PHYSICAL REVIEW A VOLUME 34, NUMBER 6 DECEMBER 1986

Brochard-Leger wall in liquid crystals

Wang Xin-Yi
*International Center for Theoretical Physics, P.O. Box 536, I-3100 Trieste, Italy and Physics Department, Suzhou University, Suzhou, China**

(Received 27 June 1986)

Theoretical problems related to the Brochard-Leger wall in liquid crystals have been further explained. It has been shown that there exists a critical region in liquid crystals beyond which the Brochard-Leger wall does not exist. The relaxation behavior of the wall has been discussed, and the relaxation time has been calculated. The speed of the Brochard-Leger wall has been investigated for the case for which the tilt angle of the external field is equal to a critical angle.

I. INTRODUCTION

Liquid crystals are delicate nonlinear systems. When submitted to an external field, they exhibit many novel patterns and interesting phenomena, such as walls,[1] the Williams domain,[2] transient periodic structures,[3] optical bistability, and chaos.[4] The study of these phenomena and structures are very attractive topics in the physics of liquid crystals and nonequilibrium statistics.

The Freedericsz transition occurs when a uniformly aligned nematic liquid-crystal film is subjected to a magnetic field $H > H_c$. Above the threshold field H_c, two equivalent tilted configurations are separated by a wall, which was called the "Brochard-Leger wall" in a previous work.[5] It has been pointed out that the Brochard-Leger wall in liquid crystals displays an important nonlinear structure.[5] I have shown that the Brochard-Leger wall provides an excellent example of nonequilibrium phase transition, which can be described exactly by a solitary wave. Furthermore, the Brochard-Leger wall shows an impressive analogy with nerve propagation in neurobiology. It is clear that there exist many interesting problems in the Brochard-Leger wall. The purpose of this work is to further explain the properties of the B-L wall and some of the related problems. Three points will be briefly addressed: (i) There exists a critical region in a nematic slab, inside which the transition of the director field is continuous, so the Brochard-Leger wall is formed; outside which the transition of the director field becomes discontinuous, so the Brochard-Leger wall does not exist; (ii) the relaxation behavior of the B-L wall is discussed, the relaxation time τ_0 of the B-L wall is given by $\tau_0 = 4\gamma_1 d^2 \times (a-1)^{1/2}/(3\pi^2 a^2 k\phi)$; (iii) in the critical case, the speed C of the Brochard-Leger wall becomes $C = k\chi_a H/(3\gamma_1) \times (dH/\pi\sqrt{\chi_a/k} - 1)^{1/2}$.

II. CRITICAL LENGTH OF THE B-L WALL

The relevant equation (see Ref. 5) for describing the B-L wall in the liquid crystals is as follows:

$$\frac{\partial\tilde{\theta}}{\partial\tau} - \frac{\partial^2\tilde{\theta}}{\partial x^2} = F(\tilde{\theta}) , \tag{1}$$

where

$$F(\tilde{\theta}) \equiv [\sin(2\tilde{\theta}\theta_M) - \tilde{\theta}\sin(2\theta_M) - \varepsilon]/\theta_M^3 . \tag{2}$$

It should be pointed out that when we examine the process of establishing Eq. (1) in detail, we see that the basic physical consideration for establishing Eq. (1) is valid not only for the middle thin layer of the nematic slab, but also for each thin layer of the considered slab. Here the important condition is that every considered layer of the nematic slab must be kept very thin. Bearing this in mind, we see that the form of Eq. (1) is appropriate to each thin layer of the nematic slab in our problem. In the general case, we should of course substitute $\theta_0(x_3)$ for θ_M in Eq. (1). So the general equation now reads

$$\frac{\partial\tilde{\theta}}{\partial\tau} - \frac{\partial\tilde{\theta}}{\partial x^2} = \{\sin[2\tilde{\theta}\theta_0(x_3)] - \tilde{\theta}\sin[2\theta_0(x_3)] - \varepsilon\}/\theta_0^3(x_3) , \tag{1$'$}$$

where $\varepsilon = \phi\sin(2\theta_0)/\theta_0$ and ϕ is the tilt angle of the external field. For the definitions of other quantities in Eq. (1′), the reader is referred to Ref. 5. Adopting the discussions parallel to that of Ref. 5, we can obtain the following similar inequality:

$$\phi \le (8\sqrt{3}/27)\theta_0^4(x_3)/\sin[2\theta_0(x_3)] \equiv \phi_c . \tag{3}$$

