Quasiclassical ground-state energies for quarkonia potentials

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Ground-state energies as well as corresponding stability thresholds characterizing several theoretical or phenomenological potentials which are of interest in the description of quarkonia have been established and discussed. For this purpose a nonperturbative quasiclassical method for the evaluation of the ground-state energy proposed previously has been used. We are then led to establish ground-state energies for nonrelativistic as well as for relativistic spin-$\frac{1}{2}$ particles. In this respect, Coulomb inverse-logarithmic potentials, linear plus Coulomb potentials, power potentials, and superpositions of power potentials have been considered. In addition, the ground-state energy characterizing the relativistic Dirac bag model has also been established and discussed.

I. INTRODUCTION

A quasiclassical nonperturbative method for the evaluation of the ground-state energy (GSE) proposed previously[1] has been validated for a wide range of quantum-mechanical (QM) systems.[2] Such results open the way to perform a general quasiclassical description of the QM-GSE problem, now for several momentum-independent or momentum-dependent potentials, as well as for several Hamiltonians. So far this method enables us to establish quite simply relevant orders of magnitudes, qualitative properties, stability thresholds, and analytical forms characterizing the GSE's for nonrelativistic and relativistic Hamiltonians. In particular, such a quasiclassical description has also been checked for Coulomb and Yukawa potentials.[3] Here we shall analyze GSE's and stability thresholds for certain potential models which are of interest in the description of quarkonia. We are then led to establish new and useful results concerning concrete as well as general stability properties of corresponding Hamiltonians.

Basic assumptions concerning the present quasiclassical approach are presented in Sec. II. The Coulomb inverse-logarithmic potentials and the linear plus Coulomb potentials are discussed in Secs. III and IV. Stability thresholds for general superpositions of power potentials have also been established. So far the Hamiltonians are nonrelativistic ones. Next one considers relativistic power potentials in Sec. V, whereas the GSE for the relativistic linear plus Coulomb potential is analyzed in Sec. VI. The GSE characterizing the Dirac bag model is established in Sec. VII. The conclusions are presented in Sec. VIII. Appendixes are included. Units for which $c=1$ will be used.

II. PRELIMINARIES

We shall begin by presenting some preliminary remarks. For the nonrelativistic two-body ($m_1=m_2=m_0$) Schrödinger Hamiltonian $H(r,p)=p^2/m_0+V(r)$, the GSE problem can be converted into the algebraic evaluation of the minima

$$E=\min H\left[r,\frac{\hbar d_0}{r}\right]=\min\left[\frac{\hbar^2 d_0^2}{m_0 r^2}+V(r)\right], \tag{2.1}$$

or equivalently,

$$E=\min H\left[\frac{\hbar d_0}{p},p\right]=\min\left[\frac{p^2}{m_0}+V\left[\frac{\hbar d_0}{p}\right]\right], \tag{2.2}$$

within coordinate ($r=|\mathbf{x}|$) and momentum ($p=|\mathbf{p}|$) representations, respectively. Here $d_0\sim 1$ denotes the underlying phase-space quantum which is subject to suitable eigenvalue conditions.[2] For the three-dimensional configuration space ($j=3$) one has, in general, $0<d_0<\frac{3}{2}$. In the relativistic case one proceeds similarly. Next let us remember that the energy "dispersion" $\delta H(r)=H(r,\hbar d_0/r)$ comes from the quasiclassical limit of $\hat{H}\psi$ under the conditions of which the usual *QM* state function $\psi=\psi(r)$ approaches the limiting case of the powerlike probe function $\phi_a\sim r^{-id_0}$. It is understood that the $\psi\to\phi_a$ limit should be supplemented with $\hat{H}\to\widetilde{H}$, where $\widetilde{H}$ (tilde) denotes the quasiclassical counterpart of the usual $\hat{H}$ operator.[2] Therefore $\widetilde{H}\phi_a=\delta H\phi_a$, as emphasized previously. The main point is that the above choice of ϕ_a can be interpreted as the analytic continuation of a concrete $\psi\to\psi_a$ limit, where $\psi_a\sim r^d$ expresses the dominant behavior of the irregular state function near the origin. For potentials which are not too singular at the origin, $d=d(l)$ is a parameter which depends on the angular momentum but which is independent of the coupling. So far one has $d(l)\in(-\frac{3}{2},0)$ on general grounds with respect to the matching of $\langle\hat{a}\rangle_\psi$ and $\langle\hat{a}\rangle_{\psi_a}$ averages. Here $\hat{a}=i\hbar\mathbf{x}\cdot\partial/\partial\mathbf{x}$ denotes the non-Hermitian constituent of the usual dilation generator. Restricting ourselves to the GSE, we have to take $l=0$, so that $d_0=|d(0)|$. This yields, in combination with Eq. (2.1), a useful prescription for the evaluation of the d_0 parameter. More definitely, the Schrödinger equation for the radial state function $\psi(r)=R_l(r)$ produces the intermediary algebraic equation

