Spontaneous generation of coherent optical beats

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We represent two groups of excited atoms which radiate at slightly different frequencies by means of a pair of inverted harmonic oscillators coupled to the radiation field. The radiation emitted spontaneously by these oscillators is amplified exponentially and the field generated by each exerts a strong dynamical influence on the other. For the regime in which the amplification rate is large compared to the frequency difference of the uncoupled oscillators, the indirect coupling via the field tends to lock the two systems in phase so that they radiate in unison. When the amplification rate is small compared to the frequency difference, however, each oscillator undergoes, in addition to its own spontaneous oscillation, a partially coherent oscillation forced at the frequency of the other oscillator. These mutually induced oscillations give rise to intensity beats in the radiated field. Such beats have predetermined phases which are independent of the random initial phases of the oscillator amplitudes. The beats can be present and detectable in statistical terms even before a single photon has been emitted on the average.

I. INTRODUCTION

Beating is one of the most familiar of oscillatory phenomena. Whenever two sources of monochromatic radiation have slightly different frequencies there is a possibility of observing in their superposed fields a periodic modulation of intensity with frequency given by the difference of the two source frequencies. While the condition stated is a necessary one for the observability of beats, it is not in general a sufficient one, since there is also a rather subtle condition of coherence to be satisfied. Beats, when they are observable, often have an arbitrary phase of oscillation, a phase that depends on the precise initial conditions of both source oscillators. If the preparation of those systems is described by an ensemble that averages over those initial conditions, then the beats will typically be averaged out in phase and disappear from the mean output field. That is the reason, expressed in classical terms, why no beats are observed as a rule in the field generated by two atoms radiating at slightly different frequencies. Beats can ordinarily be observed in such a field only by exciting the atoms in a way which preserves some special phase relation between their excitation amplitudes.

In view of the foregoing properties of beats, considerable interest is attached to the observation that they can occur in fact with fixed phases, that is to say coherently, even when the initial atomic excitations have no special phase relations. That is the case, for example, in the quantum-beating phenomena discovered in superfluorescent pulses by Vrehen, Hikspoors, and Gibbs,[1] and it is our purpose to explain the mechanism by which it takes place. In their experiment the two systems of atoms radiating at slightly different frequencies are quite unrelated in the initial phases of their oscillations. Still, a periodic modulation of the field intensity develops spontaneously and oscillates with a predetermined phase. That happens because the photons emitted spontaneously by each group of atoms induce in the other group of atoms an oscillation at a frequency slightly foreign to it. The oscillations induced in each group are partially coherent with their own spontaneous oscillations and can thus lead to beats with well-defined phases.

The interactions we are describing are part of the natural process by which such a compound system radiates spontaneously. Their mathematical treatment can be made quite transparent by introducing a simplified model of the compound system which nonetheless retains its essential physical features.

Let us consider a large collection of identical two-level atoms each of which radiates at a frequency ω. If, as in the superfluorescence experiments, the atoms are all or nearly all excited, the energy levels of the system extend downward from a certain maximum value in integer multiples of $\hbar\omega$. In this sense the system of atoms is rather like a harmonic oscillator of frequency ω with the sign of its energy inverted.[2] When the atoms are all coupled symmetrically to another system such as a radiation field, the analogy is further enhanced.[3,4] If the number of atoms is sufficiently large the dynamical behavior of the total atomic dipole moment can be shown to be identical (in permutation-symmetrical states) to the behavior of a suitably defined coordinate for the inverted harmonic oscillator.[5] All that one loses then, by using an inverted oscillator to describe the system of radiating atoms, is a description of the eventual depletion of excited atoms due to the finiteness of their number. In practice that means that only the initial stages of a radiation process can be described by the inverted-oscillator model, but these stages, in fact, allow sufficient time for the field strengths to grow to classical magnitudes and to become easily detectable. To deal with the later stages of the process by taking depletion into account, for example, would require dealing with systems of nonlinear equations for fields that are no longer of quantum-mechanical magnitude. The harmonic-oscillator model, while it is limited to describing the initial phases of superfluorescent radiation leads to a system of linear equations on the other hand, which we can easily solve accurately.

To describe the beats that can occur in spontaneous emission we require two inverted harmonic oscillators of slightly different frequencies; one to represent each homogeneous group of atoms. When these oscillators are coupled to the field, they radiate spontaneously and their radiation field amplifies exponentially. At the same time they exert a strong dynamical influence on one another through the medium of the radiated field. This influence can be felt, as we shall show, either through relative phase locking or through beating of their amplitudes. Which of these takes place depends principally on the ratio of the amplification constant for the field to the difference of frequencies of the uncoupled oscillators. When the amplification rate is large relative to the frequency difference, for example, the oscillators tend to lock together in phase and behave as a single one. When the amplification rate is slow, on the other hand, relative to the frequency difference, the two oscillators induce beating oscillations in one another and these are rendered visible as beats in the radiated field. We shall demonstrate these behaviors and discuss them in detail in the sections that follow.

II. THE MODEL: HEISENBERG-PICTURE SOLUTION

We first briefly review the properties of a single inverted harmonic oscillator[2–4] without coupling to an electromagnetic field. The Hamiltonian for this case is simply

$$H=-\hbar\omega a^\dagger a\ .$$

The operators a and $a^\dagger$ satisfy the Bose commutation relation

$$[a,a^\dagger]=1\ .$$

The Hamiltonian possesses the same eigenfunctions as the familiar harmonic oscillator, but the spectrum is bounded from above rather than below since both kinetic and potential energy are reversed in sign. The vacuum state $|0\rangle$ of the oscillators, defined by $a\,|0\rangle=0$, corresponds to the state of highest energy. The physical interpretations of the operators a and $a^\dagger$ are interchanged relative to those of the usual oscillator: Application of operator a excites the inverted oscillator to a higher energy level. The operator $a^\dagger a$ may be viewed as the number operator for deexcitations of the inverted oscillator. When coupled appropriately to a field, the inverted oscillator, as we shall see, may serve as a simple model for a linear amplifier.[3,4]

We generalize this model to include two species of radiatively coupled inverted oscillators. The contribution of the electromagnetic field energy is expressed as a sum over normal Bose creation and annihilation operators, $b_{\mathbf{k}}^\dagger$ and $b_{\mathbf{k}}$, respectively, one for each mode $\mathbf{k}$ of energy $\hbar\omega_{\mathbf{k}}$.

The total Hamiltonian for our system is then

$$H=-\sum_{j=1}^{2}\hbar\omega_j a_j^\dagger a_j+\sum_{\mathbf{k}}\hbar\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}+H_I\ , \tag{1}$$

where

$$H_I=\hbar\sum_{j=1}^{2}\sum_{\mathbf{k}}(\lambda_{j\mathbf{k}}^* a_j^\dagger b_{\mathbf{k}}^\dagger+\text{H.c.})\ . \tag{2}$$

In Eq. (1), ω_j is the transition frequency of the jth species of the inverted oscillators $(j=1,2)$ and $\lambda_{j\mathbf{k}}$ is the dipole matrix element which determines the strength of the coupling of the jth inverted oscillator with the $\mathbf{k}$ mode of the electromagnetic field. The field-atom coupling H_I retains only the terms which play a dominant role in the rotating-wave approximation. Antiresonant terms which would make much smaller contributions have been omitted.