The inequality (3) restricts the behavior of the B-L wall and shows the existence of the critical angle ϕ_c. The definite restriction (3) is of significance in physics. One of the direct consequences of this inequality is that there is a critical length d_c in the B-L wall, for which the B-L wall does not exist in the region $d_c < |x_3| < d$ where d is the thickness of the nematic slab.

The function $\theta_0(x_3)$ is the solution of the equation $k\partial^2\theta/\partial^2 x_3 + \frac{1}{2}\chi_a H^2\sin(2\theta) = 0$ with the boundary condition $\theta_0(-d/2) = \theta_0(d/2) = 0$. It is known[6] that $\theta_0(x_3)$ can be expressed as follows:

$$a \equiv \frac{H}{H_c} = \frac{2}{\pi}\int_0^{\theta_M}\left[\frac{1}{\sin^2\theta_M - \sin^2\theta_0}\right]^{1/2} d\theta_0 = 1 + \tfrac{1}{2}\theta_M^2 + \cdots , \tag{4}$$

$$\begin{aligned}\frac{x_3}{d} + \frac{1}{2} &= \frac{1}{\pi a}\int_0^{\theta_0}\left[\frac{1}{\sin^2\theta_M - \sin^2\theta_0}\right]^{1/2} d\theta_0 \\ &= \frac{1}{\pi}\sin^{-1}\frac{\theta_0}{\theta_M} \\ &\quad - \theta_0(\theta_M^2 - \theta_0^2)^{1/2}\frac{1+3+\cdots}{12\pi(1+\frac{1}{2}\theta_M^2+\cdots)} .\end{aligned} \tag{5}$$

In the case $H \gtrsim H_c$, θ_M is small. According to the above expressions of (4) and (5), $\theta_0(x_3)$ can be expressed approximately by

$$\theta_0(x_3)=2(a-1)^{1/2}\cos\left[\frac{\pi}{d}x_3\right] . \quad (6)$$

Using (6), we can obtain the following inequality from (3):

$$|x_3|\leq\frac{d}{\pi}\cos^{-1}[0.787(a-1)^{-1/2}\phi]=d_c . \quad (7)$$

The inequality (7) explicitly demonstrates the existence of the critical length d_c for the B-L wall. Inequality (7) shows that $d_0\propto d$. d_c increases with decreasing the tilt angle ϕ and increasing the external field a. The expression (7) also shows that $d_c=d$, if and only if $\phi=0$. The meaning of the results obtained above is very clear.

Because the width ξ and length d_c of the B-L wall are expressed explicitly, the elastic energy F of the wall can be calculated according to the following integration:

$$F=\tfrac{1}{2}k\int_{-\xi/2}^{\xi/2}\int_{-d_c/2}^{d_c/2}\left[\left[\frac{\partial\theta(x_1,x_3,0)}{\partial x_1}\right]^2+\left[\frac{\partial\theta(x_1,x_3,0)}{\partial x_3}\right]^2\right]dx_1dx_3 , \quad (8)$$

where

$$\theta(x_1,x_3,0)=\theta_0(x_3)\theta_1(x_1,x_3,0)=\theta_0(x_3)\{\sin(\beta+\pi/3)[1+\tanh(\xi^{-1}x_1)]+2\sqrt{3}/3\cos(\beta+2\pi/3)+\phi/\theta_0\} ,$$

