Onset of the first instability in hydrodynamic flows: Effect of parametric modulation

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We treat the effect of parametric modulation on the onset of instability in Rayleigh-Bénard and Taylor-Couette geometries. Closed-form solutions are obtained for the case of realistic rigid-boundary conditions. The high-frequency limit of the effect in the Taylor-Couette flow differs from that of Hall.

I. INTRODUCTION

The problem of the effect of modulation of the control parameter on the onset of the first instability in various hydrodynamic systems is currently receiving considerable attention, spurred by new generation of experimental measurements which will accurately determine the magnitude of the effect. The theoretical treatment[1-9] of this effect is based either on the full hydrodynamic equations[1-4] or a few-mode truncation thereof.[5-9] While the few-mode truncation always yields a closed-form expression for the effect, the hydrodynamic equations seemed to yield closed-form solutions only when the idealized free-boundary conditions were used. In this paper, we show how accurate closed-form expressions for the effect can be obtained, by using the *proper rigid-boundary conditions.* We treat both the Rayleigh-Bénard (RB) and the Taylor-Couette (TC) problems.

We follow a technique used by Chandrasekhar[10] to obtain the critical Rayleigh and Taylor numbers in closed form with the proper rigid-boundary conditions. The hydrodynamic problem reduces to a pair of coupled partial differential equations in two variables: the vertical component of velocity and the temperature in the RB problem and the radial and vertical velocities in the narrow-gap TC problem. The rigid boundary conditions are that both fields vanish at the edges, while it is required that one of the fields has a vanishing first derivative at the edges. Chandrasekhar's approximation consists of Fourier expanding the field on which the only requirement is the vanishing at the edges and truncating the expansion at the first term. The remaining field is then exactly determined from one of the differential equations and the critical Rayleigh or Taylor number determined self-consistently from the other. The procedure yields these numbers to within 1% as shown in Secs. II and III. It is this approximation that we use to study the modulation effects in both RB and TC geometry. The result for the Rayleigh-Bénard geometry agrees very well with the numerical work of Rosenblat and Herbert.[2] The result for the Taylor-Couette flow differs qualitatively from Hall's pioneering investigation[11] of this problem in that at high frequencies we find a stabilization as opposed to the destabilization reported by Hall.[11]

II. RAYLEIGH-BÉNARD PROBLEM

The linearized hydrodynamical equations for this system in the standard form are

$$(D^2-a^2)\left[D^2-a^2-\frac{\partial}{\partial t}\right]w=\theta\ , \tag{2.1a}$$

$$\left[D^2-a^2-\sigma\frac{\partial}{\partial t}\right]\theta=-Ra^2w\ . \tag{2.1b}$$

The symbols have their usual meaning: w is the Z component of velocity (gravity is in the Z direction), θ is the deviation of the temperature from the steady conduction state profile, $D\equiv d/dz'$, where $z=Z/d$ (d being the separation between plates), a is the dimensionless wave number of the convection rolls, t is the time in dimensionless units, σ is the Prandtl number and $R=\alpha(\Delta T)gd^3/\lambda\nu$ is the Rayleigh number, $\Delta T=T_1-T_2$ the temperature difference between the plates, with T_1 and T_2 being the temperatures of the lower and upper plates, respectively. The conduction temperature profile T_s in the steady state is

$$T_s=T_1-\Delta T(z+\tfrac{1}{2}) \tag{2.2}$$

and θ is defined as $T(\mathbf{r},t)-T_s$.

We now consider the system with the lower plate temperature modulated as $T=T_1+\mathrm{Re}\,\epsilon(\Delta T)e^{i\omega t}$. The steady-state temperature profile is now given by

$$T_s=T_1-\Delta T\left[z+\frac{1}{2}\right]+\epsilon\,\mathrm{Re}\Delta T\frac{\sinh\alpha d(\frac{1}{2}-z)e^{i\omega t}}{\sinh\alpha d} \tag{2.3}$$

where $\alpha^2=-i\omega/\lambda$. The linearized equations of modulated hydrodynamics now become

$$(D^2-a^2)\left[D^2-a^2-\frac{\partial}{\partial t}\right]w=\theta\ , \tag{2.4a}$$

$$\left[D^2-a^2-\sigma\frac{\partial}{\partial t}\right]\theta=-Ra^2[1+\epsilon\mathrm{Re}f(z)e^{-i\omega t}]w\ , \tag{2.4b}$$

where

$$f(z)=\frac{\alpha d}{\sinh(\alpha d)}\cosh[\alpha d(\tfrac{1}{2}-z)]\ . \tag{2.5}$$

