Even-harmonic generation in free-electron lasers

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A harmonic-generation mechanism that relies on the coupling modulation between an electron orbit and an electromagnetic mode is proposed. Applications of this mechanism to harmonic generation in linearly polarized free-electron lasers predicts the emission of even-harmonic radiation in the forward direction primarily in the TEM_{01} mode.

I. INTRODUCTION

Second and higher harmonics of electromagnetic waves are generated in oscillators and amplifiers as a consequence of the nonlinear properties of the gain mechanism. Harmonic production can be viewed as a parasitic effect or as a useful mechanism by which higher frequencies can be generated directly. In high-power experiments harmonics have caused mirror damage[1] while the possible uses for harmonics include the precision measurement of frequency and/or time and as diagnostic indicators on fundamental frequency experiments. Regardless of their usefulness, the ubiquity of harmonics necessitates a thorough understanding of their origins so that their appearance can be predicted or manipulated as desired.

A mechanism involving transverse electromagnetic-field inhomogeneities is proposed for the generation of second (even) harmonics in free-electron lasers (FEL's). It uses an even-harmonic transverse current whose space and time modulations are *not* resonant with vacuum-propagating electromagnetic waves. Although this transverse current cannot couple to a plane wave, when a transverse spatial mode exists that is traversed by the electrons (which produce the radiation), the beating of the mode-orbit field with the transverse current produces a resonant excitation of the vacuum-propagating mode. The efficiency of coupling into the mode varies with the position of the electrons, the wiggle-orbit amplitude, and the transverse mode shape. The transverse modes must be selected by the boundary conditions. This interaction process is hereafter referred to as the mode-orbit differential efficiency mechanism (MODEM). In a FEL one manifestation of MODEM is even-harmonic radiation into modes with odd transverse symmetry.

This paper describes a detailed application of MODEM to a FEL. In Sec. II a *Gedankenexperiment* is presented to show how the MODEM for a linearly polarized FEL can be understood through examination of an electron's resonance condition. Section III describes the mode amplitude equations for a linearly polarized FEL, including the MODEM terms. A physical interpretation of the analytical MODEM results, including a comparison of the MODEM coupling coefficients with those of the odd harmonics and with those due to misalignments, is conducted in Sec. IV. The conclusions are given in Sec. V.

II. ELECTRON ENERGY-LOSS MECHANISM

When a wave and electron are near resonance, the electron will exchange energy with the electromagnetic wave. Energy conservation requires the energy lost by the electron to be equal to that gained by the wave (or vice versa). In the classical FEL model, resonance is established when one wavelength of electromagnetic radiation passes over an electron as it executes one wiggle oscillation. Figures 1(a)–1(d) demonstrate the relationship between a test electron's position, velocity, and observed electric-field radiation at resonance. Here the amplitude of the transverse velocity of the electron caused by the wiggler always has the same sign as the electric-field radiation it samples. Therefore, the change in the electron energy with time, given by

$$\frac{d\gamma}{dt} = -\frac{e}{mc^2}\mathbf{v}\cdot\mathbf{E}\ , \tag{1}$$

is always negative so that the electron continually loses energy to the radiation field.

This simple model can now be extended to explain second (even) -harmonic emission. The mechanism is most easily understood if one assumes the electron wiggles in and out of an infinitely sharp-edged second-harmonic radiation field as depicted in Fig. 2. Here the cross-hatched area represents the region of space where no optical field is present. Since the electron samples the $2\omega_s$ field only when outside the cross-hatched area, its $\mathbf{v}\cdot\mathbf{E}$ product will always be greater than zero. The fourth plot of Fig. 2 shows this. Thus, the spatial variation in the optical field has modified the temporal characteristics of $\mathbf{v}\cdot\mathbf{E}$ such that its time average is no longer zero.

Extension of this concept to a system where the electrons sample a finite gradient in the transverse radiation field amplitude can now be explained as follows. The energy lost by an electron as it passes through the high-radiation-field region of its trajectory will be greater than the energy gained during its passage through the low-field region. As a result, the electron will lose a net amount of energy to the radiation field.