At $t=0$ we assume the system to be in the state with zero occupation numbers for all its modes:

$$|0\rangle_T=|0\rangle_{A_1}|0\rangle_{A_2}|\{0\}\rangle_F\ . \tag{3}$$

The subscripts A_1 and A_2 indicate the inverted-oscillator species and the subscript F denotes the product state vector of all the electromagnetic field modes. Both oscillator states in Eq. (3) represent states of maximum energy, while the field state is the vacuum state.

It is not difficult to verify the conservation of the operator:

$$\hat{N}=\sum_{j=1}^{2}a_j^\dagger a_j-\sum_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\ . \tag{4}$$

Thus, if initially the system is in state $|0\rangle_T$, then the operator $\hat{N}$ has the constant eigenvalue zero. We shall discuss the subsequent time-dependent behavior of the system in this section by making use of the Heisenberg picture.

We identify the operators which appear in Eqs. (1) and (4) with their values at $t=0$, i.e., $a_j=a_j(0)$, etc. Their values at other times are then given by, e.g.,

$$a_j(t)=U^\dagger(t)a_j(0)U(t)\ , \tag{5}$$

where $U(t)$ is the unitary time-evolution operator and $U^\dagger(t)$ is its adjoint operator.

If the Heisenberg state of the system is $|0\rangle_T$, it follows from the constancy of $\hat{N}$ that at all times the average number of photons equals the average number of deexcitations:

$$n(t)\equiv {}_T\left\langle 0\left|\sum_{\mathbf{k}}b_{\mathbf{k}}^\dagger(t)b_{\mathbf{k}}(t)\right|0\right\rangle_T$$

$$={}_T\left\langle 0\left|\sum_{j=1}^{2}a_j^\dagger(t)a_j(t)\right|0\right\rangle_T\ . \tag{6}$$

The Heisenberg equations of motion for the operators $a_j(t)$ and $b_{\mathbf{k}}^\dagger$ are

$$\dot{a}_j(t)=i\omega_j a_j(t)-i\sum_k \lambda_{j\mathbf{k}}^* b_{\mathbf{k}}^\dagger(t) \ , \quad (7)$$

$$\dot{b}_{\mathbf{k}}^\dagger(t)=i\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger(t)+i\sum_{j=1}^{2}\lambda_{j\mathbf{k}} a_j(t) \quad (8)$$

along with the corresponding equations for the adjoint operators. In its time-integrated form, Eq. (8) is

$$b_{\mathbf{k}}^\dagger(t)=e^{i\omega_{\mathbf{k}}t}b_{\mathbf{k}}^\dagger(0)+i\sum_{j=1}^{2}\lambda_{j\mathbf{k}}\int_0^t dt'\,a_j(t')e^{i\omega_{\mathbf{k}}(t-t')} \ , \quad (9)$$

and when this relation is substituted into Eq. (7), we find

$$\dot{a}_j(t)=i\omega_j a_j(t)+\sum_{j'}\int_0^t dt'\sum_{\mathbf{k}}\lambda_{j\mathbf{k}}^*\lambda_{j'\mathbf{k}}e^{i\omega_{\mathbf{k}}t'}a_{j'}(t-t')$$
$$-i\sum_{\mathbf{k}}\lambda_{j\mathbf{k}}e^{i\omega_{\mathbf{k}}t}b_{\mathbf{k}}^\dagger(0) \ . \quad (10)$$

It is convenient at this point to introduce the Laplace transform of the operators:

$$\bar{a}_j(s)\equiv\int_0^\infty dt\,e^{-st}a_j(t) \ . \quad (11)$$

Expressed as an equation for the transform of $\bar{a}_j$, Eq. (10) becomes

$$(s-i\omega_j)\bar{a}_j(s)=\sum_{j'}\Gamma_{jj'}(s)\bar{a}_{j'}(s)+A_j \quad (j,j'=1,2) \quad (12)$$

in which we have introduced the abbreviations

$$\Gamma_{jj'}(s)\equiv\sum_{\mathbf{k}}\frac{\lambda_{j\mathbf{k}}^*\lambda_{j'\mathbf{k}}}{s-i\omega_{\mathbf{k}}} \quad (13a)$$

and

$$A_j\equiv a_j(0)-i\sum_{\mathbf{k}}\frac{\lambda_{j\mathbf{k}}^*}{s-i\omega_{\mathbf{k}}}b_{\mathbf{k}}(0) \ . \quad (13b)$$

The solution to the pair of linear equations in (12) may be rewritten as

$$\begin{bmatrix}\bar{a}_1(s)\\ \bar{a}_2(s)\end{bmatrix}=\frac{1}{(s-\tilde{s}_+)(s-\tilde{s}_-)}\times\begin{bmatrix}s-i\omega_2-\Gamma_{22}(s) & \Gamma_{21}(s)\\ \Gamma_{12}(s) & s-i\omega_1-\Gamma_{11}(s)\end{bmatrix}\begin{bmatrix}A_1(s)\\ A_2(s)\end{bmatrix} \ , \quad (14)$$

where we have used

$$\tilde{s}_\pm\equiv\tfrac{1}{2}[\Gamma_{11}(s)+\Gamma_{22}(s)]+i\bar{\omega}\pm\left[\Gamma_{12}(s)\Gamma_{21}(s)-\left[\Delta-i\frac{\Gamma_{11}(s)-\Gamma_{22}(s)}{2}\right]^2\right]^{1/2} \ , \quad (15)$$

$$\bar{\omega}\equiv\tfrac{1}{2}(\omega_1+\omega_2), \quad \Delta\equiv\tfrac{1}{2}(\omega_1-\omega_2) \ . \quad (16)$$

In the following we have chosen $\omega_1>\omega_2$ so that Δ is positive. This choice, of course, involves no loss of generality. If each of the inverted oscillators were coupled to a subset of field modes and these subsets had no modes in common, then from Eq. (13a) we would have $\Gamma_{jj'}=0$ for $j\neq j'$. It is clear from Eq. (14) that in that case the system would consist of two completely decoupled inverted-oscillator modes and the properties of such systems have already been described in detail.[4]