$$d^2+d-l(l+1)-m_0r^2V(r)/\hbar^2=-m_0r^2E_S/\hbar^2\,, \tag{2.3}$$

insofar as $R_l(r) \sim \psi_a$, where E_S denotes the energy eigenvalue. However, d_0 estimates, which are functions of E_S, are not compatible with the intrinsic minimization attributes characterizing Eq. (2.1). The simplest choice is then given by $E_S = 0$. Then Eq. (2.3) becomes

$$d^2 + d - l(l+1) = m_0 r^2 V(r)/\hbar^2 , \tag{2.4}$$

which exhibits r roots for suitable d_0 values only. In this sense Eq. (2.4) exhibits the meaning of an eigenvalue condition for the d_0 parameter, too. Next we mention that for nonsingular potentials $r^2 V(r)$ vanishes at the origin (and conversely). So the d_0 parameter is independent of the coupling and takes the dominant value $d_0 = 1$. Of course, one has $d_0 > 1$ ($d_0 < 1$) in all the cases in which $r^2 V(r) > 0$ [$r^2 V(r) < 0$]. This coupling independence of d_0 implies certain symmetry properties, such as the convexity property[4] of the GSE with respect to the coupling constant (see Appendix A). Next we have to notice that under certain special conditions the usual QM virial equation itself, $2\langle \hat{p}^2/m_0 \rangle = \langle \bar{V} \rangle$, can be subject to nontrivial $\psi \to \phi_a$ and $\psi \to \psi_a$ limits. This leads to[5]

$$2\langle \hbar^2 d_0^2 / m_0 r^2 \rangle_{\phi_a} = \langle \bar{V} \rangle_{\phi_a} , \tag{2.5}$$

and similarly for $\psi \to \psi_a$. In the preceding, $\bar{V} = r(dV/dr)$, as usual. One realizes that suitable combinations of such limits are able to produce nontrivial constraints on the admissible d_0 values. Taking as an example the attractive power potentials $V_n(r) = C_n / r^n$, one sees immediately that Eq. (2.1) works only for $n < 2$. Then Eq. (2.5) yields $1 < d_0 < \frac{3}{2}$ and $0 < d_0 < 1$ for $n < 0$ and $0 < n < 2$, respectively.[6] On the other hand, Eq. (2.4) exhibits well-defined r roots if $d^2 + d > 0$ ($n < 0$) and $d^2 + d < 0$ ($0 < n < 2$). This leads to $1 < d_0 < \frac{3}{2}$ ($n < 0$) and $0 < d_0 < 1$ ($0 < n < 2$), which agrees with Eq. (2.5). This agreement provides evidence that the choice of d_0 proposed above is subject to actual relevance. However, we want to remark that a quickly tractable theoretical alternative to Eq. (2.5) is highly desirable. For the sake of generality we shall then restrict ourselves to the qualitative interpretation of Eq. (2.4).

The relativistic generalization of Eq. (2.4) is straightforward. So the Klein-Gordon (KG) equation becomes

$$d(d+1) - l(l+1) = r^2[m_0^2 - (\mathscr{E} - V)^2]/\hbar^2 , \tag{2.6}$$

insofar as $R_l(r) \sim \psi_a$. The relativistic Hamiltonian $\mathscr{H}(r,p) = p_0 + V(r)$ with a vectorial potential has been considered.[7] Accordingly, $p_0 = (p^2 + m_0^2)^{1/2}$. Now the GSE has the property $\mathscr{E} \sim m_0$, so that $\mathscr{E} \to 0$ if $m_0 \to 0$. Then the relativistic counterpart of (2.4) is

$$d(d+1) - l(l+1) = -r^2 V^2(r)/\hbar^2 < 0 , \tag{2.7}$$

in which the $m_0 \to 0$ limit has been performed. This yields $d(d+1) < l(l+1)$, so that $0 < d_0 < 1$. For the relativstic scalar potentials one has

$$d(d+1) - l(l+1) = r^2 V^2(r)/\hbar^2 > 0 , \tag{2.8}$$

instead of Eq. (2.7). This means that the general d_0 interval characterizing such potentials is $1 < d_0 < \frac{3}{2}$. For equally mixed vectorial and scalar potentials one would then have $d(d+1) = l(l+1)$, so that $d_0 = 1$. We have to notice that Eqs. (2.7) and (2.8) refer specifically to the KG equation. In this respect the GSE for the Dirac Hamiltonian is still established by minimizing the corresponding KG Hamiltonian, but the d_0 parameters need to be established with the help of the Schrödinger Hamiltonian. This Schrödinger Hamiltonian is, of course, the nonrelativistic limit of the KG one. This prescription is confirmed by the exact GSE for the Dirac-Coulomb problem, as well as by the GSE's for Dirac Hamiltonians with equally mixed scalar and vectorial potentials.[8] For this purpose we have to use the property that the Dirac equation with such equally mixed potentials can be rewritten equivalently as a Schrödinger equation.[9] On the other hand, the exact nonrelativistic GSE for the harmonic oscillator ($j = 3$) is given in terms of $d_0 = \frac{3}{2}$. These results lead to the admissible proposal $d_0 = 1$ ($0 < n < 2$) and $d_0 = \frac{3}{2}$ ($n < 0$), which agrees with Eq. (2.4). It is understood that such proposals are subject to further refinements.