Radiation due to MODEM is essentially unavoidable (where $\omega_s \simeq k_s c$) due to the transverse field gradients present in most FEL devices. To understand how the transverse field profile affects the MODEM interaction

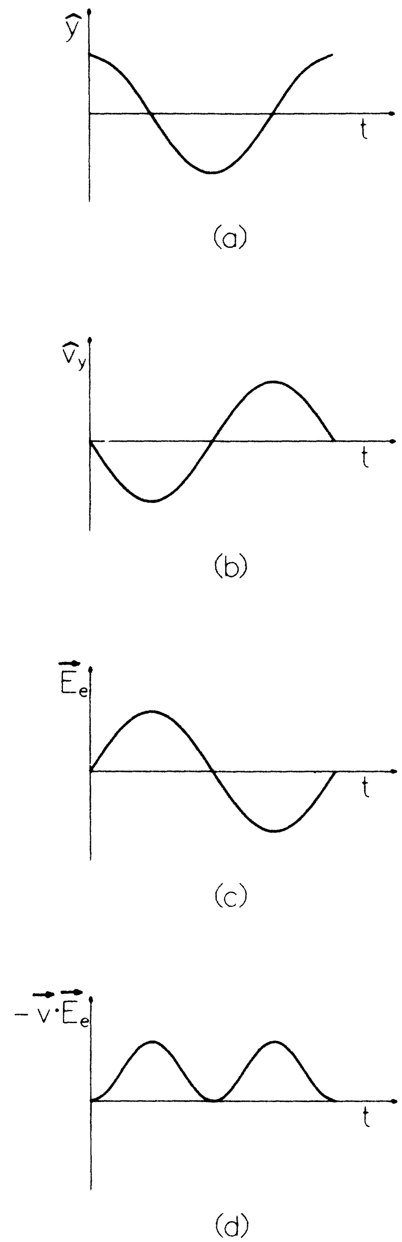

FIG. 1. Parameters for a test electron near fundamental resonance over one wiggle period: (a) transverse position, (b) transverse velocity, (c) observed fundamental electromagnetic field, and (d) product of the transverse velocity and observed electric field which is proportional to the electron's rate of change of energy.

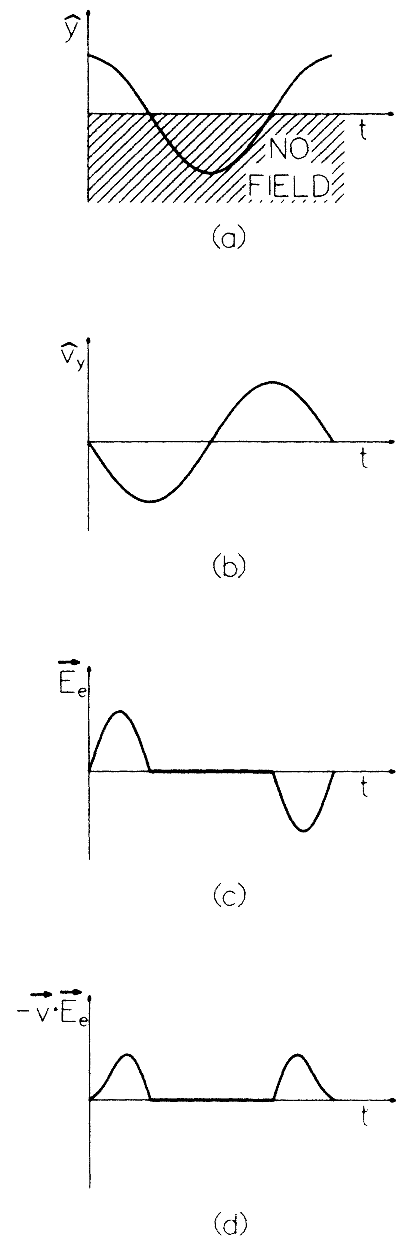

FIG. 2. Parameters for a test electron wiggling in and out of a second-harmonic field: (a) transverse position showing electromagnetic-field boundary, (b) transverse velocity, (c) observed second-harmonic electromagnetic field, and (d) product of the transverse velocity and observed second-harmonic electric field which is proportional to the electron's rate of change of energy.

we shall examine how electrons on either side of the optical axis interact with a perfectly aligned optical beam. If the optical mode has even transverse symmetry, the electrons will sample equal but opposite field gradients on either side of the optical axis. Considering the resonance model above, an electron on one side of the optical axis will radiate (at the second harmonic) exactly out of phase from the corresponding electron on the opposite side of the optical axis. This can be seen schematically in Fig. 3. The electrons' trajectories caused by the wiggler magnetic field are identical except for a transverse displacement. Therefore, while the electron on one side of the optical axis traverses the high-field region, the other electron is in the low-field region. It follows that the resonant electric-field phases for the two electrons differ by 180°, as depicted in Fig. 3(b), so that their contributions will not reinforce the imposed even transverse field profile. Alternatively, if one assumes an odd transverse mode profile, as shown in Fig. 4(a), the resonant electric fields for the two electrons, given in Fig. 4(b), are again 180° out of phase,

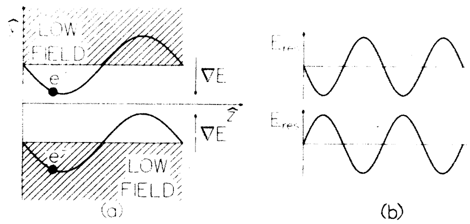

FIG. 3. (a) Symmetrically displaced test electrons wiggling in and out of either side of a cylindrical TEM_{00}-type mode; (b) resonant second-harmonic fields for each electron for optimum deceleration.