Because it is precisely the effect of coupling between the inverted oscillators that interests us most here, we shall assume that both of the inverted oscillators are coupled to all of the field modes. To secure some simplification in notation, we shall assume additionally that the coupling constants $\lambda_{j\mathbf{k}}$ are independent of j, so that both inverted oscillators are coupled equally to any field mode. Thus, the matrix elements $\Gamma_{jj'}$ all become identical and we drop their subscripts:

$$\Gamma(s)\equiv\sum_{\mathbf{k}}\frac{|\lambda_{\mathbf{k}}|^2}{s-i\omega_{\mathbf{k}}} \ , \quad (17)$$

as a consequence the roots $\tilde{s}_\pm$ in Eq. (15) reduce to

$$\tilde{s}_\pm=i\bar{\omega}+\Gamma(s)\pm[\Gamma(s)^2-\Delta^2]^{1/2} \ . \quad (18)$$

To transform the solutions back to the time domain we must invert the Laplace transforms $\bar{a}_j(s)$. This is a problem of the type which has already been extensively discussed in connection with the radiative damping of a single harmonic oscillator. That problem is characteristically simplified by assuming that the amplitude variations of the operators a_j are slow compared to the fundamental oscillations of frequency ω_j, and by thus making an approximation analogous to the Weisskopf-Wigner approximation of radiation damping theory.[5,6] We shall limit ourselves then to considering cases in which the number of field modes is so large as to be effectively infinite. We shall assume further that the frequency difference Δ is quite small compared to the mean frequency $\bar{\omega}$. We can then make the corresponding approximation in the context of our amplification process by neglecting the s dependence of the function $\Gamma(s)$ and replacing it by a suitably chosen complex constant.

The choice that we make corresponds to the summation of Eq. (17) at a point $s=i\bar{\omega}+\epsilon$ and taking the limit as ϵ goes to positive zero:

$$\kappa+i\eta\equiv\lim_{\epsilon\to 0}\Gamma(i\bar{\omega}+\epsilon) \ , \quad (19)$$

where

$$\eta=-P\sum_{\mathbf{k}}\frac{|\lambda_{\mathbf{k}}|^2}{(\bar{\omega}-\omega_{\mathbf{k}})} \quad (20)$$

and

$$\kappa=\pi\sum_{\mathbf{k}}|\lambda_{\mathbf{k}}|^2\delta(\bar{\omega}-\omega_{\mathbf{k}}) \ . \quad (21)$$

The latter summation for κ may also be written as

$$\kappa=\pi|\lambda_{\bar{\omega}}|^2 g(\bar{\omega}) \ , \quad (22)$$

where $g(\bar{\omega})$ is the spectral density of the field modes at frequency $\bar{\omega}$ and $\lambda_{\bar{\omega}}$ is the coupling strength at that frequency. It is clear from the structure of the roots $\tilde{s}_\pm$ as given by Eq. (18) that the constant Γ introduces both amplification and frequency shifts. Were $|\Gamma|$ small compared to Δ, those would be the only changes it introduces, and κ would be the characteristic amplification constant and η the frequency shift. We are interested in considering situations, however, in which the amplification constant is comparable in magnitude to the frequency difference Δ and cannot therefore completely neglect the role of Γ in the last term of Eq. (18) for the roots $\tilde{s}_\pm$. The difference of the imaginary parts of roots $\tilde{s}_\pm$ will correspond, as we shall see, to the beat frequency. It follows then that Γ plays a role in determining the beat frequency as well.

In experiments on damping and amplification there is usually no way of determining the frequency shift which accompanies the process, nor is there any need to determine it in order to apply existing theory. In such cases it suffices either to work with a slightly shifted value of the fundamental frequency, or to neglect the frequency shift altogether. In the coupled-oscillator problem that we are considering, the first and most obvious effect of having η, i.e., the imaginary part of Γ, different from zero, can be seen from Eq. (18) to be a shift of the frequency $\bar{\omega}$. Higher-order effects of η include a shift of the frequency difference Δ and slight attenuations of the amplification rate. Since these effects are all difficult to observe or imperceptible, we shall simplify most of the calculations which follow by assuming $|\eta|$ to be negligible in magnitude compared to Δ. This simplification retains all of the essential features of the amplification and beating process; we shall nonetheless reexamine it in Sec. III where we use a different approach to consider explicitly the limit $\Delta \rightarrow 0$.

The general structure of the solutions given by Eq. (14) suggests the introduction of functions $\bar{u}_{jj'}(s)$ and $\bar{v}_{j\mathbf{k}}(s)$ such that

$$\bar{a}_j(s)=\sum_{j'} \bar{u}_{jj'}(s)a_{j'}(0)+\sum_{\mathbf{k}} \bar{v}_{j\mathbf{k}}(s)b_{\mathbf{k}}^\dagger(0) . \tag{23}$$

These functions, which play a major role in the formalism, are defined by ($j=1,2$)

$$\bar{u}_{jj}(s)=\frac{(s-i\omega_{j'})}{(s-s_+)(s-s_-)}-\bar{u}(s) \quad (j'\neq j) , \tag{24a}$$

where

$$\bar{u}(s)=\frac{\kappa}{(s-s_+)(s-s_-)}=\bar{u}_{jj'}(s) \quad (j\neq j') , \tag{24b}$$

and

$$\bar{v}_{j\mathbf{k}}(s)=-\frac{i\lambda_{\mathbf{k}}^*(s-i\omega_{j'})}{(s-s_+)(s-s_-)(s-i\omega_{\mathbf{k}})} \quad (j'\neq j) . \tag{24c}$$

The roots $s_\pm$ are

$$s_\pm=i\bar{\omega}+\kappa\pm(\kappa^2-\Delta^2)^{1/2} . \tag{25}$$

To invert Eqs. (23) to the time domain we make use of the Mellin inversion formula

$$a_j(t)=\frac{1}{2\pi i}\int_c e^{st}\bar{a}_j(s)ds \tag{26}$$

in which the integration path is parallel to the imaginary axis and lies to the right of all singularities of $\bar{a}_j(s)$. By introducing the functions $u_{jj'}(t)$, $u(t)$, and $v_{j\mathbf{k}}(t)$ as inversions of $\bar{u}_{jj'}(s)$, $\bar{u}(s)$, and $\bar{v}_{j\mathbf{k}}(s)$, respectively, we can write the expression for $a_j(t)$ as

$$a_j(t)=\sum_{j'} u_{jj'}(t)a_{j'}(0)+\sum_{\mathbf{k}} v_{j\mathbf{k}}(t)b_{\mathbf{k}}^\dagger(0) . \tag{27}$$

If we introduce as an abbreviation

$$R\equiv(\kappa^2-\Delta^2)^{1/2} , \tag{28}$$

then from Eqs. (24) and (25) we find

$$u_{jj}(t)=e^{(i\bar{\omega}+\kappa)t}\left[\cosh(Rt)-i\frac{(-1)^j\Delta}{R}\sinh(Rt)\right] \tag{29}$$

and

$$u(t)=\frac{\kappa}{R}e^{(i\bar{\omega}+\kappa)t}\sinh(Rt) . \tag{30}$$

We note that these solutions take the initial values required, $u_{jj'}(0)=\delta_{jj'}$.

