Centre for Mathematics and Science, Department of Mathematics, TAIF, P.O. Box 1070, Saudi Arabia.

[1]D. F. Walls, Nature **306**, 141 (1983); **280**, 451 (1979); D. F. Walls and G. J. Milburn, in *Quantum Optics, Experimental Gravitation and Measurement Theory*, edited by P. Meystre and M. O. Scully (Plenum, New York, 1981).
[2]D. Stoler, Phys. Rev. Lett. 33, 1397 (1974).
[3]R. Loudon, Phys. Bull. **27**, 21 (1976); Rep. Prog. Phys. **43**, 58 (1980); Y. Kano, J. Phys. Soc. Jpn. **50**, 163 (1981); L. Mandel, Opt. Commun. **42**, 437 (1982); Phys. Rev. Lett. **49**, 136

(1982); R. Short and L. Mandel, *ibid.* **51**, 384 (1983); Surendra Singh, Opt. Commun. **44**, 254 (1983); Z. Ficek, R. Tanas, and S. Kielich, Phys. Rev. A **29**, 2004 (1984); R. Loudon and T. J. Shepherd, Opt. Acta **31**, 1243 (1984).

[4]H. Paul, Rev. Mod. Phys. **54**, 1061 (1982).

[5]D. Stoler, B. E. A. Saleh, and M. Teich, Opt. Acta **32**, 345 (1985).

[6]R. Simon and M. V. Satyanarayana (unpublished).

[7]R. Loudon, *The Quantum Theory of Light* (Clarendon, Oxford, 1979).

[8]M. Boiteux and A. Levelut, J. Phys. A **6**, 589 (1973); S. M. Royand Virendra Singh, Phys. Rev. D **25**, 3413 (1982).

[9]M. Venkata Satyanarayana, Phys. Rev. D **32**, 400 (1985). A detailed history of $|n,\alpha\rangle$ is given in this paper. Equation (9) of this paper is the same as Eq. (2.26) of C. M. Caves, K. S. Thorne, R. W. P. Drever, V. D. Sandberg, and M. Zimmerman, Rev. Mod. Phys. **52**, 341 (1980). See also I. Bialynicki-Birula and Z. Bialynicka-Birula, Phys. Rev. A **8**, 3146 (1973). Also later we learned that the GCS are used in the "generator coordinate method" in nuclear physics. Equation (13) of this paper is explicitly given in the book P. Ring and P. Schuck, *The Nuclear Many Body Problem* (Springer-Verlag, New York, 1980).

[10]D. Stoler, Phys. Rev. D **1**, 3217 (1970); **4**, 1925 (1971).

[11]H. P. Yuen, Phys. Rev. A **13**, 2225 (1976).

[12]M. M. Nieto, Los Alamos Report No. LA-UR-84-2773, 1984 (unpublished).

[13]D. J. Rowe, Can. J. Phys. **56**, 442 (1978). This paper has not received the attention which it deserves.