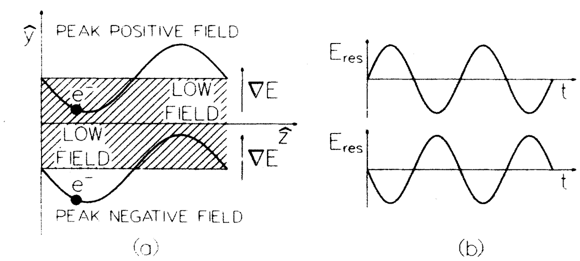

FIG. 4. (a) Symmetrically displaced test electrons wiggling in and out of either side of a cylindrical TEM_{01}-type mode; (b) resonant second-harmonic fields for each electron for optimum deceleration.

but in this case they reinforce the assumed transverse field profile. Therefore, these intuitive arguments suggest that the MODEM radiation will appear in odd transverse modes of the system.

III. ANALYTICAL TREATMENT

To express the even-harmonic interaction analytically we begin with the vector potential form of Ampere's law given by

$$\left[\frac{\partial^2}{\partial z^2}+\nabla_\perp^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathbf{A}_\perp=-\frac{4\pi}{c}\mathbf{J}_\perp\ , \tag{2}$$

where we have assumed a transverse electromagnetic wave and current density. We expand the electromagnetic vector potential in the form

$$\mathbf{A}_\perp=\hat{\mathbf{y}}\sum_{f=1}^{\infty}\frac{a_f(\mathbf{r},z,t)}{2}e^{if(k_s z-\omega_s t)}+\text{c.c.}\ , \tag{3}$$

where $a_f(\mathbf{r},z,t)$ is slowly varying and f is the harmonic number. Assuming the gain is small over an optical wavelength ($\partial_z\mathbf{A}_\perp \ll k_s\mathbf{A}_\perp$) we can make the paraxial approximation, yielding

$$e^{if(k_s z-\omega_s t)}\left[2ifk_s\left[\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right]+\nabla_\perp^2\right]\frac{a_f(\mathbf{r},z,t)}{2}+\text{c.c.}=-\frac{4\pi}{c}\mathbf{J}_\perp\ , \tag{4}$$

where we assumed $\omega_s \simeq k_s c$. Factoring the transverse dependences of $a_f(\mathbf{r},z,t)$ into Gauss-Hermite modes we define

$$a_f(\mathbf{r},z,t)=\sum_{n,m}a_{nm}^f(z,t)A_n(x,z)A_m(y,z)\ , \tag{5}$$

where[2]

$$A_l(x,z)=\frac{1}{\sqrt{w}}\frac{(2/\pi)^{1/4}}{\sqrt{2^l l!}}\left[\frac{w}{w_0}\frac{z_r}{z_r+iz}\right]^{l+1/2}H_l\left[\frac{\sqrt{2}x}{w(z)}\right]e^{-(1-iz/z_r)x^2/w^2} \tag{6}$$

$$z_r=\frac{\pi w_0^2}{\lambda}=\frac{w_0^2 k_s}{2}\ , \tag{7}$$

$$w^2=w_0^2(1+z^2/z_r^2)\ , \tag{8}$$

and $H_l(\alpha)$ are the Hermite polynomials. By definition, each of the modes satisfy the free-space paraxial wave equation and are orthonormal

$$\int_{-\infty}^{\infty}dx\ A_n(x,z)A_m^*(x,z)=\delta_{nm}\ , \tag{9}$$

where δ_{ij} represents the Kronecker δ function. Substituting the new expression for $a(\mathbf{r},z,t)$ into the paraxial wave equation and projecting out the gth harmonic by multiplying through by $e^{-ig(k_s z-\omega_s t)}$ and averaging over one optical period, gives

$$A_n(x,z)A_m(y,z)\delta_{gf}\left[\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right]a_{nm}^f-A_n^*(x,z)A_m^*(y,z)\delta_{-gf}\left[\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right]a_{nm}^{*f}=\frac{i2}{f}\int_t^{t+2\pi/\omega_s}\mathbf{J}_\perp e^{-ig(k_s z-\omega_s t)}dt\ , \tag{10}$$

where we assumed a_{nm}^f relatively constant over the integration interval.