Equation (2.1) is also able to produce well-defined minima for certain combinations containing singular potentials. Such potentials require a special treatment as d_0 ceases to be independent of the coupling. In the nonrelativistic case such potentials are given by $V(r) = V_2(r) +$ other terms. Potentials for which $r^2 V(r)$ exhibits nontrivial (nonzero and finite) upper or lower bounds can also be considered. For the KG equation this happens for $V(r) = V_1(r) +$ other terms. So the Coulomb potential is a singular potential for the KG equation, in contradistinction to the Dirac equation. Among the special peculiarities mentioned above we note the onset of relationships like $d_0^2 \gtrless |d(0;C)|$ competing with $d_0 \gtrless |d(0;C)|$. Here $d(l=0;C)$ comes from the dominant behavior of Eq. (2.4) near the origin, whereas C stays for the relevant coupling.[2] The onset of d_0^2 is favored by the agreement with exact results, as, e.g., with the exact nonrelativistic GSE for the Kratzer potential.[10] A similar conjecture has also been proved for the KG equation with a Coulomb potential. In this respect the nonrelativistic superposition of $V_2(r)$ with other power potentials will be analyzed in Appendix B. Coming back to (2.2) we also have to mention that this equation meets the upper or lower energy bounds established specifically within envelope representations for hydrogenlike systems.[11] Accordingly, $V(\hbar d_0 / p)$ reproduces so-called kinetic potentials involved in this context. Using these theoretical developments we can now make detailed calculations concerning GSE's for several potentials.

III. THE COULOMB INVERSE-LOGARITHMIC POTENTIAL

Proofs have been given that quarkonia can be treated satisfactorily considering quarks as nonrelativistic fermions interacting through some simple phenomenological and/or quantum-chromodynamically (QCD) motivated potentials. An interesting example is the theoretical Coulomb inverse-logarithmic potential[12]

$$V(r) = \hbar\alpha / r \ln(\mu r / \hbar) , \tag{3.1}$$

in which $\alpha=8\pi/(33-2n_f)$ for QCD with n_f flavors. Above $\mu\equiv\gamma m_0$ is a mass scale characterizing the renormalization-group subtraction point. Using Eq. (2.1) leads to a negative GSE:

$$0>E>E_{\rm LB}=-m_0\alpha_0^2/16\ , \tag{3.2}$$

which is the root of the algebraic equation

$$(2\gamma d_0/\alpha_0)\exp(\beta\,{\rm sgn}\alpha)=\beta^{-1}-({\rm sgn}\alpha)\beta^{-2}\ . \tag{3.3}$$

Above $|\alpha|=\alpha_0 d_0$, whereas

$$\beta=\beta(E)=\left[\frac{1}{2}+\frac{1}{2}\,{\rm sgn}E\left[1+\frac{16E}{m_0\alpha_0^2}\right]^{1/2}\right]^{-1/2}>\sqrt{2}\ . \tag{3.4}$$

The above lower-bound $E_{\rm LB}$ is four times smaller than the GSE characterizing the attractive Coulomb potential $-\hbar\alpha_0 d_0/r$. First let us consider that $\alpha>0$. Then Eq. (3.3) produces the required solution only for supercritical values of the coupling:

$$\alpha_0>\alpha_c^{(+)}=4\gamma d_0\xi_{(+)}\cong 39.721\mu d_0/m_0\ , \tag{3.5}$$

or, equivalently, for supercritical values

$$m_0>m_c^{(+)}(n_f)=4d_0\mu\xi_{(+)}/\alpha_0\cong 39.721\mu d_0/\alpha_0\ , \tag{3.6}$$

of the quark mass. Here $\xi_{(+)}=(\exp\sqrt{2})(\sqrt{2}-1)\cong 9.930$. Starting with a fixed coupling, one would then have to consider Eq. (3.6). This shows that the present nonrelativistic Hamiltonian is subject to stability if $m_0>m_c^{(+)}(n_f)$ only. In contradistinction, one would have

$$\alpha_0<\alpha_c^{(-)}=4\gamma d_0\xi_{(-)}\cong 0.4028\mu d_0/m_0\ , \tag{3.7}$$

for $\alpha<0$, which can be rewritten equivalently as

$$m_0<m_c^{(-)}(n_f)=4d_0\mu\xi_{(-)}/\alpha_0\cong 0.4028\mu d_0/\alpha_0\ . \tag{3.8}$$