For the case $\Delta>\kappa$ in which R is purely imaginary, we let

$$r\equiv(\Delta^2-\kappa^2)^{1/2} . \tag{31}$$

The temporal behavior of Eqs. (29) and (30) is then given by

$$u_{jj}(t)=e^{(i\bar{\omega}+\kappa)t}\left[\cos(rt)-i(-1)^j\frac{\Delta}{r}\sin(rt)\right] \tag{32}$$

and

$$u(t)=\frac{\kappa}{r}e^{(i\bar{\omega}+\kappa)t}\sin(rt) . \tag{33}$$

If we introduce the abbreviations

$$u_\pm(t)\equiv e^{\pm irt}, \quad c_\pm\equiv\frac{1}{2}\left[1\pm\frac{\Delta}{r}\right], \quad d\equiv\frac{\kappa}{2ir} ,$$

then we can write the time-dependent solutions for a_1 and a_2 as

$$\begin{aligned}a_1(t)=e^{(i\bar{\omega}+\kappa)t}\{&[c_+u_+(t)+c_-u_-(t)]a_1(0)\\&+d[u_+(t)-u_-(t)]a_2(0)\}\\&+\sum_{\mathbf{k}} v_{1\mathbf{k}}(t)b_{\mathbf{k}}^\dagger(0) ,\end{aligned} \tag{34a}$$

$$\begin{aligned}a_2(t)=e^{(i\bar{\omega}+\kappa)t}\{&d[u_+(t)-u_-(t)]a_1(0)\\&+[c_-u_+(t)+c_+u_-(t)]a_2(0)\}\\&+\sum_{\mathbf{k}} v_{2\mathbf{k}}(t)b_{\mathbf{k}}^\dagger(0) .\end{aligned} \tag{34b}$$

The dependence of the $\{a_j(t)\}$ on their initial values thus factorizes into an amplified oscillation with frequency $\bar{\omega}$ and a slowly varying modulation with frequency r. The dynamics of the inverted oscillators is strongly influenced by the mixing term $d(u_+-u_-)$, which enters the two solutions in a symmetrical way. These coupling terms vanish periodically at times $t_n=n\pi/r$. The effect of this periodic decoupling will be evident in the statistical properties of the system.

The functions $\bar{v}_{j\mathbf{k}}(s)$ are linearly related to the $\bar{u}_{jj'}(s)$, as can be seen from Eqs. (24):

$$\bar{v}_{j\mathbf{k}}(s)=-\frac{i\lambda_{\mathbf{k}}^*}{(s-i\omega_{\mathbf{k}})}\sum_{j=1}^{2}\bar{u}_{jj'}(s)\ . \tag{35}$$

To complete the solution of the system of coupled Eqs. (7) and (8) we may introduce functions $y_{\mathbf{k}j}(t)$ and $x_{\mathbf{k}\mathbf{k}'}(t)$ to describe the temporal evolution of the field amplitude operators $b_{\mathbf{k}}^{\dagger}$:

$$b_{\mathbf{k}}^{\dagger}(t)=\sum_{j}y_{\mathbf{k}j}(t)a_j(0)+\sum_{\mathbf{k}'}x_{\mathbf{k}\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}(0)\ . \tag{36}$$

The Laplace transforms of the functions $x_{\mathbf{k}\mathbf{k}'}(t)$ and $y_{\mathbf{k}j}(t)$ may be found from the transformed version of Eq. (9). They are

$$\bar{y}_{\mathbf{k}j}(s)=i\frac{\lambda_{\mathbf{k}}}{s-i\omega_{\mathbf{k}}}\sum_{j'}\bar{u}_{jj'}(s)\ , \tag{37}$$

$$\bar{x}_{\mathbf{k}\mathbf{k}'}(s)=\frac{1}{(s-i\omega_{\mathbf{k}})}\left[\delta_{\mathbf{k}\mathbf{k}'}+i\lambda_{\mathbf{k}}\sum_{j}\bar{v}_{j\mathbf{k}'}(s)\right]\ . \tag{38}$$

All the functions necessary for the explicit calculation of oscillator and field averages are thus provided by the functions $\bar{u}_{jj'}(s)$ and $\bar{v}_{j\mathbf{k}}(s)$.

Our solutions must satisfy the fundamental commutation relations for the operators $\{a_j\}$ and $\{b_{\mathbf{k}}\}$; these relations provide us with some useful identities. By substituting Eq. (27) and its adjoint into the commutation relations $[a_j(t),a_{j'}^{\dagger}(t)]=\delta_{jj'}$, we find

$$\sum_{i=1}^{2}u_{ji}(t)u_{j'i}^*(t)-\sum_{\mathbf{k}}v_{j\mathbf{k}}(t)v_{j'\mathbf{k}}^*(t)=\delta_{jj'}\ . \tag{39}$$

Similarly, by using the field commutation relation $[b_{\mathbf{k}}(t),b_{\mathbf{k}'}^{\dagger}(t)]=\delta_{\mathbf{k}\mathbf{k}'}$ and Eq. (36), we find

$$\sum_{\mathbf{l}}x_{\mathbf{k}\mathbf{l}}^*(t)x_{\mathbf{k}'\mathbf{l}}(t)-\sum_{j}y_{\mathbf{k}j}^*(t)y_{\mathbf{k}'j}(t)=\delta_{\mathbf{k}\mathbf{k}'}\ . \tag{40}$$

These identities are used in the following sections to help evaluate the statistical properties of the system.

III. THE AVERAGE PHOTON NUMBER AND RELATED QUANTITIES

We are interested in determining the average number of photons in the electromagnetic field at a time t when the system was initially in the vacuum state $|0\rangle_T$ given by Eq. (3).

According to Eq. (6) we may calculate this average from

$$n(t)={}_T\left\langle 0\left|\sum_{j}a_j^{\dagger}(t)a_j(t)\right|0\right\rangle_T\ . \tag{41}$$

By inserting Eq. (27) and its adjoint, we obtain

$$n(t)=\sum_{j}\sum_{\mathbf{k}}|v_{j\mathbf{k}}(t)|^2\ . \tag{42}$$

By using Eq. (39), we have

$$\sum_{\mathbf{k}}|v_{j\mathbf{k}}(t)|^2=\sum_{i=1}^{2}|u_{ji}(t)|^2-1\geq 0\ . \tag{43}$$

As can be seen from Eq. (24), $\sum_i |u_{ji}|^2$ does not depend on j. Thus, each inverted oscillator contributes $\frac{1}{2}n(t)$ to the average photon number.