What we need to find is the critical value of R at which the conduction state $w=\theta=0$ is destabilized. We do this within perturbation theory by expanding w, θ, and R in powers of ϵ. The resulting equations to $O(\epsilon^2)$ are

$$(D^2-a^2)\left[D^2-a^2-\frac{\partial}{\partial t}\right]w_0=\theta_0\ , \tag{2.6a}$$

$$\left[D^2-a^2-\sigma\frac{\partial}{\partial t}\right]\theta_0=-R_0a^2w_0\ , \tag{2.6b}$$

$$(D^2-a^2)\left[D^2-a^2-\frac{\partial}{\partial t}\right]w_1=\theta_1\ , \tag{2.7a}$$

$$\left[D^2-a^2-\sigma\frac{\partial}{\partial t}\right]\theta_1=-R_0a^2w_1-R_1a^2w_0 -R_0a^2\mathrm{Re}(fw_0e^{i\omega t})\ , \tag{2.7b}$$

and

$$(D^2-a^2)\left[D^2-a^2-\frac{\partial}{\partial t}\right]w_2=\theta_2\ , \tag{2.8a}$$

$$\left[D^2-a^2-\sigma\frac{\partial}{\partial t}\right]\theta_2=-R_0a^2w_2-R_2a^2w_0 -R_0a^2\mathrm{Re}(fw_1e^{i\omega t}) -R_1a^2\mathrm{Re}(fw_0e^{i\omega t})\ , \tag{2.8b}$$

where

$$w=w_0+\epsilon w_1+\epsilon^2w_2+\cdots \tag{2.9a}$$

$$\theta=\theta_0+\epsilon\theta_1+\epsilon^2\theta_2+\cdots \tag{2.9b}$$

$$R=R_0+\epsilon R_1+\epsilon^2R_2+\cdots\ . \tag{2.9c}$$

In the zeroth order the solution to Eqs. (2.6a) and (2.6b) have to be obtained under the realistic boundary conditions

$$\theta_0=0,\ \text{at } z=-\tfrac{1}{2} \text{ and } +\tfrac{1}{2} \tag{2.10a}$$

and

$$w_0=0=\frac{dw_0}{dz},\ \text{at } z=-\tfrac{1}{2} \text{ and } +\tfrac{1}{2}\ . \tag{2.10b}$$

We will look for solutions which are time independent (principle of exchange of stabilities) and symmetric because they are the ones that are obtained at a lower Rayleigh number. The approximation will consist of a single-mode truncation of the Fourier expansion for $\theta_0(z)$, i.e.,

$$\theta_0(z)=A\cos(\pi z)\ . \tag{2.11}$$

Inserting the above in Eq. (2.6a) the solution for w_0 with the boundary conditions (2.10b) becomes

$$w_0=B\cosh(az)+Cz\sinh(az)+\frac{A\cos(\pi z)}{(\pi^2+a^2)^2}\ , \tag{2.12}$$

where

$$2B=-C\tanh(a/2) \tag{2.13}$$

and

$$C=\frac{2\pi A\cosh(a/2)}{(\pi^2+a^2)^2(a+\sinh a)}\ . \tag{2.14}$$

Inserting Eqs. (2.12) and (2.11) in Eq. (2.6b) with $(\partial/\partial t)$ ( ) term absent, integrating from $z=-\frac{1}{2}$ to $\frac{1}{2}$ after multiplying by $\cos(\pi z)$ yields

$$R_0=\frac{(\pi^2+a^2)^3}{a^2}\left[1-\frac{16\pi^2a\cosh^2(a/2)}{(\pi^2+a^2)^2(a+\sinh a)}\right]^{-1}\ . \tag{2.15}$$

Minimizing $R_0(a)$ with respect to a yields $a_0=3.12$ and $R_0\simeq1712$ within 1% of the exact numerical solution.