Now $\xi_{(-)}=\exp(-\sqrt{2})(\sqrt{2}+1)\cong 0.1007$. In addition, one would also have a positive GSE which is given by the algebraic equation

$$(2\gamma d_0/\alpha_0)\exp\beta=\beta^{-2}-\beta^{-1}\ , \tag{3.9}$$

insofar as $\alpha<0$, now irrespective of α_0 and m_0. This time $\beta<\sqrt{2}$ because $E>0$. Further, we have to consider $d_0<1$ ($d_0>1$) for $E<0$ ($E>0$). Taking, for the moment, $n_f=3$ and $\mu=400$ MeV, yields $\alpha\cong 0.9308$ and $m_c^{(+)}(3)\cong 17.069$ GeV, where $d_0=1$. So far, this $m_c^{(+)}(3)$ evaluation approaches the magnitude order of the t-quark mass. More refined numerical estimates are also of further interest. Note that $m_c^{(+)}(n_f)$ decreases linearly with n_f for all underlying $n_f<16.5$ values. So one has $m_c^{(+)}(n_f)/\mu\in[48.994,1.580]$ for $n_f\in[1,16]$. In other words, the nonrelativistic Hamiltonian with the quark potential (3.1) exhibits stability for sufficiently heavy quarks only. This is a quite relevant result which comes from the QCD background of (3.1). Next $m_c^{(-)}(n_f)$ increases linearly with n_f, now for $n_f>16.5$ and with an appreciable smaller slope parameter. In particular, $m_c^{(+)}(16.5)=m_c^{(-)}(16.5)=0$.

Note that Eq. (3.7) is also of special interest in order to explore the logarithmic approximation to the polarization effects in quantum electrodynamics (QED). This time $\alpha=-3\pi/2$, whereas $\mu=m_0\delta\exp(3\pi/2\alpha_e)$. Here $\delta\cong 4.0982$,[13] whereas α_e denotes the usual Coulomb coupling of the electron-positron interaction. Then Eq. (3.7) yields the upper bound

$$\alpha_e<\alpha_c(\delta)=3\pi/2\ln(3\pi/8\xi_{(-)}d_0^2\delta)\cong 4.4925\ , \tag{3.10}$$

where $d_0=1$. Other relativistic evaluations of the critical Coulomb coupling should also be noticed.[14] In this respect $\alpha_c(\delta)$ is approximately 2.5 times larger than some concrete critical Coulomb couplings ($\alpha_c\sim 2$) relying on the two-body problem. Assuming, however, that $\delta=1$ leads to $\alpha_c(1)\cong 1.9160$ instead of Eq. (3.10). Note that for the one-body problem the critical couplings established above become two times smaller.

IV. THE LINEAR PLUS COULOMB POTENTIAL

Another potential used frequently in the description of quarkonia is the linear plus Coulomb potential[15]

$$V(r)=\kappa r-(\hbar\alpha/r)\ , \tag{4.1}$$

in which a typical choice is $\alpha\cong 0.48$ and $\hbar\kappa\cong 0.182$ GeV^2. Equation (2.1) shows that the GSE cannot be defined if both κ and α take negative values. In the other cases the GSE is given by the algebraic equation

$$\frac{E}{m_0}+\frac{2}{9}\alpha_0^2+({\rm sgn}\alpha)\frac{2}{9\alpha_0}\left[\alpha_0^2+3\frac{E}{m_0}\right]^{3/2}={\rm sgn}(\alpha\kappa)\frac{3\gamma_0}{\alpha_0}\ , \tag{4.2}$$

in which $|\kappa|=m_0\gamma_0/l_1$ ($\gamma_0\equiv\gamma_{(-1)}^0$) and $l_1=\hbar d_0/m_0$. For the sake of simplicity the above Coulomb coupling has been denoted as before. An eigenvalue equation which is similar to Eq. (4.2) has also been obtained using the method of potential envelopes.[16] However, Eq. (4.2) is more simple. Using the virial equation one sees that the energy dispersion can also be written as[17]

$$\delta H=m_0\left[\frac{3\gamma_0}{\alpha_0}\frac{1}{u^2}-2({\rm sgn}\alpha)\frac{(\alpha_0\gamma_0)^{1/2}}{u}\right]\ , \tag{4.3}$$

in which $r/l_1=u(\alpha_0/\gamma_0)^{1/2}$. Next the minima of the energy dispersion are located at

