Comments

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Comment on "Renormalization-group analysis of the global structure of the period-doubling attractor"

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It is shown how to calculate any generalized fractal dimension of the Feigenbaum attractor from the trajectory scaling function. The Hausdorff dimension is determined.

Bensimon *et al.*[1] have developed a renormalization technique to compute, using universal quantities, the generalized fractal dimensions of the attractor at the accumulation point in period doubling, first calculated by Grassberger.[2,3] We wish to point out that these numbers can be calculated in a much simpler way using the Feigenbaum trajectory scaling function[4] and the thermodynamic analogy developed by Vul *et al.*,[5,6] and we compute one of these dimensions, the Hausdorff dimension. The input in our calculations is Lanford's[7] high-accuracy approximation to the Feigenbaum function.

The formula for the generalized fractal dimension[1-3] D_q, at level N, as a sum over intervals i, with lengths l_i and weights p_i

$$\sum_{i=1}^{2^N} p_i^q / l_i^{\tau_q} = 1; \quad \tau_q = (q-1)D_q \tag{1}$$

can, for the period-doubling attractor, be rewritten[5,6] as

$$\sum_{i=1}^{2^N} e^{-\beta W(i)} = 2^{-NF(\beta)} , \tag{2}$$

where $\beta = -\tau_q$, $F(\beta) = -q$, $p_i = 2^{-N}$ for all intervals on level N and $i = i_1 2^{N-1} + \cdots + i_N$; $i_j = 0,1$;

$$W(i) = -\ln | l(i_1, \ldots, i_N) |$$

$$\approx -\ln | \sigma(i_1, \ldots, i_N)\sigma(i_2, \ldots, i_N, 0) \ldots l_0 | \ . \tag{3}$$

Here $\sigma(i_1, \ldots, i_N)$ is the Feigenbaum trajectory scaling function with argument $\sigma(i/2^{N+1})$[4]. Hence $|\sigma|^\beta$ plays the role of a transfer matrix,[6] and $F(\beta) = -\ln\lambda(\beta)/\ln 2$, where $\lambda(\beta)$ is the largest eigenvalue of the matrix. The Hausdorff dimension $D_0 = \beta_0$ is defined by $F(\beta_0) = 0$. It is therefore the value of the parameter β for which the largest eigenvalue of the transfer matrix $|\sigma|^\beta$ takes the value 1.

We compute successive approximations to D_0 by increasing the number of significant bits in σ, or, equivalently, by truncating the transfer matrix at higher orders, and we extract the leading eigenvalue by Gaussian elimination. The results are summarized in Table I.

The deviation in β_0 from the asymptotic value is proportional to the error in σ, and one may show that the error due to not taking the full structure of σ into account goes to zero with level n at least as fast as 2^{-n}. In practice the errors are much smaller, the change in β_0 at a given level being roughly -6 times the change in the next. To find a better approximation to the asymptotic value, we apply a series of Shanks transformations.[8]

We have thus the following estimate of the asymptotic value of the Hausdorff dimension of the Feigenbaum attractor: $\beta_0 = 0.538\,045\,143\,5$ (1). The computation is essentially limited by the finite accuracy in the polynomial expansion of the universal function $g_{pd}(x)$. The errors thus introduced in g_{pd} grow, as many iterates of the function g_{pd} have to be computed to find σ in level n, roughly as β^n, where β is a universal constant[5,9-11] with numerical value ≈ 6.6. This induces an error in σ roughly $(\beta\alpha_{pd})^n$ where α_{pd} is the Feigenbaum constant. As Lanford's[7] data are good to one part in 10^{-30} we are limited to something like 15 levels and 15 digits in the calculation of β_0.

TABLE I. The approximation to the Hausdorff dimension as a function of level index.

Significant bits in σ	$\beta_0 = D_0$
2	0.537 843 517 84...
3	0.538 103 284 69...
4	0.538 037 608 94...
5	0.538 046 713 78...
6	0.538 044 902 91...
7	0.538 045 188 26...
8	0.538 045 136 24...
9	0.538 045 144 88...

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