The perturbative corrections R_1, R_2, etc. which are compatible with the R_0 may be obtained by requiring that Eqs. (2.7a) and (2.7b) and Eqs. (2.8a) and (2.8b) must have solutions compatible with the solution of Eqs. (2.6a) and (2.6b), obtained above. The solvability criterion leads to $R_1=0$. To obtain the form of w_1 and θ_1 within the one-Fourier-mode approximation described above, we note that the time dependence of w_1 and θ_1 are of the form $e^{i\omega t}$ and that the spatial dependence can be obtained by first Fourier expanding fw_0 and retaining the $\cos(\pi z)$ term. With this in mind, we write

$$fw_0\simeq F\cos(\pi z)\ , \tag{2.16}$$

where

$$\begin{aligned} F&=2\int_{-1/2}^{1/2}fw_0\cos(\pi z)dz\\ &=2\left\{\frac{A}{(\pi^2+a^2)^2}\frac{2\pi^2}{4\pi^2+(\alpha d)^2}+\frac{B\alpha d}{\sinh(\alpha d)}\pi\cosh\left[\frac{\alpha d}{2}\right]\left[\frac{\cosh[\frac{1}{2}(a-\alpha d)]}{(a-\alpha d)^2+\pi^2}+\frac{\cosh[\frac{1}{2}(a+\alpha d)]}{(a+\alpha d)^2+\pi^2}\right]\right.\\ &\quad+\frac{C\alpha d}{\sinh(\alpha d)}\pi\cosh\left[\frac{\alpha d}{2}\right]\left[\frac{\sinh[(a-\alpha d)/2]}{(a-\alpha d)^2+\pi^2}+\frac{\sinh[(a+\alpha d)/2]}{(a+\alpha d)^2+\pi^2}\right.\\ &\quad\left.\left.-\frac{4(a-\alpha d)}{[(a-\alpha d)^2-\pi^2]^2}\cosh\left[\frac{a-\alpha d}{2}\right]-\frac{4(a+\alpha d)}{[(a+\alpha d)^2+\pi^2]^2}\cosh\left[\frac{a+\alpha d}{2}\right]\right]\right\}\ . \end{aligned} \tag{2.17}$$

We proceed, as for the zeroth-order solution, by making the ansatz

$$\theta_1 = A_1 \cos(\pi z) \ . \tag{2.18}$$

The solution for w_1 is obtained from Eq. (2.7a) with the boundary conditions $w_1 = dw_1/dz = 0$ at $z = -\frac{1}{2}$ and $\frac{1}{2}$ as

$$w_1 = B_1 \cosh(az) + C_1 \cosh(bz) + \frac{A_1 \cos(\pi z)}{(\pi^2 + a^2)(\pi^2 + b^2)} \ , \tag{2.19}$$

with

$$B_1 \cosh(a/2) = -C_1 \cosh(b/2) \ , \tag{2.20}$$

$$C_1 = A_1 \frac{\pi \cosh(a/2)}{b \sinh(b/2)\cosh(a/2) - a \sinh(a/2)\cosh(b/2)} \frac{1}{\pi^2 + a^2} \frac{1}{\pi^2 + b^2} \ , \tag{2.21}$$

and

$$b^2 = a^2 + i\omega \ . \tag{2.22}$$

From Eq. (2.7b) we now obtain the amplitude A_1 as

$$A_1 = \frac{R_0 a^2 F}{\pi^2 + a^2 + \sigma i \omega - \frac{R_0 a^2}{(\pi^2 + a^2)(\pi^2 + b^2)} \left[1 + \frac{4^2 (a^2 - b^2) \cosh(a/2) \cosh(b/2)}{(\pi^2 + a^2)(\pi^2 + b^2)[b \sinh(b/2) \cosh(a/2) - a \sinh(a/2) \cosh(b/2)]} \right]} \tag{2.23}$$