$$u=u(d_0,\gamma_0)=\frac{2}{\sqrt{3}}\times\begin{cases}\sinh(\phi/3), & \sinh\phi=\beta_0;\ \alpha>0,\ \kappa>0\\ \cosh(\phi/3), & \cosh\phi=\beta_0>1;\ \alpha<0,\ \kappa>0\\ \cos(\phi/3), & \cos\phi=\beta_0<1;\ \alpha<0,\ \kappa>0\\ -\cos[(\phi+4\pi)/3], & \cos\phi=\beta_0<1;\ \alpha>0,\kappa<0\end{cases} \tag{4.4}$$

where $\beta_0=3^{3/2}\gamma_0^{1/2}/\alpha_0^{3/2}$. Of course, we also can eliminate the u variable from the virial equation and from Eq. (4.3), thereby obtaining Eq. (4.2). Starting with $\beta_0<1$, one sees that the present GSE exhibits the limits $E\to 0$ ($\alpha<0$) and $E\to -m_0\alpha_0^2/4$($\alpha>0$) for $\gamma_0\to 0$. Similarly, the $\alpha\to 0$ limit produces the GSE $3m_0(\gamma_0/2)^{2/3}$ characterizing the attractive linear potential κr, now for $\beta_0>1$. Concerning the d_0 choice one should have $d_0>1$ for $\alpha<0$($\kappa>0$), $d_0 \gtrless 1$ for $\alpha>0$ combined with $\kappa>0$, whereas $d_0<1$ for $\kappa<0$ ($\alpha>0$).[18] Numerical estimates can also be performed. Using the data mentioned above gives $E\cong 0.6229$ GeV for $d_0=\frac{3}{2}$, where we have considered that $m_0=1$ GeV.

Further, one has the GSE bounds

$$\begin{aligned} &0>E/m_0>-\alpha_0^2/3, \quad \beta_0<2 , \\ &3(\gamma_0/2)^{2/3}>E/m_0>0, \quad \beta_0>2 , \end{aligned} \tag{4.5}$$

for $\alpha>0$ and $\kappa>0$. Further $E<0$ if $\kappa<0$, so that

$$\begin{aligned} &-\alpha_0^2/6-9\gamma_0/2\alpha_0>E/m_0>-\alpha_0^2/3, \quad \sqrt{3}/2<\beta_0<1 , \\ &-\alpha_0^2/6-9\gamma_0/2\alpha_0>E/m_0>-3\gamma_0/\alpha_0-\alpha_0^2/4 , \\ &\qquad\qquad \sqrt{2}/2<\beta_0<\sqrt{3}/2 , \\ &-\alpha_0^2/4>E/m_0>-3\gamma_0/\alpha_0-\alpha_0^2/4, \quad \beta_0<\sqrt{2}/2 , \end{aligned} \tag{4.6}$$

If $\alpha<0$ the GSE takes positive values which are subject to

$$\begin{aligned} &2(\alpha_0\gamma_0)^{1/2}+3\gamma_0/\alpha_0>E/m_0>2(\alpha_0\gamma_0)^{1/2}, \quad \beta_0<\beta_0' , \\ &3(\gamma_0/2)^{2/3}+2\alpha_0(\gamma_0/2)^{1/3}>E/m_0>3(\gamma_0/2)^{2/3} \\ &\qquad\qquad +\alpha_0(\gamma_0/16)^{1/3}, \quad \beta_0>\beta_0' , \end{aligned} \tag{4.7}$$

in which $\beta_0'=3\sqrt{3}/2$. Such upper and/or lower bounds come simply from inequalities characterizing the locations of the minima. So one has $x<2/\alpha_0$ or $x<(2/\gamma_0)^{1/3}$ if $\alpha>0$ and $\kappa>0$, $2/\alpha_0<x<3/\alpha_0$ for $\kappa<0$, whereas $x>(2/\gamma_0)^{1/3}$ or $x>(\alpha_0/\gamma_0)^{1/2}$ for $\alpha<0$. Here $x=r/l_1$.

General results concerning the superposition of the Coulomb potential with other power potentials can also be noticed. So the superposition of an attractive Coulomb potential with repulsive power potentials exhibits a well-defined GSE if

$$\gamma_n^0<\gamma_c^{(1)}(n)=\frac{1}{|n|}\left[\frac{|1-n|}{2}\right]^{1-n}\left[\frac{\alpha_0}{|2-n|}\right]^{2-n}, \tag{4.8}$$

for $n\leq 1$, whereas $\gamma_n^0\in(0,\infty)$ for $n>1$. The present results show the possibility to perform a theoretical synthesis concerning $\gamma_c^{(1)}(n)$ for several n values. Next we notice that the parametrization $C_n=m_0\gamma_n l_1^n$ ($\gamma_n^0=|\gamma_n|$) is the same as that used in Appendix B. The superposition of an attractive Coulomb potential with other attractive power potentials can also be considered. Now the GSE is well defined if $\gamma_n^0<\gamma_c^{(1)}(n)$ for $n\geq 2$, whereas $\gamma_n^0\in(0,\infty)$ for $n<2$. Further, the repulsive Coulomb potential is subject to superpositions with attractive $n<2$ potentials only. In this case one should have

$$\gamma_n^0>\gamma_c^{(1)}(n) , \tag{4.9}$$

for $1\leq n<2$, whereas $\gamma_n^0\in(0,\infty)$ for $n<1$. If $n=1$, Eq. (4.9) reads simply $\gamma_1^0>\alpha_0$. Above $V(r)=-\alpha\hbar/r+V_n(r)$.