$$\xi=\frac{\sqrt{3}d}{2\pi a\sqrt{a-1}} .$$

From the phase-transition point of view the critical length d_c is an interesting quantity. For $|x_3|<d_c$ the transition of the director field is continuous. For $|x_3|>d_c$ the transition of the director field becomes discontinuous. d_c discriminates between the two different transitions of the director field in the nematic slab. Figure 1 shows the dependence of the critical length d_c on the reduced magnetic field $a\equiv H/H_c$. Here we take $\phi=0.05$ ($\sim 3°$) and $\beta\equiv d_c/d$. Note that the condition (7) is of universality, as it is independent of the concrete parameters of liquid crystals.

In a previous work[7] the theoretical aspect of the electrohydrodynamic instability of the Williams domain under inclined external field was investigated. One of the conclusions made was that there also exist a critical angle $\theta_c(\omega)$ (ω is the frequency of the field) and critical length d_c' in the Williams domain (see Ref. 7). At $\theta>\theta_c(\omega)$ the Williams domain does not exist. Note that this conclusion is similar to that of the present work. In the case of the inclined external field, the symmetries of both systems are broken. There will appear some new physical phenomena connected with the symmetry breaking of the Brochard-Leger wall and the Williams domain. Here we have seen that in spite of the different physical mechanism under the Williams domain and the Brochard-Leger wall, their responses to the inclined fields show some similar features.

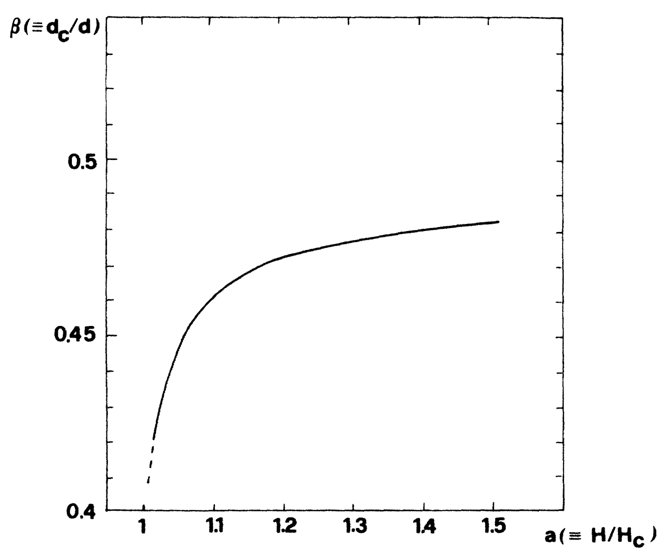

FIG. 1. The dependence of the reduced critical length $\beta\equiv d_c/d$ of the Brochard-Leger wall on the deduced magnetic field $a\equiv H/H_c$. $\phi=0.05$ ($\sim 3°$). Note that when $H\rightarrow H_c$, the inequality (3) is destroyed.

III. RELAXATION BEHAVIOR OF B-L WALL

The dynamical behavior of the director field was investigated by Pieranski, Brochard, and Guyon,[8] when the magnetic field is switched on and off suddenly. Their results give an ideal of the order of magntiude of the relaxation time for many of liquid-crystal devices. The relaxation behavior of the Brochard-Leger wall is also of importance. By using the results obtained in Ref. 5 the investigation of this problem becomes quite simple and clear.

If $\phi\neq 0$, the directors in the Brochard-Leger wall go down continuously from one stable side to the other. In the latter stable side the directors make the larger tilt angle have lower energy. It is very clear that the transient regime of the director field in our problem is the width ξ of the B-L wall (i.e., the correlation length for the phase transition). If the B-L wall passes the characteristic distance ξ in the time τ_0, so we can naturally define the time τ_0 as the relaxation time of the B-L wall. That is,

$$\tau_0\equiv\frac{\xi}{C}=\frac{\sqrt{3}\gamma_1}{(\frac{1}{2}-\cos\beta)\chi_a H^2\theta_M^2} . \quad (9)$$