We turn now to the $O(\epsilon^2)$ equations. For Eqs. (2.8a) and (2.8b) to be solvable under the condition that Eqs. (2.6a) and (2.6b) admit the solutions described above, we take the time average and require that the left vectors of Eq. (2.6a) be orthogonal to the right-hand side (rhs) of Eq. (2.8b). This leads to

$$\begin{aligned} R_2 &= -R_0 \mathrm{Re} \frac{\langle \theta_0 | f w_1 \rangle}{2 \langle \theta_0 | w_0 \rangle} \ , \\ &= -R_0 \mathrm{Re} \frac{\langle (\pi^2 + a^2) \theta_0 | f w_1 \rangle}{2 \langle (\pi^2 + a^2) \theta_0 | w_0 \rangle} \ , \\ &= -R_0 \mathrm{Re} \frac{\langle (D^2 - a^2) \theta_0 | f w_1 \rangle}{2 \langle (D^2 - a^2) \theta_0 | w_0 \rangle} \ , \\ &= -R_0 \mathrm{Re} \frac{\langle w_0 | f w_1 \rangle}{2 \langle w_0 | w_0 \rangle} = -R_0 \mathrm{Re} \frac{\langle w_1 | f w_0 \rangle}{2 \langle w_0 | w_0 \rangle} \ . \end{aligned} \tag{2.24}$$

Straightforward algebra leads to

$$R_2 = \frac{R_0^2 a^2 |F|^2}{4 \langle w_0 | w_0 \rangle} \mathrm{Re} \frac{G(\omega)}{-(\pi^2 + a^2)(\pi^2 + a^2 + i\omega)(\pi^2 + a^2 + i\sigma\omega) + R_0 a^2 G(\omega)} \tag{2.25}$$

where

$$G(\omega) = 1 + \frac{4\pi^2 (a^2 - b^2) \cosh(a/2) \cosh(b/2)}{(\pi^2 + a^2)(\pi^2 + b^2)[b \sinh(b/2) \cosh(a/2) - a \sinh(a/2) \cosh(b/2)]} \ . \tag{2.26}$$

It is useful to look at the asymptotic behaviors of R_2/R_0, the fractional correction for $\omega \to 0$ and $\omega \to \infty$. In the zero-frequency limit, we expand $G(\omega)$ in powers of $i\omega$ and find $R_0 a^2 G'(\omega) \ll (\pi^2 + a^2)^2 (\sigma + 1)$ $[R_0 a^2 G'(\omega) \simeq 0.2]$. The second derivative contributes even less. The zero-frequency limit turns out to be

$$\frac{R_2(\omega = 0)}{R_0} \simeq \frac{1}{2} \frac{\sigma}{(\sigma + 1)^2} \ . \tag{2.27}$$

For very high frequency, $G(\omega) \to 1$, $|F|^2 \sim \omega^{-3}$, and thus

$$\frac{R_2}{R_0} \simeq \frac{1}{\omega^5} \ . \tag{2.28}$$

The correction is positive definite and monotonic. With the idealized boundary conditions this dependence turns out to be ω^{-6}

III. TAYLOR-COUETTE FLOW

We work in the approximation that the separation d between the cylinders is much less than their radii (r_1 and r_2). We will restrict ourselves to the linearized equations and use the notation of Chandrasekhar[10] and Hall.[11] In the small-gap approximation, there is only one variable, $z=(r-r_1)/d$, which ranges from 0 to 1. Using a dimensionless time variable, the hydrodynamic equations for the modulated flow can be written, exactly as in Hall,[11] as

$$\left[\frac{\partial^2}{\partial z^2}-a^2-\frac{\partial}{\partial t}\right]\left[\frac{\partial^2}{\partial z^2}-a^2\right]u=\left[1-z+\epsilon\mathrm{Re}\frac{\sinh[\alpha d(1-z)]}{\sinh(\alpha d)}e^{i\omega t}\right]v \tag{3.1a}$$

$$\left[\frac{\partial^2}{\partial z^2}-a^2-\frac{\partial}{\partial t}\right]v=-Ta^2u\left[1+\epsilon\mathrm{Re}\frac{\alpha d}{\sinh(\alpha d)}\cosh[\alpha d(1-z)]e^{i\omega t}\right], \tag{3.1b}$$

where u and v are the dimensionless radial and vertical components of the velocity field, $\alpha^2=i\omega/\nu$, a is the dimensionless wave number of the roll, ω is the modulation frequency of the rotation rate Ω of the inner cylinder (while the outer cylinder is at rest), and T is the dimensionless Taylor number

$$T=\frac{2\Omega^2 d^3\bar{R}}{\nu^2}, \tag{3.2}$$

where $\bar{R}=(r_1+r_2)/2$.