For the remainder of the paper we make use of a scaled time variable

$$\tau\equiv\kappa t\ , \tag{44}$$

and the dimensionless quantity ϵ, defined as

$$\epsilon\equiv\frac{\Delta^2}{\kappa^2}\ . \tag{45}$$

If we evaluate $n(\tau)$ according to Eqs. (43) and (29)–(33), we find the result

$$n(\tau)=2(e^{2\tau}-1)+4e^{2\tau}\times\begin{cases}\dfrac{\sinh^2(\sqrt{1-\epsilon}\,\tau)}{1-\epsilon}, & \epsilon<1\\[2mm] \tau^2, & \epsilon=1\\[2mm] \dfrac{\sin^2(\sqrt{\epsilon-1}\,\tau)}{\epsilon-1}, & \epsilon>1\end{cases} \tag{46}$$

for the different possible values of ϵ. It is interesting to note that $(d/d\tau)n(\tau)>0$ for all τ.

Had we considered only one inverted oscillator coupled to the field,[4] the average photon number in the field after τ would have been $(e^{2\tau}-1)$. The first term on the right-hand side of Eq. (46) corresponds, therefore, to the output of two completely decoupled inverted oscillator-field systems. The second term on the right-hand side is thus due solely to the radiative interaction between the two inverted oscillators. It is clear from the three forms for that term, or from $s_{\pm}$ given by Eq. (25), that beating occurs only for Δ exceeding the threshold value κ; i.e., $\epsilon>1$.

Let us first discuss in greater detail the case $\epsilon\gg 1$, in which Δ greatly exceeds the amplification constant. In that case, the indirect coupling between the inverted oscillators which is provided by the electromagnetic field is effectively quite weak. The inverted oscillators radiate independently for all practical purposes. The field occupation number rises as the sum of the amplified noise outputs of the two separate oscillators. That behavior of the field intensity which contains no beating effects is shown in Figs. 1 and 2 by the full curves which correspond to $\epsilon\rightarrow\infty$.

As Δ is decreased, and allowed to become comparable to κ, the radiation field emitted by each of the inverted oscillators begins significantly to influence the oscillation of the other. The oscillation which is induced in each of them is proportional to the initial amplitude of the other and contains components of the frequencies centered on both ω_1 and ω_2. It is this mutual forcing of the oscillations at both frequencies that leads to the presence of intensity beats with nonrandom phase. Because the time dependence of the beats is fixed by the dynamics, field strengths of classical magnitude are not required for their detection. They may still be quite detectable in statistical terms when the ensemble-average photon numbers are smaller than unity. This structure is demonstrated in

Figs. 1 and 2 by the short-dashed curves for $\epsilon=10$ and 100. In Fig. 2 the beats, indicated by the departure from the solid curve, are apparent even though the photon number is still too small to apply the classical correspondence principle.

At the particular times $\tau_n=n\pi/\sqrt{\epsilon-1}$, the average photon number is $n(\tau_n)=2(e^{2\tau_n}-1)$, which is precisely the average number of photons emitted by two completely decoupled oscillators. We show in the next section that at these times, τ_n, the radiatively induced correlation between the two oscillators disappears.

When ϵ is decreased to the value unity, that is, when the detuning is matched to the amplification constant, $\Delta=\kappa$, the beats disappear. That is true as long as the sum η given by Eq. (20), which acts principally as a frequency shift, is negligibly small in magnitude compared to Δ.

To decrease ϵ further corresponds then to letting Δ become smaller than κ. To go to the limit $\epsilon=0$, where the frequencies become degenerate, $\Delta=0$, we can no longer assume the absolute value of η much less than Δ. To consider that limit we can return to the equations of motion (12) and note that they are easily diagonalized by rewriting them as equations of motion for $a_+\equiv a_1+a_2$ and $a_-\equiv a_1-a_2$:

$$\dot{a}_+=i\bar{\omega}a_+ +2(\kappa+i\eta)a_+ -2i\sum_{\mathbf{k}}\lambda_{\mathbf{k}}^* e^{i\omega_{\mathbf{k}}t}b_{\mathbf{k}}(0)\ , \qquad (47)$$

and

$$\dot{a}_-=i\bar{\omega}a_-\ . \qquad (48)$$

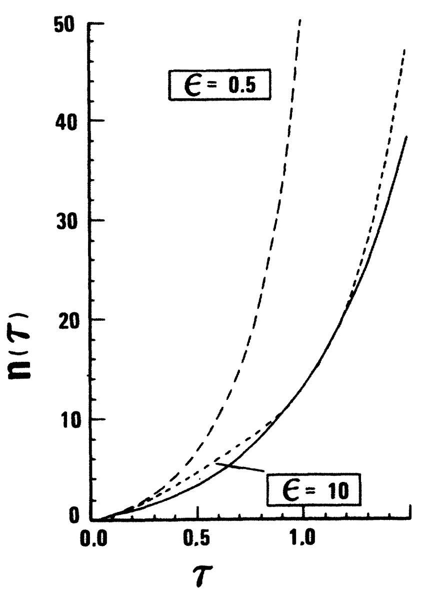

FIG. 1. Time dependence of the average photon number, $n(\tau)$; the time is scaled by the amplification constant, Eq. (44). The solid line represents $n(\tau)$ for two uncoupled inverted oscillators; the curve with $\epsilon=10$ [see Eq. (45)] shows the beating structure as a periodic change of the slope. For $\epsilon=0.5$ the increase of $n(\tau)$ is considerably more rapid than it is for the decoupled oscillators and has no periodic structure.

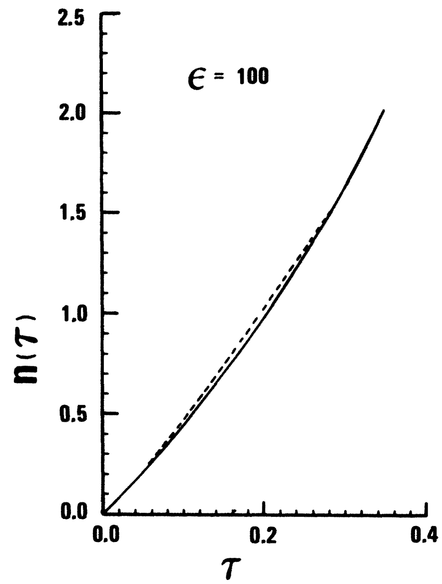

FIG. 2. The time development of $n(\tau)$ for $\epsilon=100$ $(\Delta/\kappa=10)$. Note that the beating structure is identifiable even when there is on the average less than a single photon emitted. The solid curve is the same as in Fig. 1.

The transition frequency of the sum mode has been slightly shifted from $\bar{\omega}$ by an amount 2η; the coupling of the electromagnetic field to the inverted oscillators thus lifts their degeneracy. Any resulting beats would be quite difficult to observe since the mode a_- remains unamplified and the mode a_+ then completely dominates the time development.