Now, since we are only interested in waves with k vectors in the direction of the electron-beam propagation, both f and g must be positive integers, thereby eliminating the complex-conjugate term. Multiplying through by $A_n^*A_m^*$, integrating over transverse space, and invoking the above orthonormal relation yields

$$\left[\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right]a^f_{nm}(z,t)=\frac{i2}{f}\int_t^{t+2\pi/\omega_s}dt\int dx\int dy\, A^*_n(x,z)A^*_m(y,z)\mathbf{J}_\perp e^{-if(k_s z-\omega_s t)} . \tag{11}$$

To evaluate the integrals on the right-hand side we express $\mathbf{J}_\perp$ explicitly as

$$J_\perp=-en(x,y,z,t)v_y(z,t) , \tag{12}$$

where $n(x,y,z,t)$, $v_y(z,t)$, and e are the electron density, transverse velocity, and charge, respectively. Assuming a linearly polarized wiggler field, $\mathbf{B}_w=-\hat{\mathbf{x}}B_w\sin(k_w z)$, the electrons wiggle in the y direction as they travel down the z axis. Conservation of canonical angular momentum dictates

$$v_y=\frac{eA_y}{mc\gamma}\simeq-\frac{e}{mck_w\gamma}B_w\cos(k_w z) , \tag{13}$$

where $\sum_{f=1}^{\infty}E_f\propto A_s\ll A_w$ has been implied. This expression can be integrated to give an explicit expression for y in terms of z, given by

$$y-y_0=-\chi\sin(k_w z) , \tag{14}$$

where y_0 is the electron's transverse position at $z=0$, $\chi\simeq\kappa/(\gamma k_w)$, and $\kappa=eB_w/(mc^2k_w)$. The higher-order terms in an exact expansion of $y-y_0$ give rise to odd-harmonic terms in k_w which are small ($O\{[\chi/w_e]^{2n+1}\}$) for the conditions considered here. The parameter χ has physical significance since it is the maximum transverse deviation of the electron from its wiggle axis.

Since the electrons wiggle in the $\hat{\mathbf{y}}$ direction, the electron density along the $\hat{\mathbf{y}}$ axis is also a function of z. Assuming normalized Gaussian dependences in the transverse directions and a parametric displacement w_1 in the $\hat{\mathbf{y}}$ direction, we write

$$n(x,y_0,z,t)=\frac{n_z(z,t)}{\pi w_e^2}\exp\{-[x^2+(y_0-w_1)^2]/w_e^2\} , \tag{15}$$

where $n_z(z,t)$ is the axial electron density defined

$$n_z(z,t)=\sum_i\delta(z_i(t)-z) . \tag{16}$$

Inserting the explicit expression for y_0 into the density and expressing $\mathbf{J}_\perp$ in terms of v_y and n in the wave equation gives

$$\left[\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right]a^f_{nm}(z,t)=\frac{i2e\kappa c}{\pi w_e^2 f}F_n(z)G_m(z)\cos(k_w z)\int_t^{t+2\pi/\omega_s}\frac{e^{-if(k_s z-\omega_s t)}}{\gamma}\sum_i\delta(z_i(t)-z)dt . \tag{17}$$

where we have expressed the transverse averages as

$$F_n(z)=\int dx\, e^{-x^2/w_e^2}A^*_n(x,z) , \tag{18}$$

and

$$G_m(z)=\int dy\exp\{-[y+\chi\sin(k_w z)-w_1]^2/w_e^2\}A^*_m(y,z) , \tag{19}$$

both having the dimensions of $\mathrm{cm}^{1/2}$. Converting the δ function electron distribution into an explicit time form

$$\delta(z_i(t)-z)\simeq\frac{1}{\bar\beta c}\delta(t_i(z)-t) , \tag{20}$$

where $\bar\beta$ is the normalized axial velocity averaged over a wiggler period, given by

$$\bar\beta=\beta_0\left[1-\frac{\kappa^2}{4\gamma_0^2}\right] , \tag{21}$$

the integral in Eq. (17) becomes trivial, yielding

$$\left[\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right]a^f_{nm}(z,t)=\frac{i2e\kappa}{\bar\beta\pi w_e^2 f}F_n(z)G_m(z)\cos(k_w z)\sum_{i=1}^{N_w}\frac{\exp\{-if[k_s z-\omega_s t_i(z)]\}}{\gamma_i} , \tag{22}$$

where N_w is the number of electrons that pass through z during one optical period.