The difference from the hydrodynamic equations for the modulated convection problem lies in the extra terms on the rhs of Eq. (3.1a) as compared to the rhs of Eq. (2.4a). While the modulation affects the second equation above in the convection problem, here, the effect shows up in both equations and that causes a qualitative difference in the final result.

We proceed perturbatively, as in Sec. II by employing the modulation amplitude ϵ as the small parameter. Introducing the expansions

$$u=u_0+\epsilon u_1+\epsilon^2 u_2+\cdots, \tag{3.3a}$$

$$v=v_0+\epsilon v_1+\epsilon^2 v_2+\cdots, \tag{3.3b}$$

$$T=T_0+\epsilon T_1+\epsilon^2 T_2+\cdots, \tag{3.3c}$$

and inserting in Eq. (3.1a) and (3.1b) and equating identical powers of ϵ, we get up to $O(\epsilon^2)$

$$\left[D^2-a^2-\frac{\partial}{\partial t}\right](D^2-a^2)u_0=(1-z)v_0, \tag{3.4a}$$

$$\left[D^2-a^2-\frac{\partial}{\partial t}\right]v_0=-T_0a^2u_0, \tag{3.4b}$$

$$\left[D^2-a^2-\frac{\partial}{\partial t}\right](D^2-a^2)u_1=(1-z)v_1+\mathrm{Re}g(z)v_0e^{i\omega t}, \tag{3.5a}$$

$$\left[D^2-a^2-\frac{\partial}{\partial t}\right]v_1=-T_0a^2u_1-T_1a^2u_0-T_0a^2u_0\mathrm{Re}f(z)e^{i\omega t}, \tag{3.5b}$$

and

$$\left[D^2-a^2-\frac{\partial}{\partial t}\right](D^2-a^2)u_2=(1-z)v_2+\mathrm{Re}g(z)v_1e^{i\omega t}, \tag{3.6a}$$

$$\left[D^2-a^2-\frac{\partial}{\partial t}\right]v_2=-T_0a^2u_2-T_2a^2u_0-T_1a^2u_1-T_0a^2u_1\mathrm{Re}f(z)e^{i\omega t}, \tag{3.6b}$$

where

$$g(z)=\frac{\sinh\alpha d(1-z)}{\sinh\alpha d}, \tag{3.7a}$$

and

$$f(z)=\frac{\alpha d}{\sinh(\alpha d)}\cosh[\alpha d(1-z)]. \tag{3.7b}$$

We begin with the $O(1)$ equations [Eqs. (3.4a) and (3.4b)], which correspond to the unmodulated system. The boundary conditions require that v be zero at the boundaries $z=0$ and 1, and $u=0=du/dz$ at $z=0$ and 1. At this order, $v_0=0=u_0=du_0/dz$ at $z=0$ and 1. As before, we proceed by making the ansatz

$$v_0=\sin(\pi z). \tag{3.8}$$

This is the first term of a Fourier expansion for v_0 and the approximation consists of truncating the expansion at this order. Using Eq. (3.4b), we now solve for $u_0(z)$ to obtain

$$u_0(z)=A_0\sinh(az)+B_0\cosh(az)+C_0z\sinh(az)+D_0z\cosh(az)+\frac{(1-z)\sin(\pi z)}{(\pi^2+a^2)^2}-\frac{4\pi}{(\pi^2+a^2)^3}\cos(\pi z), \tag{3.9}$$

where

$$A_0=\frac{-4\pi(1+\cosh a)}{(\sinh a-a)(\pi^2+a^2)^3}+\frac{\pi a}{(\pi^2+a^2)^2(\sinh^2 a-a^2)}, \tag{3.10a}$$

$$B_0=\frac{4\pi}{(\pi^2+a^2)^3}, \tag{3.10b}$$

$$C_0=\frac{\pi}{(\pi^2+a^2)^2}\frac{\sinh a\cosh a-a}{\sinh^2 a-a^2}-\frac{4\pi a\sinh a}{(\pi^2+a^2)^3(\sinh a-a)}\,, \quad (3.10c)$$

and

$$D_0=\frac{4\pi a(1+\cosh a)}{(\sinh a-a)(\pi^2+a^2)^3}-\frac{\pi\sinh^2 a}{(\pi^2+a^2)^2(\sinh^2 a-a^2)}\,. \quad (3.10d)$$