The roots $\tilde{s}_\pm$ according to Eq. (18) take the general form

$$\tilde{s}_\pm=i(\bar{\omega}+\eta)+\kappa\pm[\kappa^2-(\Delta^2+\eta^2)+2i\kappa\eta]^{1/2} \qquad (49)$$

for $\eta\neq0$. Large values of the frequency shift η could thus alter the threshold value of Δ, which separates what we have identified as the beat regime from the pure amplification regime.

Further insight is gained by considering the average photon number in the kth mode of the field as a function of the energy of the emitted photons $\omega_{\mathbf{k}}$. This quantity is defined as

$$n_{\mathbf{k}}(\tau)\equiv{}_T\langle 0|b_{\mathbf{k}}^\dagger(\tau)b_{\mathbf{k}}(\tau)|0\rangle_T =\sum_j |y_{\mathbf{k}j}(\tau)|^2=\sum_j |v_{\mathbf{k}j}(\tau)|^2\ . \qquad (50)$$

In the regime $\Delta>\kappa$, the asymptotic form of this quantity for large times is

$$n_{\mathbf{k}}(\tau)=2\frac{|\lambda_{\mathbf{k}}|^2}{\kappa^2}e^{2\tau}\frac{f_+(\tau)x_{\mathbf{k}}^2+f_-(\tau)\epsilon}{(x_{\mathbf{k}}^2-\epsilon)^2+4x_{\mathbf{k}}^2} \qquad (51)$$

with

$$x_{\mathbf{k}} \equiv \frac{\bar{\omega}-\omega_{\mathbf{k}}}{\kappa} ,$$

$$f_{\pm}(\tau) \equiv \frac{\epsilon}{\epsilon-1}\sin^2(\sqrt{\epsilon-1}\tau) + \left[\frac{\sin(\sqrt{\epsilon-1}\tau)}{\sqrt{\epsilon-1}} \pm \cos(\sqrt{\epsilon-1}\tau)\right]^2 .$$

For $\Delta \gg \kappa$ the denominator has its minima at $\omega_{\mathbf{k}}=\bar{\omega}\pm\Delta$ and each of the corresponding peaks in $n_{\mathbf{k}}(\tau)$ has a half-width 2κ. An example for the frequency-dependent part of the function for $\epsilon=100$ has been plotted in Fig. 3 for the particular values of τ at which $f_{\pm}(\tau)=1$ on the assumption that $\lambda_{\mathbf{k}}$ is frequency independent over the range shown. For $\Delta\to\infty$ the overlap between the modes becomes so weak that the inverted oscillators amplify independently. As the detuning is decreased, the overlap between the modes increases; this trend continues until $\Delta=\kappa$, at which point the two peaks are no longer resolved.

For the regime $\Delta<\kappa$, the asymptotic expression, Eq. (51), is no longer valid; the asymptotic result for large times in that case is

$$n_{\mathbf{k}}(\tau) = \left[\frac{|\lambda_{\mathbf{k}}|^2}{\kappa^2} e^{2(1+\sqrt{1-\epsilon})\tau} \frac{1+\sqrt{1-\epsilon}}{1-\epsilon}\right] \times \frac{x_{\mathbf{k}}^2+(\epsilon/1+\sqrt{1-\epsilon})^2}{(x_{\mathbf{k}}^2-\epsilon)^2+4x_{\mathbf{k}}^2} . \quad (52)$$

The shape of this spectral function continues the trend noted earlier. It has a single maximum which decreases in width as ϵ decrease. An example of the spectral function for $\epsilon=0.1$ is shown in Fig. 3. For $\epsilon\to 0$ the width of the function $n_{\mathbf{k}}(\tau)$ tends to 4κ, which is characteristic of two identical inverted oscillators coupled to the same field, described by Eqs. (47) and (48).

We close this section by evaluating a particular fourth-order field correlation function. The total transition rate of the field due to the simultaneous absorption of two photons from the same mode by an ideal detector is given by

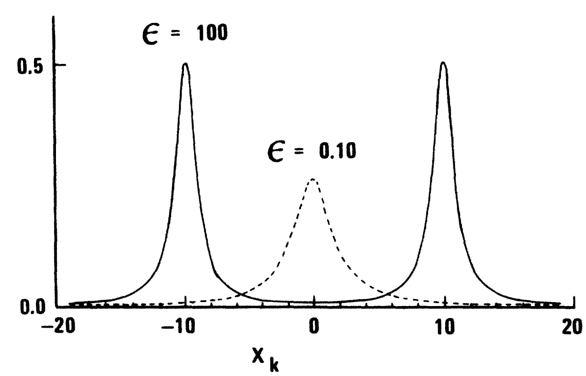

FIG. 3. Spectral distribution of photon numbers $n_{\mathbf{k}}(\tau)$ as a function of $x_{\mathbf{k}}=(\bar{\omega}-\omega_{\mathbf{k}})/\kappa$. Solid curve for $\epsilon=100$, and dashed curve for $\epsilon=0.1$ at times $\tau_n=n\pi/\sqrt{\epsilon-1}$.

$$w_{\mathbf{k}}(t) = \sum_f |\langle f | b_{\mathbf{k}} b_{\mathbf{k}} U(t) | 0\rangle_T|^2 = {}_T\langle 0 | b_{\mathbf{k}}^\dagger(t) b_{\mathbf{k}}^\dagger(t) b_{\mathbf{k}}(t) b_{\mathbf{k}}(t) | 0\rangle_T . \quad (53)$$

By inserting Eq. (36) and its adjoint into Eq. (53), we obtain

$$w_{\mathbf{k}}(t) = 2\left[\sum_j |y_{\mathbf{k}j}(t)|^2\right]^2 = 2\,{}_T\langle 0 | b_{\mathbf{k}}^\dagger(t) b_{\mathbf{k}}(t) | 0\rangle_T^2 = 2n_{\mathbf{k}}^2(t) . \quad (54)$$

This result illustrates the intrinsically Gaussian statistics of the radiation generated by our linear model.

IV. THE DENSITY OPERATOR FOR THE TWO-OSCILLATOR SYSTEM

In this section we consider the reduced Schrödinger density operator for the radiatively coupled inverted-oscillator system. As before, we take

$$\hat{\rho}_T(\tau=0) = |0\rangle_T\,{}_T\langle 0| \quad (55)$$

to be the initial density operator for the entire system of coupled oscillators.

Our interest will be centered principally on the statistical properties of the two oscillators and for that reason we shall calculate only the part of the time-dependent density operator which refers to them. The reduced density operator for these oscillators is defined as the trace of $\hat{\rho}_T(\tau)$ taken over the space of the b modes:

$$\hat{\rho}_{A_1,A_2}(\tau) \equiv \mathrm{Tr}_B \hat{\rho}_T(\tau) .$$

We shall solve for this reduced density operator in the P representation[5] by writing it as

$$\hat{\rho}_{A_1,A_2}(\tau) = \int d^2\gamma_1 d^2\gamma_2 P_{A_1,A_2}(\gamma_1,\gamma_2;\tau) \times |\gamma_1,\gamma_2\rangle\langle\gamma_1,\gamma_2| , \quad (56)$$

in which the integrations extend over the complex planes of γ_1 and γ_2. The states $|\gamma_1,\gamma_2\rangle$ are products of coherent states $|\gamma_j\rangle$, belonging to the jth oscillator ($j=1,2$). The task then is to determine the weight function P_{A_1,A_2}, which expresses the information about the quantum state of the coupled oscillators in terms of coherent states. At this stage our notation for P_{A_1,A_2} does not yet explicitly indicate the initial state of the system.