V. ATTRACTIVE POWER POTENTIALS

A simple potential motivated by the $c\bar{c}$ and $b\bar{b}$ data is the attractive power potential[19]

$$V_n(r)=C_n/r^n , \tag{5.1}$$

where $nC_n<0$. For such potentials the nonrelativistic GSE's have been established explicitly within the usual WKB approach,[20] as well as within the present quasiclassical method.[2] Relativistic power potentials have also been discussed. The main point is that the quasiclassical GSE's characterizing the Dirac Hamiltonian can be established minimizing the dispersion

$$\delta\mathcal{H}(r)=\left[\frac{\hbar^2 d_0^2}{r^2}+m_0^2\right]^{1/2}+V(r) , \tag{5.2}$$

of the relativistic Hamiltonian $\mathcal{H}(r,p)=p_0+V(r)$, where a vectorial potential has been considered. Then the Dirac GSE is given by $\mathcal{E}=\min\delta\mathcal{H}$, insofar as d_0 is subject to the nonrelativistic choice, as mentioned in Sec. II. One proceeds similarly with respect to scalar potentials as well as with respect to combinations of scalar and vectorial potentials. We shall then use this opportunity to analyze in more detail the relativistic GSE's characterizing spin-$\frac{1}{2}$ particles interacting through V_n potentials. First the minimization of (5.2) leads to the algebraic equation

$$|n|\gamma_n^0|1-n|^{1-n}=f_n(y) , \tag{5.3}$$

in which $y=\mathcal{E}^2/m_0^2$ and

$$f_n(y)=[yn^2+n(1-n)]^{(2-n)/2}[yn^2+(1-n)]^{-1/2} . \tag{5.4}$$

In general $1<d_0<\frac{3}{2}$ and $0<d_0<1$ for $n<0$ and $0<n<2$, respectively. Remember that a useful particular choice reads $d_0\cong 1$ ($0<n<2$) and $d_0\cong\frac{3}{2}$ ($n<0$). Using inequality byproducts of Eq. (5.4), one realizes that

$$\begin{aligned} &\mathcal{E}_{\rm LB}^2<\mathcal{E}^2<\mathcal{E}_{\rm UB}^2, \quad 1<n<2 , \\ &\mathcal{E}_{\rm UB}^2<\mathcal{E}^2, \quad n<0 , \end{aligned} \tag{5.5}$$

whereas $\mathcal{E}^2\in(0,\infty)$ for $n\in(0,1)$. Above $\mathcal{E}_{\rm LB}^2=m_0^2(n-1)/n$ and $\mathcal{E}_{\rm UB}^2=2m_0^2(n-1)/n^2$. On the other hand, $f_n(y)>f_n(0)$ and $f_n(y)<f_n[2(n-1)/n^2]$ if $0<n<1$ and $1<n<2$, respectively. Then Eq. (5.3) yields the critical couplings

$$\gamma_n^0>\gamma_c'(n)=n^{-n/2}(1-n)^{(n-1)/2} , \tag{5.6}$$

for $0<n<1$, whereas[2]

$$\gamma_n^0<\gamma_c(n)=\frac{1}{n}(n-1)^{(n-1)/2}(2-n)^{(2-n)/2} , \tag{5.7}$$

if $1<n<2$. Note that Eq. (5.7) can also be extrapolated towards $n=1$ and $n=2$. If $n<0$ such constraints cease to be involved, so that $\gamma_n^0\in(0,\infty)$. In other words the relativistic Hamiltonian p_0+V_n is subject to stability for supercritical ($0<n<1$) and subcritical ($1<n<2$) values of the coupling, respectively. Equations (5.6) and (5.7) can also be rewritten equivalently in terms of the upper rest-mass bounds

$$m_0 < m_c'(n) = (1-n)^{1/2} n^{n/2(1-n)} M_n \ , \tag{5.8}$$

acting for $0 < n < 1$, whereas

$$m_0 < m_c(n) = n^{-1/(n-1)}(n-1)^{1/2} \times (2-n)^{(2-n)/2(n-1)} M_n \ , \tag{5.9}$$

for $1 < n < 2$, in which $M_n = \hbar d_0(\hbar d_0/|C_n|)^{1/(n-2)}$. Of special interest are also the GSE's characterizing the relativistic two body Hamiltonian $\mathscr{H}(r,p) = 2p_0 + V_n$. Such GSE's can be established with the help of Eq. (5.3), now using the rescalings $\mathscr{E} \to \mathscr{E}'/2$ and $\gamma_n^0 \to \gamma_n^0/2$. This leads to the eigenvalue equation

$$|n|\gamma_n^0|1-n|^{1-n} = 2f_n(\mathscr{E}'^2/4m_0^2) \ , \tag{5.10}$$

in which $\mathscr{E}'$ denotes, this time, the GSE of $2p_0 + V_n$. Accordingly, Eqs. (5.6) and (5.7) become $\gamma_n^0 > 2\gamma_c'(n)$ and $\gamma_n^0 < 2\gamma_c(n)$.