Inserting this solution in Eq. (3.4b), multiplying by $\sin\pi z$, and integrating from 0 to 1, we obtain

$$T_0=\frac{2(\pi^2+a^2)^3}{a^2\{1-16a\pi^2\cosh^2(a/2)[(\pi^2+a^2)^2(a+\sinh a)]^{-1}\}}\,. \quad (3.11)$$

Minimizing T_0 with respect to a, we find the critical Taylor number for the onset of instability. The result, as stated in the Introduction, is within 1% of the exact numerical answer. The one-mode truncation is thus found to be a very good approximation and can be used effectively for studying the modulated system.

To proceed with the $O(\epsilon)$ equations as shown in Eqs. (3.5a) and (3.5b), we first introduce the Fourier expansion of the functions $g(z)v_0$ and $f(z)u_0$ as

$$g(z)v_0=G\sin(\pi z) \quad (3.12)$$

and

$$f(z)u_0=F\sin(\pi z)\,, \quad (3.13)$$

where

$$G=2\int_0^1 g(z)v_0\sin(\pi z)dz=\frac{4\pi^2}{4\pi^2+(\alpha d)^2}\left|\frac{\cosh(\alpha d)-1}{\alpha d\sinh(\alpha d)}\right|\,, \quad (3.14)$$

and

$$F=2\int_0^1 f(z)u_0\sin(\pi z)dz\,. \quad (3.15)$$

For $\omega\to 0$,

$$F(\omega=0)=\frac{\pi^2+a^2}{T_0a^2}\,, \quad (3.16)$$

while for $\omega\to\infty$,

$$F(\omega)\sim\frac{1.5}{(\alpha d)^3}\,. \quad (3.17)$$

The solution of Eqs. (3.5a) and (3.5b) proceeds by making the ansatz

$$v_1(z)=\tilde{A}_1\sin(\pi z)\,, \quad (3.18)$$

obtaining u_1 from Eq. (3.5a) by using the proper boundary conditions, and finally determining $\tilde{A}_1$ from Eq. (3.5b). The time dependence has the form $e^{i\omega t}$. Straightforward, but tedious, algebra leads to

$$u_1=P\cosh(az)+Q\sinh(az)+R\sinh(bz)+S\cosh(bz)+\frac{\tilde{A}_1(1-z)\sin(\pi z)}{(\pi^2+a^2)(\pi^2+b^2)}-\frac{2\tilde{A}_1(2\pi^2+a^2+b^2)\cos(\pi z)}{(\pi^2+a^2)^2(\pi^2+b^2)^2}+\frac{G\sin(\pi z)}{(\pi^2+a^2)(\pi^2+b^2)}\,, \quad (3.19)$$

where

$$b^2=a^2+i\omega\,, \quad (3.20)$$

$$P\Delta=X[b^2\sinh a\sinh b+ab(\cosh b+1)(1-\cosh a)]+Y(a\cosh a\sinh b-b\sinh a\cosh b)+z(a\sinh b-b\sinh a)\,, \quad (3.21)$$

$$Q\Delta=X[ab\sinh a(1+\cosh b)-b^2\sinh b(1+\cosh a)]+Y(b\cosh a\cosh b-a\sinh a\sinh b-b)+Z(b\cosh a-b\cosh b)\,, \quad (3.22)$$

$$R\Delta=X(ab\sinh b+ab\sinh b\cosh a-a^2\sinh a\cosh b-a^2\sinh a)+Y(a\cosh a\cosh b-b\sinh a\sinh b-a)+Z(a\cosh b-a\cosh a)\,, \quad (3.23)$$

$$S\Delta=X[a^2\sinh a\sinh b-ab(1+\cosh a)(\cosh b-1)]+Y(b\sinh a\cosh b-a\cosh a\sinh b)+Z(b\sinh a-a\sinh b)\,, \quad (3.24)$$