It can be shown that for $\chi/w_e\ll 1$, $G_m(z)$ can be split into two integrals so that

$$G_m(z)\simeq G_m^{(1)}(z)+G_m^{(2)}(z)$$
$$=G_m^{(1)}(z)-\frac{\chi}{w}\sin(k_w z)I_m(z) . \tag{23}$$

where

$$G_m^{(1)}(z)=\int_{-\infty}^{\infty}dy\, A^*_m(y,z)e^{-(y-w_1)^2/w_e^2} \tag{24}$$

and

$$I_m(z)=w\frac{\partial}{\partial w_1}G_m^{(1)}(z) . \tag{25}$$

Evaluation of the wave equation for the $G_m^{(1)}(z)$ term yields fundamental and odd-harmonic mode amplitudes while the $\sin(k_w z)$ component in $G_m^{(2)}(z)$ gives rise to even-harmonic mode amplitudes. Concentrating on the even-harmonic portion of the wave equation and defining the total derivative

$$\frac{d}{dz}=\left[\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right] \tag{26}$$

gives

$$\frac{da_{nm}^{ef}}{dz}=-\frac{ie\kappa}{\pi w_e^2 f\bar{\beta}}\frac{\chi}{w}\sin(2k_w z)F_n(z)I_m(z)\times\sum_{i=1}^{N_w}\frac{\exp\{-if[k_s z-\omega_s t_i(z)]\}}{\gamma_i}, \tag{27}$$

where the ef superscript signifies that these are the even-harmonic mode coefficients.

The time it takes an electron to reach a particular z position can be divided into a component proportional to the average axial velocity, denoted $\bar{t}_i$ (which is dependent on the electron's final and initial positions), and a component that describes the variation from this average time, Δt, dependent only on axial position, or

$$t_i(z)=\bar{t}_i+\Delta t\ . \tag{28}$$

These quantities are defined in the high-γ limit as[3,4]

$$\Delta t=\frac{\xi}{\omega_s}\sin(2k_w z) \tag{29}$$

and

$$\bar{t}_i=\frac{z-z_{0i}}{\bar{\beta}c}-\frac{\xi}{\omega_s}\sin(2k_w z_{0i})\ , \tag{30}$$

where

$$\xi=\frac{\kappa^2}{4(1+\kappa^2/2)}\ , \tag{31}$$

and z_{0i} labels the initial position of the ith electron. Substituting the expression for t_i into the wave equation and expanding the Δt portion of the exponential along with the $\sin(2k_w z)$ term in an exponential series of the form

$$\sin(2k_w z)e^{if\omega_s\Delta t}=\sum_{l=-\infty}^{\infty}D_l e^{i2lk_w z}\ , \tag{32}$$

where

$$D_l=\frac{i}{2}[J_{l+1}(f\xi)-J_{l-1}(f\xi)] \tag{33}$$

yields

$$\frac{da_{nm}^{ef}}{dz}=\frac{e\kappa\chi/w}{2\pi f w_e^2\bar{\beta}}F_n(z)I_m(z)\sum_{l=-\infty}^{\infty}[J_{l+1}(f\xi)-J_{l-1}(f\xi)]e^{i2lk_w z}\sum_{i=1}^{N_w}\frac{\exp\{-if[k_s z-\omega_s\bar{t}_i(z)]\}}{\gamma_i}\ . \tag{34}$$

Defining the slowly varying phase for the ith electron in the optical and wiggler fields as

$$\psi_i=(k_w+k_s)z-\omega_s\bar{t}_i\ , \tag{35}$$

the wave equation can be written

$$\frac{da_{nm}^{ef}}{dz}=\frac{e\kappa\chi/w}{2\pi f w_e^2\bar{\beta}}F_n(z)I_m(z)\sum_{l=-\infty}^{\infty}[J_{l+1}(f\xi)-J_{l-1}(f\xi)]e^{i(2l+f)k_w z}\sum_{i=1}^{N_w}\frac{e^{-if\psi_i}}{\gamma_i}\ . \tag{36}$$

Specializing to interactions over many wiggler periods, we select out the nonoscillating term in the l summation so that

$$\frac{da_{nm}^{ef}}{dz}=-\frac{e\kappa\chi/w}{\pi f w_e^2\bar{\beta}}F_n(z)I_m(z)(-1)^{f/2}J'_{f/2}(f\xi)\sum_{i=1}^{N_w}\frac{e^{-if\psi_i}}{\gamma_i}\ , \tag{37}$$

where f must be an even integer.

