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Narrow fingers in the Saffman-Taylor instability

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Saffman-Taylor fingers with a relative width much smaller than the classical limit $\lambda=0.5$ are found when a small isolated bubble is located at their tip. These solutions are members of a family found by Saffman and Taylor neglecting superficial tension. Recent theories have shown that when capillary forces are taken into account an unphysical cusplike singularity would appear at the tip of all the fingers with $\lambda<0.5$. Conversely, here the replacement of the tip by a small bubble makes these solutions possible. At large velocity these fingers show dendritic instability.

When between two narrowly spaced glass plates, a fluid of small viscosity drives a fluid of large viscosity, their interface is affected by the Saffman-Taylor instability. Two geometrical configurations are used for its study. In the original Saffman-Taylor experiment[1–4] the fluids are confined in a linear channel of width w. In the axisymmetric geometry introduced by Paterson[5] the viscous fluid is contained between two circular glass disks and the less viscous fluid is injected at the center. In both geometries the initial destabilization is characterized by its length scale $l_0 \sim b(T/\mu V)^{1/2}$ where T is the surface tension, μ the dynamical viscosity, b the thickness of the cell, and V the velocity of the interface. The subsequent evolution of the instability is very different in each geometry.

In the axisymmetric case[5] the fingers tend to remain a size of the order of l_0. As they move outwards they undergo a series of irregular tip splitting and side branching. In a previous experiment[6] performed in this geometry we showed that the influence of an isolated bubble at the tip created an artificial sharper point to a growing finger. Because of the higher curvature at the tip, the pressure gradient is increased and so is the finger velocity. These faster fingers take a parabolic shape which, for larger velocities, is affected by dendritic instabilities very similar to those observed in crystal growth.

In the present paper we study the influence of a perturbing bubble in the linear geometry. We will first recall the main classical results in this case. Because of the geometrical configuration, the growing fingers tend to screen each other off, so that only the fastest continue to grow. This gives rise to a stable finger shape where the length scale of the width w of the cell imposes itself. The finger is characterized by the ratio λ of its width to that of the cell. In the original paper by Saffman and Taylor λ was studied as a function of the capillary number $C=\mu V/T$. More recently a new parameter[4,7] was used, $1/B=12[(w/b)^2 \times(\mu V/T)]$ which for equal values gave comparable λ in cells of various geometrical dimensions. In a particular cell for a given fluid the control parameter is the velocity V which can be chosen by changing the applied pressure.

The classical puzzling result is that with increasing $1/B$ the relative width λ of the finger first decreases from 1 to 0.5, at which it remains in a very large range of values of $1/B$.

Saffman and Taylor, assuming that the capillary forces should become negligible at large values of $1/B$, solved analytically the two-dimensional potential flow problem in the absence of surface tension. They found a family of solutions parametrized by λ which could take all values $0<\lambda<1$,

$$x=\frac{1-\lambda}{\pi}\ln\frac{1}{2}\left(1+\cos\frac{\pi y}{\lambda}\right) . \qquad (1)$$

The origin is taken at the tip of the finger; $0x$ and $0y$ are, respectively, parallel to the length and to the width of the cell and the half width $w/2$ of the cell is taken as unit length. No physical argument indicates which solution is selected. Furthermore at low velocity the observed fingers do not correspond to the predicted shape, only the particular $\lambda=0.5$ solution corresponds to the fingers observed at large $1/B$. No reason could be found at first why the solutions $\lambda<0.5$ were not observed at all.

Pitts,[2] using an empirical law on the local curvature at a point of the profile, found a modified relation

$$x=\frac{\lambda}{\pi}\ln\frac{1}{2}\left(1+\cos\frac{\pi y}{\lambda}\right) . \qquad (2)$$

This fitted well the observed fingers in the range $0.5<\lambda<0.80$, where all the experimental profiles are homothetics so that they can all be reduced to a single curve by magnification of the axes.

The first attempt to take surface tension into account in this problem was made by McLean and Saffman.[3] Following their work, recent progress in this problem has been made by Vanden-Broek[8] using numerical techniques, and by Dombre, Hakim, and Pomeau,[9] Shraiman,[10] and Hong and Langer[11] using analytical calculation. They showed that the taking into account of surface tension in the Saffman-Taylor solutions leads to finding a cusplike singularity at the tip of most of them. Only for a family of solutions the cusp's amplitude is reduced to zero. These solutions have width $\lambda>\frac{1}{2}$ and all tend toward $\lambda=\frac{1}{2}$ for vanishing surface tension.