VI. THE VECTORIAL LINEAR PLUS COULOMB POTENTIAL

Other concrete examples can also be discussed. The two-body relativistic energy dispersion for the linear plus Coulomb potential reads

$$\delta\mathscr{H}(x) = m_0\left[\frac{2}{x}(1+x^2)^{1/2} - \frac{\alpha}{x d_0} + \gamma_{(-1)} x\right] . \tag{6.1}$$

We notice that (6.1) can also be written as

$$\delta\mathscr{H}(x) = m_0\left[\frac{2}{x}(1+x^2)^{1/2} + 2(\operatorname{sgn}\kappa)\gamma_0 x\right] , \tag{6.2}$$

where the virial equation has been used. The minumum of (6.1) is located at

$$x^2 = f_{(+)}(\alpha,\gamma_0) + f_{(-)}(\alpha,\gamma_0) - \frac{1}{3}\left[1 + \frac{2\alpha}{d_0\gamma_0}\right] > 0 \ , \tag{6.3}$$

in which $x = r/l_1$, $\kappa > 0$, $\alpha = \pm\alpha_0 d_0$, and $\gamma_0 \equiv |\gamma_{(-1)}|$, whereas

$$f_{(\pm)}(\alpha,\gamma_0) = \left[\frac{1}{\gamma_0}\right]^{2/3}\left\{1 \pm \left[1 + \frac{\gamma_0^2}{27}\left[\frac{\alpha}{\gamma_0 d_0} - 1\right]^3\right]^{1/2}\right\}^{2/3} . \tag{6.4}$$

In addition, one has $\alpha_0 < 2$ insofar as $\alpha > 0$. Note that the present virial equation gives the cubic equation

$$u_0^3 + (\alpha/\gamma_{(-1)} - 1)u_0 - 2/\gamma_{(-1)} = 0 \ , \tag{6.5}$$

where $u_0 = (1+x^2)^{1/2} > 1$. The present relativistic Hamiltonian is also subject to a well-defined GSE if $\kappa < 0$ $(\alpha > 0)$. This agrees with the nonrelativistic limit discussed in Sec. IV. The location of this minimum is then given by

$$x^2 = \frac{2(\alpha_0+\gamma_0)}{3\gamma_0}\cos[\tfrac{2}{3}(\phi+\pi)] + \frac{2\alpha_0-\gamma_0}{3\gamma_0} \ , \tag{6.6}$$

provided that $27\gamma_0 < (\alpha_0+\gamma_0)^3$ and $\alpha_0 < 2$, where

$$\cos\phi = 3(3\gamma_0)^{1/2}/(\alpha_0+\gamma_0)^{3/2} \ . \tag{6.7}$$

This time the $\alpha_0 < 2$ condition comes from $u_0 > 1$. Further, Eq. (6.6) is also subject to $\alpha_0 > 2\gamma_0$, $\alpha_0 + \gamma_0 < 3$, and $\gamma_0 < 1$. One realizes that $\mathscr{E}' > 0$ in all the cases discussed above. Next the GSE is subject to the bounds $\mathscr{E}'/m_0 > 2$ for $\alpha < 0$,

$$2(1-\gamma_0^2)^{1/2} > \mathscr{E}'/m_0 > (4-\alpha_0^2)/2 \ , \tag{6.8}$$

for $\kappa < 0$ and

$$\mathscr{E}'/m_0 > (4-\alpha_0^2)/2 \ , \tag{6.9}$$

for $\alpha > 0$ and $\kappa > 0$.