If we assume the axial electron density at the entrance to the wiggler varies negligibly over a ponderomotive wavelength, the electron density inside the wiggler will be quasiperiodic—with period λ_p. The average density inside each ponderomotive wavelength will be invariant as it travels with velocity $\bar{\beta}c$ through the wiggler. Denoting this average density $\bar{n}$, we have

$$\bar{n}=\frac{n_\perp}{\lambda_p}\int_z^{z+\lambda_p}dz\sum_{i=1}^{N_w}\delta(z_i(t)-z)=n_\perp N_w/\lambda_p\ , \tag{38}$$

where $n_\perp=(\pi w_e^2)^{-1}$ from the definition of the normalized density. Therefore

$$\bar{n}=\frac{N_w}{\pi w_e^2\lambda_p}\simeq\frac{N_w}{\pi w_e^2}\frac{\omega_s}{2\pi\bar{\beta}c}\ , \tag{39}$$

near resonance, so that

$$\frac{da_{nm}^{ef}}{dz}=-\frac{2e\pi\bar{n}\kappa}{fk_s}\frac{\chi}{w}F_n(z)I_m(z)\times(-1)^{f/2}J'_{f/2}(f\xi)\left\langle\frac{e^{-if\psi_i}}{\gamma}\right\rangle_{e^-}\ , \tag{40}$$

where the brackets denote an average over the N_w electrons. Defining the coupling coefficient for each even harmonic for each mode as

$$\mathscr{K}^{ef}_{nm}(\xi,z)=\frac{i\chi}{w}(-1)^{f/2}J'_{f/2}(f\xi)F_n(z)I_m(z)\ , \tag{41}$$

and the normalized electric-field mode coefficient as $\mathscr{E}^f_{nm}=ifk_s a^f_{nm}$, the wave equation becomes

$$\frac{d\mathscr{E}^{ef}_{nm}}{dz}=2\pi\bar{\rho}\kappa\mathscr{K}^{ef}_{nm}(\xi,z)\left\langle\frac{e^{-if\psi_i}}{\gamma}\right\rangle_{e^-}\ , \tag{42}$$

for the even harmonics, where $\bar{\rho}=-e\bar{n}$ is the average charge density. The analogous derivation for the odd harmonics yields

$$\frac{d\mathscr{E}^{of}_{nm}}{dz}=2\pi\bar{\rho}\kappa\mathscr{K}^{of}_{nm}(\xi,z)\left\langle\frac{e^{-if\psi_i}}{\gamma}\right\rangle_{e^-}\ , \tag{43}$$

where the odd coupling coefficients are given by

$$\mathscr{K}^{of}_{nm}(\xi,z)=(-1)^{(f-1)/2}[J_{(f-1)/2}(f\xi)-J_{(f+1)/2}(f\xi)]\times F_n(z)G^{(1)}_m(z)\ . \tag{44}$$

Following a similar line of reasoning, the energy equation for the electrons including both even and odd harmonic contributions becomes

$$\left\langle\frac{d\gamma}{dt}\right\rangle_{x,y}=\frac{e\kappa}{2\pi w_e^2 mc}\mathrm{Re}\left[\sum_{f=\mathrm{odd}}\sum_{n,m}\mathscr{K}^{of}_{nm}(\xi,z)\mathscr{E}^{*of}_{nm}\frac{e^{-i(f\psi+\varphi_f)}}{\gamma}+\sum_{f=\mathrm{even}}\sum_{n,m}\mathscr{K}^{ef}_{nm}(\xi,z)\mathscr{E}^{*ef}_{nm}\frac{e^{-i(f\psi+\varphi_f)}}{\gamma}\right]\ , \tag{45}$$

where the angular braces denote an average over transverse space. Evaluation[5] of the definitions for the transverse spatial averages yields

$$F_n(z)=\mathrm{even}(n)w_e\frac{(2\pi)^{1/4}}{(n/2)!}\left[\frac{n!}{2^n w}\right]^{1/2}\Omega\Delta^n\Lambda_n\ , \tag{46}$$

where even(n) is zero for n odd and one for n even,

$$G^{(1)}_m(z)=\frac{(2\pi)^{1/4}w_e}{i^m(m!2^m w)^{1/2}}\Omega\Delta^m\Lambda_m e^{-\Gamma w_1^2/w_e^2}H_m\left[i\sqrt{2}\frac{w_1}{w}\frac{\Omega^2}{\Delta}\right]\ , \tag{47}$$