We employ the method of normally ordered characteristic functions[5,6] to obtain P by introducing the characteristic function

$$\chi_{A_1,A_2}(\mu,\lambda;\tau) \equiv \mathrm{Tr}_T[\hat{\rho}_T(0) e^{\mu a_1^\dagger(\tau)} e^{\lambda a_2^\dagger(\tau)} \times e^{-\mu^* a_1(\tau)} e^{-\lambda^* a_2(\tau)}] . \quad (57)$$

Here μ and λ are two complex parameters. We may solve for the time-dependent P function by observing that there are two ways of evaluating the trace in Eq. (57). On the one hand we can use Eqs. (27) and (34) to evaluate the solutions $a_j(\tau)$ to the Heisenberg equations of motion and then evaluate the trace explicitly. On the other hand, Eq. (57) is equivalent in the Schrödinger picture to

$$\chi_{A_1,A_2}(\mu,\lambda;\tau)=\mathrm{Tr}_{A_1,A_2}\hat{\rho}_{A_1,A_2}(\tau)e^{\mu a_1^\dagger(0)} \times e^{\lambda a_2^\dagger(0)}e^{-\mu^* a_1(0)}e^{-\lambda^* a_2(0)} . \quad (58)$$

When the P representation of Eq. (56) is used to find the expectation value of the normally ordered product in this relation, we see that the characteristic function χ is just the four-dimensional Fourier transform of the weight function P. Inversion of this relation then yields

$$P_{A_1,A_2}(\gamma_1,\gamma_2;\tau)=\frac{1}{\pi^4}\int d^2\mu\, d^2\lambda\, \chi_{A_1,A_2}(\mu,\lambda;\tau) \times e^{\mu^*\gamma_1-\mu\gamma_1^*}e^{\lambda^*\gamma_2-\lambda\gamma_2^*} , \quad (59)$$

which is normalized to unity with respect to integrations over the γ_1 and γ_2 planes.

The solution for the χ_{A_1,A_2} as derived from (57) with the help of Eq. (27) is

$$\chi_{A_1,A_2}(\mu,\lambda;\tau)=\exp[-\mathbf{\Lambda}'\underline{V}(\tau)\mathbf{\Lambda}^*] . \quad (60)$$

Here $\mathbf{\Lambda}'$ is the transposed form of $\mathbf{\Lambda}=\binom{\mu}{\lambda}$. The matrix $\underline{V}(\tau)$ is positive definite and Hermitian for $\omega_1\neq\omega_2$. Following the usage of the probability theory it may be called the (complex) variance-covariance matrix:

$$\underline{V}(\tau)=\begin{bmatrix} v(\tau) & v_{12}^*(\tau) \\ v_{12}(\tau) & v(\tau)\end{bmatrix} \equiv v(\tau)\begin{bmatrix} 1 & \rho^*(\tau) \\ \rho(\tau) & 1\end{bmatrix} , \quad (61)$$

where

$$v(\tau)\equiv\sum_{\mathbf{k}} |v_{j\mathbf{k}}(\tau)|^2 = \tfrac{1}{2}n(\tau)=\tfrac{1}{2}\ {}_T\!\left\langle 0\left|\sum_j a_j^\dagger(\tau)a_j(\tau)\right|0\right\rangle_T \quad (62)$$

and

$$v_{12}(\tau)\equiv\sum_j u_{1j}(\tau)u_{2j}^*(\tau)={}_T\langle 0|a_1(\tau)a_2^\dagger(\tau)|0\rangle_T .$$

This shows that v_{12} determines the correlation between two inverted oscillators via virtual photon exchange, whereas v gives the variance of the weight function P. The notation introduced in Eq. (61),

$$\rho(\tau)\equiv\frac{v_{12}(\tau)}{v(\tau)} , \quad (63)$$

corresponds again to a quantity often used in conventional statistics, where the modulus of $\rho(\tau)$ represents the correlation coefficient, which obeys $|\rho(\tau)|\leq 1$ at all times. By substituting our result for the characteristic function, Eq. (60), in Eq. (59) we obtain the joint weight function for the coupled system of inverted oscillators:

$$P_{A_1,A_2}(\gamma_1,\gamma_2;\tau)=\frac{1}{\pi^2}\frac{1}{\det V(\tau)}\exp[-\mathbf{\Gamma}'\underline{V}^{-1}(\tau)\mathbf{\Gamma}^*] , \quad (64)$$

in which we have used the vector notation $\mathbf{\Gamma}\equiv\binom{\gamma_1}{\gamma_2}$. The determinant of $\underline{V}$ is $\det\underline{V}=v^2(1-|\rho|^2)$ and the weight function is real, positive, and normalized. It corresponds to a two-dimensional Gaussian quasiprobability density for the complex variables γ_1 and γ_2.

Integrating over one of the γ variables corresponds to taking the partial trace over the appropriate oscillator coordinate. When we do that we find for the reduced weight function

$$P_{A_1}(\gamma;\tau)=P_{A_2}(\gamma;\tau)\equiv P(\gamma;\tau)=\frac{1}{\pi v(\tau)}\exp\left[-\frac{|\gamma|^2}{v(\tau)}\right] . \quad (65)$$

The way in which the variance $v(\tau)=\frac{1}{2}n(\tau)$ increases with time is given by Eq. (46).

A useful way of illustrating the effect of correlation between the two inverted oscillators is to define an appropriate conditioned quasiprobability density. To do that it is convenient to split the right-hand side of Eq. (64) into two factors:

$$P_{A_1,A_2}(\gamma_1,\gamma_2;\tau)=\left[\frac{1}{\pi}\frac{1}{v(1-|\rho|^2)} \times\exp\left[-\frac{|\gamma_1-\rho\gamma_2|^2}{v(1-|\rho|^2)}\right]\right]P(\gamma_2;\tau) \equiv P_{A_1|A_2}(\gamma_1|\gamma_2;\tau)P(\gamma_2;\tau) . \quad (66)$$

The large square brackets in Eq. (66) contain a normalized weight function, which we have denoted by $P_{A_1|A_2}(\gamma_1|\gamma_2;\tau)$. For $\rho=0$, this expression is just $P(\gamma_1;\tau)$. More generally $P_{A_1|A_2}$ is the weight we have to give to coherent states of inverted oscillator A_1 with amplitudes near γ_1 at time τ, if at the same time inverted oscillator A_2 is characterized by a given amplitude γ_2.