I. EXPERIMENT SETUP

Four experimental cells were used. They were made with glass plates 1.50 m long and 1.5 cm thick separated by 0.1-cm-thick spacers. Their widths w were, respectively, 12, 6, 4, and 2 cm. The viscous fluid filling them was a silicone oil Rhodorsil 47V100 with surface tension $T=20.9$ N/m at 25° and viscosity $\mu=9.65\times10^{-2}$ kg/m s. The fluid of low viscosity was nitrogen gas. We measured finger shapes, width, and velocities either with photographs taken with a motorized camera or with video tape recordings. Small bubbles were injected into the cell before the experiment started by means of a long hypodermic needle.

II. EXPERIMENTAL RESULTS

We inject into the cell small air bubbles near the initial motionless front between nitrogen and oil. To be effective these bubbles have to be of an oblate shape between the flat plates so that their initial diameter in the cell plane must be a few times the thickness b. When pressure is applied the bubbles start moving. As their friction on the glass plate is negligible they form zones of constant pressure which distort the neighboring isobars in the oil. As the front destabilizes, one of the fingers situated behind a bubble will grow faster and catch up. The bubble then remains stable at the tip. As the velocity of this finger becomes larger (because of the increase of the curvature of the tip) it then overcomes the other fingers and forms a steady solution translating along the whole cell (Fig. 1).

Fingers with a bubble at the tip can be observed in a large range of values of velocities. However, for small velocities the bubble does not always remain stable; it drifts off on the side so that its effect vanishes. For large velocities the bubble and the fingertip are pressed against each other so that the tip of the finger and the back of the bubble are flattened and reach a stable configuration (Fig. 1).

For a given applied pressure the velocity of the new stable finger with a bubble at the tip is larger and its width λ smaller than in the usual Saffman-Taylor case. If the film that separates the bubble from the finger breaks, the finger slows down and returns to its classical shape.

The photograph of a finger with $\lambda=0.32$ obtained in the cell $w=6$ cm for $V=5$ cm/s is shown on Fig. 1. The finger profile has a well-determined shape which does not include the isolated bubble. We have superposed, excluding the bubble, points of the profiles $\lambda=0.32$ calculated, respectively, with Eqs. (1) and (2). We see immediately that this finger is not a member of the family described by Pitt's formula, but that its shape coincides with striking precision with the solution of Saffman and Taylor. The only deviation is localized behind the bubble where its influence has created a little flat part.

Figure 2 shows the evolution of the observed values of λ as a function of the capillary number in three of the four experimental cells. As in the usual fingers, λ decreases with increasing C, and a saturation occurs at a value λ_s, which depends upon the aspect ratio of the cell. We find for cells of thickness $b=0.1$ cm:

$$\lambda_s \simeq 0.46 \text{ for } w=2 \text{ cm}, \ \lambda_s \simeq 0.345 \text{ for } w=4 \text{ cm} ,$$

$$\lambda_s \simeq 0.30 \text{ for } w=6 \text{ cm}, \ \lambda_s \simeq 0.225 \text{ for } w=12 \text{ cm} .$$

These values are much smaller than the classical limit $\lambda=0.5$. Contrary to the classical situation, the results obtained in the various cells cannot be brought to near coincidence by using the dimensionless number $1/B$.

The mean diameter d of the efficient bubbles can be chosen in the range $b<d<8b$. For the same applied pressure the velocity of the finger will depend upon the size of the bubble. However, in a given geometry, all the ob-