$$I_m(z)=-2\frac{(2\pi)^{1/4}w_e}{i^m(m!2^m w)^{1/2}}\Omega\Delta^m\Lambda_m e^{-\Gamma w_1^2/w_e^2}\left[\frac{w_1 w}{w_e^2}\Gamma H_m\left[i\sqrt{2}\frac{w_1}{w}\frac{\Omega^2}{\Delta}\right]-i\sqrt{2}m\frac{\Omega^2}{\Delta}H_{m-1}\left[i\sqrt{2}\frac{w_1}{w}\frac{\Omega^2}{\Delta}\right]\right]\ , \tag{48}$$

where

$$\Omega=\left[\frac{w^2/w_e^2}{1+w^2/w_e^2+iz/z_r}\right]^{1/2}\ , \tag{49}$$

$$\Gamma=\frac{1+iz/z_r}{1+w^2/w_e^2+iz/z_r}\ , \tag{50}$$

$$\Lambda_l=\left[\frac{1+iz/z_r}{(1+z^2/z_r^2)^{1/2}}\right]^{l+1/2}\ , \tag{51}$$

and

$$\Delta=\left[\frac{1-w^2/w_e^2-iz/z_r}{1+w^2/w_e^2+iz/z_r}\right]^{1/2}\ . \tag{52}$$

It can easily be shown that the FEL model governed by Eqs. (42), (43), and (45) conserves energy.

IV. INTERPRETATION OF THE PHASE-AVERAGED EQUATIONS

A. Complex coupling coefficient

To compare the coupling of the even and odd harmonics into the various TEM modes we first note that both the complex coupling coefficients, given by Eqs. (41) and (44), depend linearly on $F_n(z)$. We can therefore omit this term from the comparison. The average over the wiggle direction yields the $G^{(1)}_m(z)$ factor in the odd-harmonic coupling coefficients and the $I_m(z)$ term in the even-harmonic coefficients. For perfect alignment, $w_1=0$, and $G^{(1)}_m(z)\Longrightarrow F_m(z)$ which is zero for odd m. Thus, only modes with even transverse symmetry can be driven at the fundamental and odd harmonics when the optical and electron beams are perfectly aligned. Setting $w_1=0$ in $I_m(z)$ gives nonzero results only for m odd. Thus, only modes with odd transverse symmetry (in the wiggle direction) can be driven at the even harmonics when the optical and electron beams are perfectly aligned.

To see why only odd transverse modes are driven by the even harmonics, and vice versa, we look at the expression for the transverse driving current given by

$$J_\perp=-env_y\ . \tag{53}$$

Using Eq. (14) in (15) we have

$$n(x,y,z)=\frac{1}{\pi w_e^2}\exp(-x^2/w_e^2)\times\exp\{-[y-w_1+\chi\sin(k_w z)]^2/w_e^2\}\ , \tag{54}$$

and with the $\chi/w_e \ll 1$ assumption

$$n(x,y,z)\simeq\frac{1}{\pi w_e^2}e^{-x^2/w_e^2}e^{-(y-w_1)^2/w_e^2}\times[1-2(y-w_1)\chi\sin(k_w z)]\ . \qquad (55)$$

The second term in brackets is the MODEM term. Substituting this term along with Eq. (13) into Eq. (53) gives

$$J_\perp\,|_{\text{MODEM}}=-\frac{e\kappa c}{\gamma}\left[\frac{(y-w_1)\chi}{\pi w_e^2}\right]\sin(2k_w z)\times\exp\{-[x^2+(y-w_1)^2]/w_e^2\}\ . \qquad (56)$$

This expression is obviously odd in y (for $w_1=0$) and therefore only modes with odd symmetry along this axis are driven. Analogously, the transverse current for the fundamental and odd harmonics, obtained using the first term of Eq. (55), yields an expression that has even transverse symmetry. Thus, only modes with even transverse symmetry are driven at these frequencies. From Eq. (56), the extrema of $J_\perp$ depend only on the electron-beam spot size and are located at $y=\pm w_e/\sqrt{2}$. Since the spot size of the source of the harmonic radiation is determined solely by the *electron*-beam radius, the excited harmonic radiation must also have a spot size determined by the electron beam. For cavities where the harmonic reflectivities prevent the formation of harmonic modes, the single-pass harmonic radiation will be observed with a spot size given by the electron-beam spot size.

For $w_1\neq 0$, all modes can be generated for both even and odd harmonics. A nonzero value of w_1 implies a transverse displacement of the electron beam in the wiggle plane. Coupling of the second harmonic into the TEM_{00} mode can then occur and will be optimized when $w_1=w_e$. The magnitude of the coupling coefficient for the optimally displaced electron beam into the TEM_{00} mode is approximately half that of the perfectly aligned system into the TEM_{01} mode.