In the case $H\gtrsim H_0$, we can express the relaxation time τ_0 by the following formula:

$$\tau_0=\frac{4\gamma_1 d^2(a-1)^{1/2}}{3\pi^2a^2k\phi} . \quad (10)$$

The above formula can also be used to determine the viscosity γ_1 in an experiment. Equation (10) shows that the relaxation time $\tau \propto \phi^{-1}$, $\tau \propto \gamma_1$. Figure 2 gives the dependence of the relaxation time τ_0 versus the reduced magnetic field a. Here we take $d = 10^2$ μm $k = 6\times10^{-7}$ dyn, $\phi = 0.05$ ($\sim 3°$) and $\gamma_1 = 0.76p$. Typically τ_0 for a MBBA (*p*-methoxybenzylidene-*p*′-*n*-butylaniline) film of $d = 10^2$ μm is about 10^2 s. Note that when $a = \frac{4}{3}$, τ_0 gets the maximum value.

When the external field is switched off suddenly from $H > H_c$ to zero, the equation describing the relaxation behavior of the Brochard-Leger wall will become the following linear diffusion equation:

$$\gamma_1 \frac{\partial \theta}{\partial t} - k \frac{\partial^2 \theta}{\partial x_1^2} = 0 \ . \qquad (11)$$

The initial value condition is

$$\theta_1(x_1,0,t)\,|_{t=0} = \sin(\beta+\pi/3)[1+\tanh(\xi^{-1}x)] + 2\sqrt{3}/3\cos(\beta+2\pi/3) + \phi/\theta_M \ . \qquad (12)$$

In this situation the B-L wall will relax from the pattern (12) to an equilibrium stationary state. Such an evolutionary process can be described by the following solution of Eq. (12):

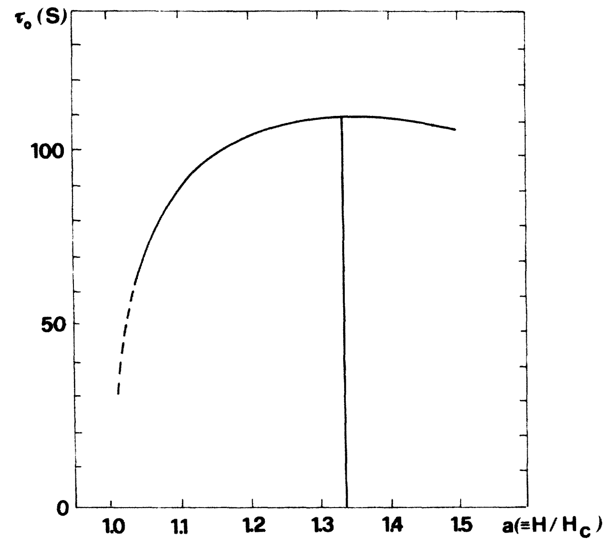

FIG. 2. The dependence of the relaxation time τ_0 vs the reduced magnetic field $a \equiv H/H_c$. $\phi = 0.05$ ($\sim 3°$). Note that when $H \rightarrow H_c$, the inequality (3) is destroyed.

$$\theta(x_1,t) = \frac{1}{2}\left(\frac{\gamma_1}{k\pi t}\right)^{1/2} \int_{-\infty}^{\infty} \{\sin(\beta+\pi/3)[1+\tanh(\xi^{-1}y)] + 2\sqrt{3}\cos(\beta+2\pi/3) + \phi/\theta_M\} \exp[-\gamma_1(y-x_1)^2/(4kt)]dy. \qquad (13)$$

A proper description of the relaxation behavior of the Brochard-Leger wall should consider the fluctuation phenomena in detail. In this aspect we have noted a series of articles by Buttick and Landauer.[9] Buttick and Landauer have studied the dynamical behavior in linear arrays of overdamped multistable systems coupled to a thermal reservior. Some of the general discussions by Buttick and Landauer are relevant to the behavior of the Brochard-Leger wall. In another aspect, the model for the Brochard-Leger wall subjected to fluctuation provides an excellent example for the Buttick-Landauer general system. Recently Sagues and San Miguel[10] have studied the dynamical behavior of the fluctuation in the Freederickz transition and obtained some interesting results. It is easy to see that the study of the B-L wall is closely related to that of Sagues and San Miguel, and it is also concerned with some of the interesting problems in nonequilibrium statistical mechanics.