In the case $\Delta<\kappa$ ($\epsilon<1$), as we shall see, the two oscillators become closely correlated, $|\rho|\to 1$ as $\tau\to\infty$. In that limit $P_{A_1|A_2}$ then becomes a δ function:

$$P_{A_1|A_2}(\gamma_1;\tau)\to\delta^{(2)}(\gamma_1-\gamma_2) ,$$

In the case of strong correlation, which is characteristic of the nonbeating regime $\epsilon<1$, the inverted oscillators are frequency locked to one another and assume the same quantum state. No beating structure can thus be seen in the emitted field. That is consistent with our conclusions in Sec. III, where we discussed the interaction from the standpoint of the field.

The explicit expressions for $|\rho(\tau)|^2$ are

$$|\rho(\tau,\epsilon<1)|^2=\frac{\left[1+\frac{2}{1-\epsilon}\sinh^2(\sqrt{1-\epsilon}\,\tau)\right]^2-1}{\left[1-e^{-2\tau}+\frac{2}{1-\epsilon}\sinh^2(\sqrt{1-\epsilon}\,\tau)\right]^2} \quad (67)$$

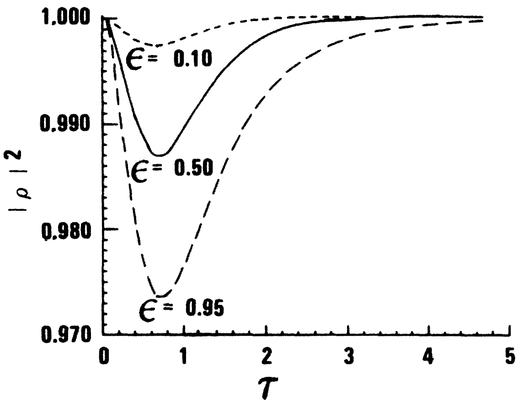

FIG. 4. The correlation coefficient $|\rho(\tau)|^2$ of Eq. (67) for three values of ϵ. After an initial decrease $|\rho(\tau)|^2$ approaches unity as $\tau \to \infty$.

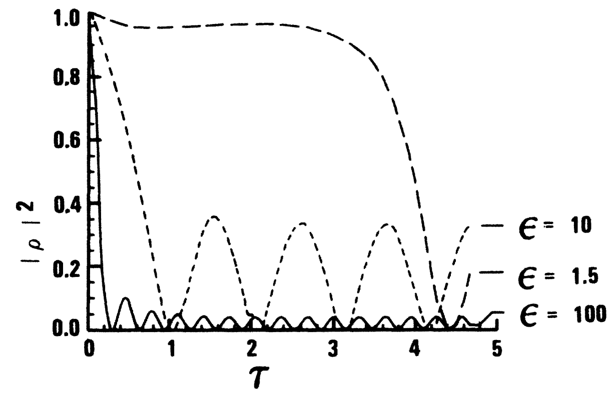

FIG. 5. The correlation coefficient $|\rho(\tau)|^2$ of Eq. (68) for three values of ϵ in the beat regime. The function vanishes periodically in time and has maxima which decrease in magnitude as ϵ increases.

and

$$|\rho(\tau,\epsilon>1)|^2=\frac{\left[1+\frac{2}{\epsilon-1}\sin^2(\sqrt{\epsilon-1}\,\tau)\right]^2-1}{\left[1-e^{-2\tau}+\frac{2}{\epsilon-1}\sin^2(\sqrt{\epsilon-1}\,\tau)\right]^2}\ . \tag{68}$$

These expressions follow from Eqs. (63) and (62) and the functions $u_{jj'}(\tau)$, given by Eqs. (32) and (33). The expressions are plotted in Figs. 4 and 5 for three values of ϵ less than 1 and three values greater than 1.

For $\tau=0$ the system is completely correlated due to our choice of initial state for the oscillators. Immediately afterwards, the correlation begins to decrease because of the spontaneous emission of photons. As soon as there are a few photons in the field the radiative coupling between the inverted oscillators overtakes the spontaneous emission process and the system thereafter behaves more deterministically. For the case $\epsilon<1$ (Fig. 4) the inverted oscillators eventually return to a completely correlated state, whereas for $\epsilon>1$ (Fig. 5) the inverted oscillators become uncorrelated periodically at the times $\tau_n=n\pi/\sqrt{\epsilon-1}$ ($n=1,2,\dots$). At these times, the photon number, Eq. (46), is given by the amplified contribution of pure spontaneous emission noise. The periodic maxima of $|\rho(\tau)|^2$ are reduced in magnitude as ϵ increases. Finally, for $\epsilon\to\infty$ the two inverted oscillators become in effect uncorrelated. This completes our discussion of the density operator $\hat{\rho}_{A_1,A_2}$ of the oscillator system, based on the initial condition $\hat{\rho}_T(0)=|0\rangle_T\,{}_T\langle 0|$.

Other initial states can be treated by similar techniques. Thus we can easily find all of the foregoing quantities when the field modes are initially in coherent states. Then by mixing such initial states we can investigate for example the case

$$\hat{\rho}_T(0)=|0\rangle_{A_1}|0\rangle_{A_2}\,{}_{A_2}\langle 0|\,{}_{A_1}\langle 0|\prod_{\mathbf{k}}\hat{\rho}_{\mathbf{k}}\ , \tag{69}$$

in which the field modes are described initially by the chaotic states

$$\hat{\rho}_{\mathbf{k}}=\frac{1}{\pi\langle n_{\mathbf{k}}\rangle}\int \exp\left[-\frac{|\beta_{\mathbf{k}}|^2}{\langle n_{\mathbf{k}}\rangle}\right]|\beta_{\mathbf{k}}\rangle_{\mathbf{k}}\,{}_{\mathbf{k}}\langle\beta_{\mathbf{k}}|\,d^2\beta_{\mathbf{k}}\ , \tag{70}$$

with mean occupation numbers $\langle n_{\mathbf{k}}\rangle$. The result for $P(\gamma,\tau)$ in that case is simply

$$P(\gamma,\tau)=\frac{1}{\pi}\frac{1}{\sum_{\mathbf{k}}(\langle n_{\mathbf{k}}\rangle+1)|v_{j\mathbf{k}}(\tau)|^2}\times\exp\left[\frac{-|\gamma|^2}{\sum_{\mathbf{k}}(\langle n_{\mathbf{k}}\rangle+1)|v_{j\mathbf{k}}(\tau)|^2}\right]\ , \tag{71}$$

which when compared with Eq. (65) shows the increase in the uncertainty of the quantum state of either of the oscillators due to the initial presence of photons in the field. The factors $\langle n_{\mathbf{k}}\rangle+1$ in Eq. (71) reflect the addition of the effects of induced emission to those of spontaneous emission already present in Eq. (65).

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