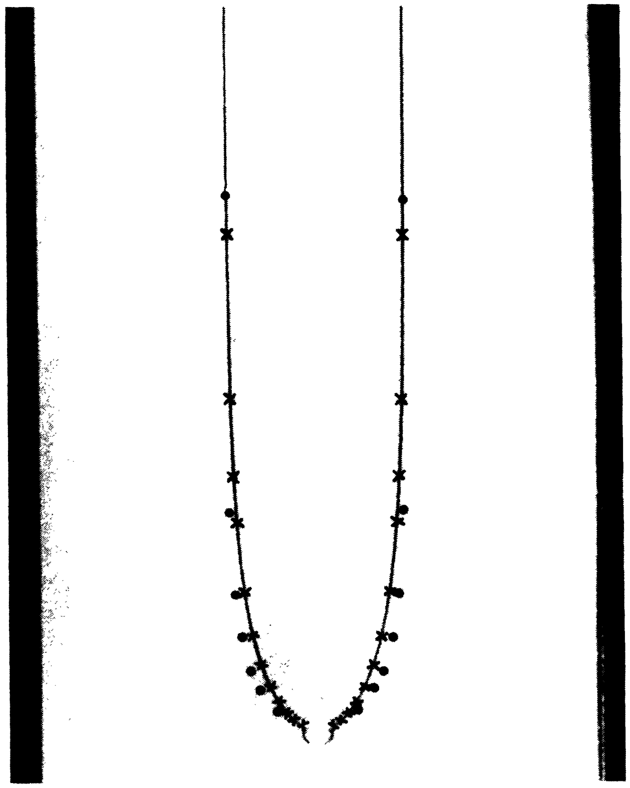

FIG. 1. Photograph of a finger obtained in the cell $w=6$ cm, $b=0.1$ cm at $V=5.3$ cm/s. The crosses are points of the theoretical profile from Saffman-Taylor Eq. (1) for $\lambda=0.32$. The dots are calculated from Pitt's Eq. (2) for $\lambda=0.32$.

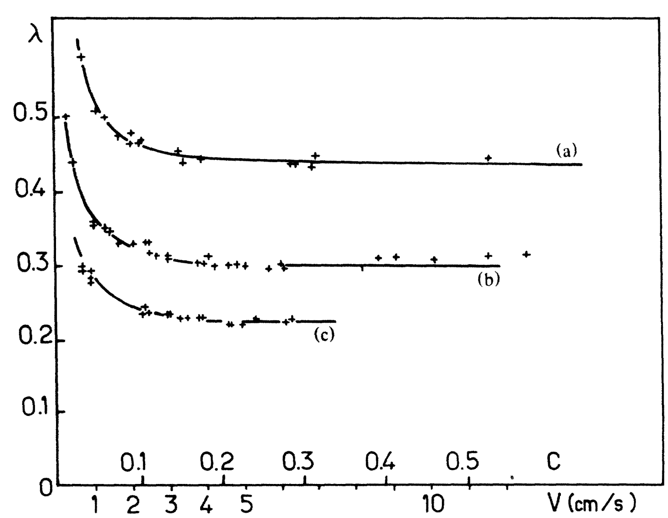

FIG. 2. Variation of λ with the velocity V (or the capillary number $C=\mu V/T$) in the three geometries: (a) $w=2$ cm, (b) $w=6$ cm, (c) $w=12$ cm.

served values of λ lie on the same curve $\lambda(V)$ (Fig. 2). This shows that the shape of a finger depends only on its velocity. This velocity is itself determined by both the applied pressure and the bubble size.

To understand these results we must remember that in the axisymmetric case the fingers, in the presence of a bubble, take a parabolic shape[6] with a curvature at the tip determined by the velocity.

In the present linear case the solutions given by Eq. (1) also have a parabolic shape near the tip. For small values of y they reduce to

$$x = \frac{\lambda - 1}{4\lambda^2}\pi y^2 \ . \qquad (3)$$

We must remember that the solutions (1) were found, $w/2$ being taken as unit lengh. In a real cell the shape of the finger near its tip can be characterized by the curvature K of the parabola at the tip

$$1/K = -\frac{\lambda^2 w}{\pi(1-\lambda)} \ . \qquad (4)$$

From the measures of λ we can deduce the curvature K as a function of the velocity V in the different cells. The result is shown on Fig. 3. It shows that at a given velocity the curvature K is the same in all the cells (of equal thickness). Then Eq. (4) gives a relation between the observed λ and the width w of the cell:

$$\lambda = \frac{1-(1-4wK/\pi)^{1/2}}{2wK/\pi} \ . \qquad (5)$$

Figure 4 shows the observed values of λ in the four experimental cells at a velocity $V = 6$ cm/s compared to the predicted dependence $\lambda(w)$ given by the relation (5). The best fitting value of K for this velocity was $K = 4.1$ cm^{-1}.

We can compare the destabilization of these new fingers to that of the classical ones. For values of $1/B > 7000$ the usual Saffman-Taylor fingers destabilize when irregular side branching and tip splitting occur.[4] With a very small

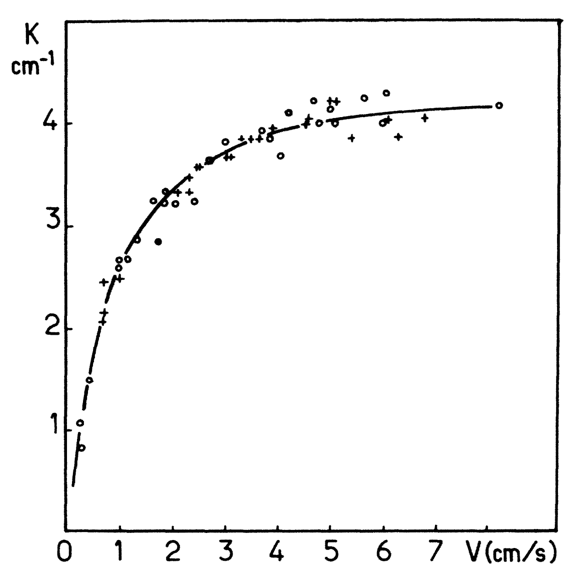

FIG. 3. The curvature of the parabolic extremity as a function of the velocity V. o: $w = 6$ cm; +: $w = 12$ cm.