$$X=\frac{2\pi A_1(2\pi^2+a^2+b^2)}{(\pi^2+a^2)^2(\pi^2+b^2)^2}\,,\quad Y=\frac{\pi(A_1+G)}{(\pi^2+a^2)(\pi^2+b^2)}\,,\quad Z=\frac{G}{(\pi^2+a^2)(\pi^2+b^2)}\,, \quad (3.25)$$

$$\Delta=(a^2+b^2)\sinh a\sinh b+2ab-2ab\cosh a\cosh b \quad (3.26)$$

and

$$\tilde{A}_1=\frac{T_0a^2F}{\pi^2+b^2}\,\frac{1+\frac{G}{F}\mathscr{L}}{1-\frac{T_0a^2}{2(\pi^2+b^2)}\mathscr{L}} \quad (3.27)$$

with

$$\mathscr{L}=\frac{1}{(\pi^2+a^2)(\pi^2+b^2)}+\frac{2\pi^2(b^2-a^2)}{[\Delta(\pi^2+a^2)^2(\pi^2+b^2)^2]}\times[a\sinh b(1+\cosh a)-b\sinh a(1+\cosh b)]\,. \quad (3.28)$$

We now turn to the $O(\epsilon^2)$ system shown in Eqs. (3.6a) and (3.6b). The solvability condition yields

$$\left\langle (u_0^\dagger \;\; v_0^\dagger) \begin{bmatrix} g(z)v_1 e^{i\omega t} \\ T_2 a^2 u_0 + T_0 a^2 u_1 f(z) e^{i\omega t} \end{bmatrix} \right\rangle = 0 , \qquad (3.29)$$

where the large angle brackets denote averaging with respect to time and $u_0^\dagger$ and v_0 are the left vectors satisfying [with $(\partial/\partial t)(\;)=0$]

$$(D^2-a^2)^2 u_0^\dagger = -T_0 a^2 v_0^\dagger ,$$
$$(D^2-a^2) v_0^\dagger = (1-z) u_0^\dagger \qquad (3.30)$$

with the boundary conditions that $v_0^\dagger=0$ at $z=0$ and 1 and $u_0^\dagger$ and $du_0^\dagger/dz=0$ at $z=0$ and 1. Equation (3.29) yields

$$T_2 a^2 (v_0^\dagger, u_0) = -\tfrac{1}{2}(u_0^\dagger, g(z)v_1(z)) - \frac{T_0}{2} a^2 (v_0^\dagger, f(z)u_1(z)) , \qquad (3.31)$$

where the factor $\frac{1}{2}$ comes from the time averaging. Thus,

$$T_2 = -\mathrm{Re}\frac{1}{2}\left[T_0 \frac{(v_0, f(z)u_1(z))}{(v_0,u_0)} + \frac{(u_0^\dagger, g(z)v_1(z))}{(v_0^\dagger,u_0)a^2} \right] . \qquad (3.32)$$

Explicit expressions can be written down for the integrals on the right-hand side of Eq. (3.32). However it is more worthwhile to examine the expression in the two limits of low and high frequencies.

In the low-frequency regime $f(z)\to 1$ and $g(z)\to 1-z$. We now make use of Eqs. (3.4), (3.5), and (3.30) to arrive at

$$\lim_{\omega\to 0} \frac{(u_0^\dagger, g(z)v_1(z))}{a^2(v_0^\dagger,u_0)} = \lim_{\omega\to 0} \frac{\int_0^1 dz u_0^\dagger(1-z)v_1(z)}{a^2 \int_0^1 dz\, v_0^\dagger(z)u_0(z)}$$
$$= \lim_{\omega\to 0} \frac{\int_0^1 (D^2-a^2)v_0^\dagger v_1(z)dz}{a^2 \int_0^1 v_0(z)u_0(z)dz}$$
$$= -\lim_{\omega\to 0} \frac{A_1}{2} T_0 \qquad (3.33)$$

and

$$\lim_{\omega\to 0} T_0 \frac{(v_0, f(z)u_1(z))}{(v_0,u_0)} = \lim_{\omega\to 0} T_0 \int_0^1 \frac{v_0(z)u_1(z)dz}{\int_0^1 v_0(z)u_0(z)dz}$$
$$= -\lim_{\omega\to 0} \frac{\int_0^1 (D^2-b^2)v_1(z)v_0(z)dz + T_0 a^2 \int_0^1 v_0(z)u_0(z)dz}{a^2 \int_0^1 v_0(z)u_0(z)dz}$$
$$= \lim_{\omega\to 0} -T_0 + \frac{(\pi^2+b^2)}{\pi^2+a^2}\tilde{A}_1 T_0 . \qquad (3.34)$$