IV. ANALOGY WITH NERVE PROPAGATION

In my previous works[5,11] I have outlined the similarity between the motion of the B-L wall and nerve propagation. Because nerve fiber shows the structure of liquid crystals, such a similarity is obviously an impressive one. Here I will briefly mention other related facts and conclusions.

The Huxley theory gives good descriptions for the behavior of nerve propagation. A large number of studies have demonstrated that (i) the Huxley theory shows the threshold phenomena in nerve propagation, (ii) the Huxley theory shows solitary wave solution, (iii) the Huxley theory shows multiple solitary wave structure, and (iv) the Huxley theory shows periodic wave structure. These conclusions agree with the experimental facts in neurobiology.

It is of importance in liquid crystals that all of these typical phenomena in neurobiology also appear in a series of experiments in liquid crystals. Leger's experiment[1] has shown the solitary wave and multiple solitary wave structures and threshold phenomena. The recent experiments performed by Guyon, Meyer, Salans,[3] and Sun and Kleman[12] and Lonberg, Fraden, Hurd, and Meyer[13] show that there also exist interesting transient periodic structures which are similar to the Williams domain in liquid crystals. When one discusses the theoretical aspect of these phenomena, the coupling between the rotation of directors and fluid velocity should be considered in detail. In this case, we can analyze Eqs. (8) in Ref. 11. By using singular perturbation theory[14] in phase space, one can reveal the various propagation phenomena of the solutions, such as periodic structure, coupled solitary waves, and multiple solitary waves. These facts show the impressive similarity once again.

Finally, I want to point out the uncertainty of the dependence of the velocity C of the B-L wall on the external field H. Such an uncertainty comes from the inequality (3). As the velocity C is sensitively dependent upon the tilt angle ϕ and ϕ is restricted by the inequality (3), so relationship between the velocity C and field H becomes variable. Note that because of the inequality (3), the divergence velocity C, as H goes to H_c, is a pseudophenomenon.[1] In the critical situation, the inequality (3) becomes the equality $\phi = (8\sqrt{3}/27)\theta_M^4/\sin(2\theta_M)$. In this case, the velocity formula for the B-L wall is transformed to the following form

$$C = \frac{k\chi_a H}{3\gamma_1}\left[d\frac{H}{\pi}\left(\frac{\chi_a}{k}\right)^{1/2} - 1\right]^{1/2} . \qquad (14)$$

The above formula shows the relationship between the velocity of the wall and the thickness d of the nematic slab in the critical situation. Note that the experimental results in neurobiology show that[15]

$$\bar{C} \sim (\bar{d})^{1/2}, \text{ or } \bar{C} \sim \bar{d} \ , \qquad (15)$$

where $\bar{C}$ is the velocity of the nerve propagation and $\bar{d}$ is the diameter of the nerve fiber. It is clear that the study of the propagation speed is a very interesting problem. There also exists another interesting problem. When the nerve signal is propagating, the coupled ion flow in the nerve fiber always follows the propagation. Correspondingly, when the B-L wall and related periodic structures are formed, the coupled convection in liquid crystals also links with the motion of these structures. One can naturally ask whether or not the convection can provide a possible manner of ion exchange in nerve fiber. This is a meaningful question.

The author would like to thank Dr. Jamil Ariai for his helpful disucssion. He would also like to thank Professor Abdus Salam, the International Atomic Energy Agency, and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

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*Permanent address: Department of Physics, Suzhou University, China.

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