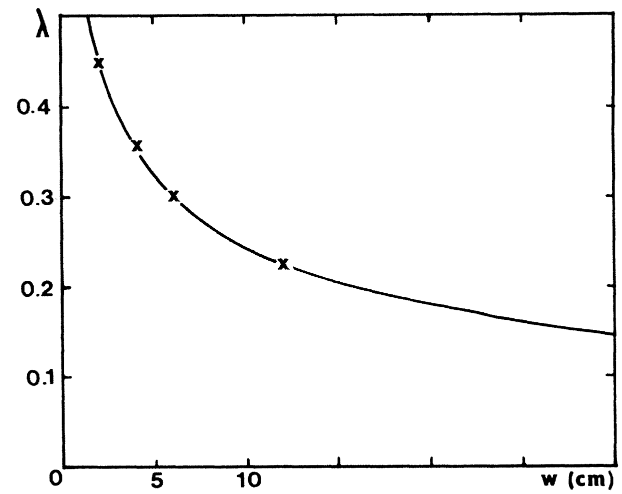

FIG. 4. Values of λ as a function of the width of the cell at velocity $V = 6$ cm/s. The continuous line is calculated from Eq. (5), the crosses are experimental.

bubble at their tip the fingers are more stable and can be observed to values as large as 20 000.

They also destabilize differently into one of two periodic behaviors. In the first one [Fig. 5(a)], observed for small bubbles, the curvature of the extremity of the finger varies periodically. The pulsating tip creates symmetrical lateral branches. When the bubble is large [Fig. 5(b)] the extremity of the finger oscillates transversely. We had ob-

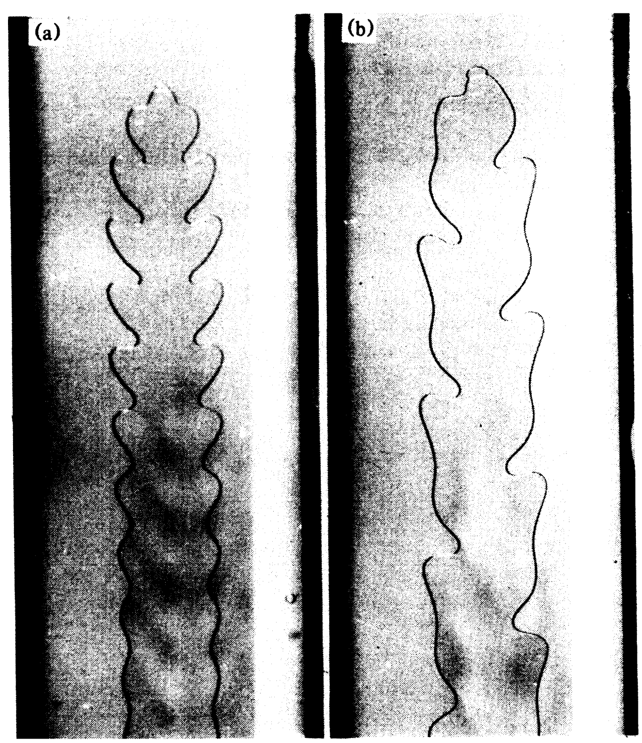

FIG. 5. (a) The pulsating tip regime of a finger with a small bubble, $w = 6$ cm, $V = 10$ cm/s, (b) the oscillating tip regime of a finger with a larger tip bubble, $w = 6$ cm, $V = 7$ cm/s.

served[6] both these instabilities in the Paterson geometry where the lateral branches could fully develop. We pointed out that the first one is identical in shape to a classical dendritic instability. In the present experiment the vicinity of the side walls limits the growth of the instability. In the widest cell, where λ is small, lateral branches can still be formed. In the narrowest the growth is practically inhibited. These instabilities will be described in more detail in a forthcoming paper.