Inserting in Eq. (3.32)

$$T_2(\omega=0) = \lim_{\omega\to 0} \frac{T_0}{2}\left[1 - \frac{i\omega\tilde{A}_1}{\pi^2+a^2} - \frac{\tilde{A}_1}{2} \right] . \qquad (3.35)$$

Using Eq. (3.27) to arrive at the limits and $a\simeq 3.12$, we find

$$\frac{T_2(\omega=0)}{T_0} \simeq -0.10 , \qquad (3.36)$$

in good agreement with Hall's value[11] of -0.07.

For high-frequency modulation we observe that the $O(\epsilon)$ term in Eq. (3.1a) is smaller than the $O(\epsilon)$ term in Eq. (3.1b) by a factor of αd where $\alpha d\to\infty$ in the high-frequency limit. Thus we expect that in the high-frequency limit the modulation effect will be present in the v equation alone and the structure would be the same as the corresponding Rayleigh-Bénard problem in Eqs. (2.4a) and (2.4b). To see this, we note that as $\omega\to\infty$, $G/F\to$const [Eqs. (3.14)–(3.17)], $\mathscr{L}\to 0$ [Eq. (3.28)], and thus $\tilde{A}_1$ has the same form as the corresponding A_1 [Eq. (2.23)] for the Rayleigh-Bénard problem. Finally we note that the first term in Eq. (3.32) dominates in the limit $\omega\to\infty$ because the wave functions v_0 and u_1 of the first term and $u_0^\dagger$ and v_1 of the second have similar structures and $G(z)$ is smaller than $F(z)$ by a factor $\omega^{1/2}$. Thus the threshold shifts in the Taylor-Couette and Rayleigh-Bénard problems have identical structures in the large frequency limit. We find

$$\frac{T_2}{T_0} \simeq \frac{1.2}{\omega^5} \qquad (3.37)$$

in disagreement with Hall's calculation ($T_2/T_0 \simeq -\omega^{-3}$), both in the sign and in the asymptotic power law. This is our main new result.

IV. CONCLUSION AND COMPARISON WITH PREVIOUS WORK

We have developed a systematic perturbation theory for treating the effect of modulation in presence of rigid boundary conditions. Our results supplement those of Venezian[1] for the ideal free-boundary conditions in the case of RB geometry. For the Taylor-Couette flow we provide an alternate approach to that of Hall.[11] Numerical results for both Rayleigh-Bénard and Taylor-Couette geometry exist for the rigid boundary conditions and we shall now compare our results with these, wherever applicable. Truncated systems[4,12] yield similar results.

For the RB problem the relevant numerical work is that of Rosenblat and Tanaka.[13] To compare we choose the modulation amplitude $\epsilon=0.4$, the value at which Ahlers *et al.*[14] have compared their results (using Lorenz-like truncated equations) with Rosenblat and Tanaka.[13] For both $\sigma=10$ and 1, our results agree with that of Ref. 13 within 5% for *all* frequencies. This clearly establishes the soundness of our perturbative approach.

We now turn to the Taylor-Couette problem and set $\epsilon=0.5$. For low frequencies ($\omega \lesssim 50$ in units of ν/d^2), the result shows a clear destabilization and the quantitative agreement with Hall is excellent. At higher frequencies, we find a stabilization. Hall[11] reports a destabilization in this range. Riley and Laurence's[15] numerical procedure is not sensitive enough to find the shift in T at such high frequencies for low ϵ. However, we note that for $\epsilon > 1$, Riley and Laurence[15] do observe a stabilization at higher frequencies in qualitative agreement with our work. The sudden disappearance of this effect from their results as ϵ is lowered is most probably due to the diminishing size of the effect and the consequent numerical difficulties. As for the experimental situation, Donnelly[16] reports stabilization at all frequencies. The later experiments of Thompson[17] show a low-frequency destabilization but a high-frequency stabilization.

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