B. Coherent spontaneous emission

The harmonics generated in FEL oscillator experiments are produced primarily by coherent spontaneous radiation of the fundamentally bunched electron beam. This mechanism dominates over the much weaker harmonic linear-gain mechanism which cannot overcome cavity mirror losses. The change in the amplitude of each harmonic is determined by its transverse-current Fourier component, scaled by the harmonic coupling coefficient.

An intuitive feel for the amplitudes of the even harmonics, relative to those of the odd harmonics, can be obtained by taking the ratio of the complex coupling coefficients for the second and third harmonics. Considering the form of $J_\perp$ in Eq. (56), we assume all the second-harmonic radiation to appear in the TEM_{01} mode, while the third harmonic is assumed to radiate completely into the TEM_{00} mode. Then

$$\left|\frac{\mathcal{K}_{01}^{(2)}}{\mathcal{K}_{00}^{(3)}}\right|=\left|\frac{\chi}{2w}\frac{I_1(z)}{G_0^{(1)}(z)}\frac{[J_0(2\xi)-J_2(2\xi)]}{[J_1(3\xi)-J_2(3\xi)]}\right|\ , \qquad (57)$$

and assuming $w_1=0$, $w_e=w$, and $z\ll z_r$, we have

$$\left|\frac{\mathcal{K}_{01}^{(2)}}{\mathcal{K}_{00}^{(3)}}\right|\simeq\left|\frac{\chi}{2w}\frac{[J_0(2\xi)-J_2(2\xi)]}{[J_1(3\xi)-J_2(3\xi)]}\right|\simeq 0.25\ , \qquad (58)$$

where we have assumed parameters consistent with those of the Los Alamos FEL oscillator ($\kappa=0.76$, $\lambda_w=2.73$ cm, $\gamma=42$, $w_e=1$ mm). Thus, the coupling into the second harmonic is one fourth that of the third harmonic.

One-dimensional computer simulations[4] of the Los Alamos FEL oscillator performed with the code ONED predict that the Fourier component at the second harmonic, due to bunching by the fundamental, is three or more times larger than the third-harmonic component (where saturated electric-field amplitudes have been assumed). Thus, one would expect equal second- and third-harmonic radiation. Preliminary measurements[6] taken for the Los Alamos FEL bear out this result, although the modal content of the two harmonics was not ascertained.

C. Comparison with misalignment effects

Even-harmonic radiation can also be excited by misalignment of the optical and electron beams. Colson *et al.*[7] have calculated a coupling coefficient for such a system assuming an imposed plane-wave electric field, with the result

$$\mathcal{K}_\theta=\sum_{n'=-\infty}^{\infty}J_{n'}(f\xi)[J_{2n'+f+1}(fZ)+J_{2n'+f-1}(fZ)]\ , \qquad (59)$$

where

$$Z=\frac{2\kappa\gamma\theta}{1+\kappa^2/2+\gamma^2\theta^2}\ , \qquad (60)$$

and θ is the angle of misalignment. To determine if this mechanism can compete with the MODEM in the Los Alamos FEL oscillator, we assume a maximum electron-beam misalignment of $y=+w_e$ at one end of the wiggler and $y=-w_e$ at the other end of the 1-m wiggler. For these conditions, $\mathcal{K}_\theta\simeq 0.1$ which is less than half that calculated for the MODEM. The coupling to even harmonics caused by misalignments is actually smaller than 0.1, since the transverse fall off of the optical electric field is not included in the plane-wave model. Also, since the even-harmonic radiation is produced primarily through coherent spontaneous emission, and the optical cavity and electron beam are aligned experimentally by maximizing fundamental power output, any misalignment will be minimized. Therefore, for the Los Alamos FEL oscillator, the MODEM will be the dominant source of even-harmonic radiation (where dispersion of the optical fields and even harmonics of the wiggler magnetic field can be neglected).

V. CONCLUSIONS

We have found a mechanism by which even harmonics are generated in plane-polarized FEL oscillators. Coupling to the even harmonics is produced by the optical-field variation sampled by each electron as it wiggles through the optical cavity. The radiation is confined to modes with odd symmetry in the wiggle direction (null on axis) resembling the angular spontaneous emission, but much narrower in angular divergence. To analyze this mechanism we have chosen to decompose the optical field into Gauss-Hermite modes of the optical cavity. We have obtained a set of one-dimensional equations with complex coupling coefficients that conserve energy and are suitable for numerical analysis. The strength of the coupling for this mechanism dominates that due to angular misalignment of the electron and optical beams. Preliminary calculations predict even-harmonic amplitudes on the order of the odd-harmonic amplitudes for the Los Alamos FEL.

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