Finally, we tried to apply to the finger another type of local perturbation of the tip. We stretched a very thin thread along the whole length of the cell and in the middle of its width. This technique was first used by Grace and Harrison[12] who showed that in a Hele Shaw cell, rising bubbles have a larger velocity when they surround a vertical wire. Except for their tip which becomes asymmetrical with respect to the wire, fingers similar to those obtained with the small bubble are observed.[13] Dendritic instability also occurs. A similar result has been obtained recently by A. Libchaber.[14]

III. CONCLUSIONS

For large velocities where viscous forces should dominate, surface tension remains important in the determination of the shape of the fingers. Recent theories have shown that this is due to its critical role in the localized region of the finger tip. Conversely, we have shown that introducing a localized perturbation at the tip suppresses the width selection described in Refs. 9-11. Other solutions (those predicted in theories neglecting surface tension) become possible. These are the parametrized Saffman-Taylor solutions here and the parabolas in the circular geometry. The curvature at the tip is determined by the velocity and selects the finger experimentally observed. Both are affected at larger velocities by dendritic instabilities.

Dendrites have usually been associated with crystalline anisotropy. In the present experiment as well as in the axisymmetric one, we show that a local perturbation of the tip creates these characteristic patterns. Anisotropy of the medium is not therefore a necessary condition to dendritic growth. It is only one of the means by which the singularity of the tip can be removed so that fingers with parabolic extremities are made possible.

Note added in proof. Recent experiments in cells of various thicknesses (b =0.05, 0.1, and 0.2 cm) show that the relevant parameter for the cruves $\lambda(V)$ of Fig. 2 is the ratio of the width over the thickness of the cell w/b. The nondimensional version of Fig. 3 is then a plot of Kb vs C and in Fig. 4 the abscissa can be scaled adimensionally in w/b.

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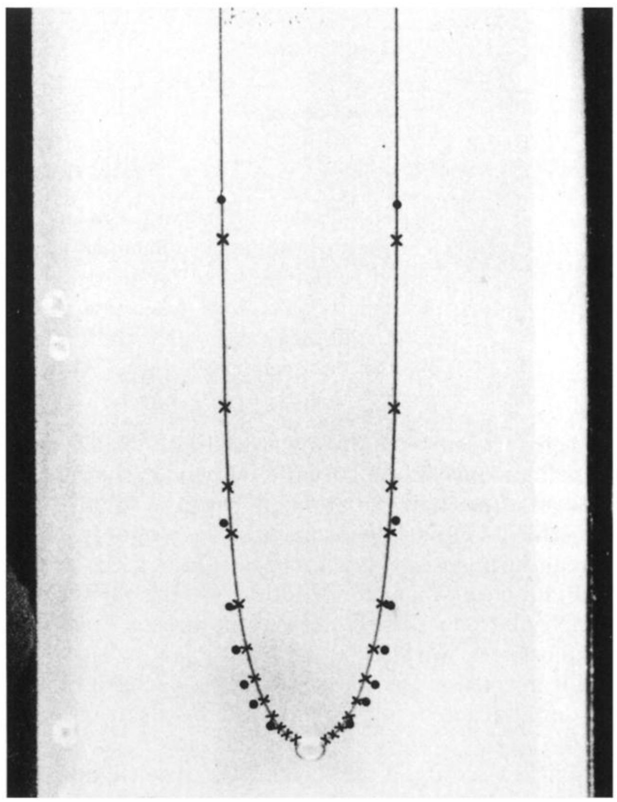

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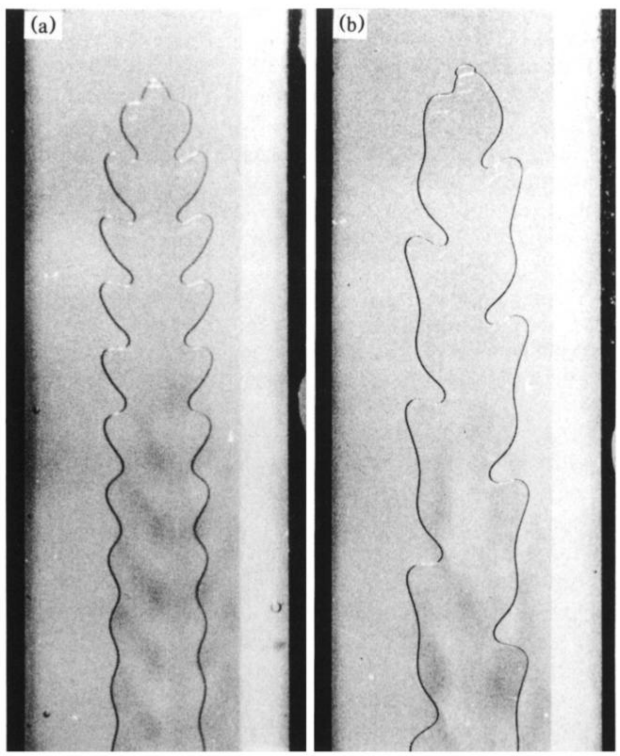

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