Now let us consider, just for instance, the data $\hbar\kappa \cong 0.177$ GeV^2 (Ref. 21) and $3\alpha/4 \cong 0.23$ (Ref. 22) as well as the masses 0.02, 0.33, 1.28, and 4.57 GeV for the u, s, c, and b quarks, respectively.[23] In addition let us use, this time, the general $d_0 = 1$ choice. Then Eqs. (6.2)–(6.4) show that the GSE's of corresponding quarkonia are given by 1.09, 1.35, 2.94, and 9.22 GeV. Note that $m_u = m_d$, so that $\mathscr{E}'_{u\bar{u}} = \mathscr{E}'_{d\bar{d}}$. For the top quarks $(m_t \cong 30$ GeV) one would have $\mathscr{E}'_{t\bar{t}} \cong 59.33$ GeV. Such results agree satisfactorily with other evaluations performed on this subject.[24]

VII. THE DIRAC BAG MODEL

Now we have the opportunity to analyze the GSE characterizing the Dirac bag Hamiltonian[25]

$$\mathscr{H}(r,p) = \alpha\hbar/2r + (p^2 + \sigma^2 r^4)^{1/2} \ , \tag{7.1}$$

in which $\alpha\hbar/2r$ expresses the Coulomb repulsion, whereas σ relies on the surface tension of the quarks confined within the bag of radius r. It is understood that r and p can be treated as usual canonical variables. Then the minimization of the energy dispersion leads to the GSE

$$\mathscr{E} = 2^{1/6}F(\alpha_0)m_0\gamma'^{1/3} = 2^{1/6}(\hbar d_0)^{2/3}\sigma^{1/3}F(\alpha_0) \ , \tag{7.2}$$

in which $\sigma = m_0\gamma' l_1^{-2} > 0$, $|\alpha| = \alpha_0 d_0$, and

$$F(\alpha_0) = \{(\alpha_0/2)\operatorname{sgn}\alpha + [1 + \tfrac{1}{2}G(\alpha_0)]^{1/2}\}/[G(\alpha_0)]^{1/6} \ . \tag{7.3}$$

Above

$$G(\alpha_0) = 1 + \frac{\alpha_0^2}{16}\left[1 + (\operatorname{sgn}\alpha)\left[1 + \frac{96}{\alpha_0^2}\right]^{1/2}\right] . \tag{7.4}$$

We notice that this time, m_0 has the meaning of an arbitrary mass scale relying on the parametrization of σ. For the sake of generality the $\alpha < 0$ case has also been considered. Accordingly, $\alpha_0 < 2$, which is a typical relativistic result characterizing the attractive Coulomb potential. An interesting property of Eq. (7.2) is the onset of the factorization with respect to the α_0 and γ' parameters. For small α_0 values $F(\alpha_0)$ behaves as

$$F(\alpha_0)=\sqrt{3/2}(1-\tfrac{7}{192}\alpha_0^2)+\operatorname{sgn}\alpha\frac{5\alpha_0}{8}+O(\alpha_0^3)\ , \qquad (7.5)$$

so that $F(0)=(3/2)^{1/2}$. Comparing the $\alpha_0=0$ limit of Eq. (7.2) with the corresponding WKB evaluation of the relativistic GSE,[25] one finds the d_0 fit:

$$d_0=d_0^{\mathrm{WKB}}=3^{5/4}\left[\frac{\pi}{2}\right]^{1/2}\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\cong 1.6725\ , \qquad (7.6)$$

which approaches reasonably the $d_0=\frac{3}{2}$ choice.

In particular, the relativistic Hamiltonian (7.1) can also be interpreted as the superposition of the vectorial Coulomb potential $\alpha\hbar/2r$ with the shifted harmonic-oscillator scalar potential $V_S(r)=\sigma r^2-m_0$. Accordingly, m_0 has the meaning of an "arbitrary" mass parameter. Then the nonrelativistic limit of (7.1) reads

$$H_{\mathrm{NR}}(r,p)=p^2/m_0+\alpha\hbar/2r+\sigma r^2-m_0(2+\gamma')/2\ , \qquad (7.7)$$

insofar as $m_0>|\sigma r^2-m_0|$. This means that $\gamma'>0$ and $x<(2/\gamma')^{1/2}$. So the virial equation works if $4(4-\gamma')\geq\alpha_0(2\gamma')^{1/2}\operatorname{sgn}\alpha$ only. Next, Eq. (2.4) reads $d_0^2-d_0>0$ if $\alpha\geq 0$ so that $d_0>1$. However, combining the right-hand side of Eq. (2.4) with the virial equation leads to $d_0>\frac{1}{4}$ for $\alpha<0$, where the concavity condition $3\alpha_0 x<16$ has also been used. Note that other interpretations of the nonrelativistic limit can also be proposed.[26] Using (7.7) gives the GSE as

$$\tfrac{9}{8}\alpha_0\gamma'\operatorname{sgn}\alpha[(y'+f_B)/3\gamma']^{1/2}=y'^2-3\gamma'+y'f_B\ , \qquad (7.8)$$

in which $y'=E/2m_0+(2+\gamma')/4$ and $f_B=(y'^2+3\gamma')^{1/2}$. One has $y'>0$ $(\alpha>0)$ and $y'>-9\alpha_0^2/256$ $(\alpha<0)$. On the other hand,

$$x^2=(y'+f_B)/3\gamma'<2/\gamma'\ , \qquad (7.9)$$

so that $y'<(12-\gamma')/4$.

VIII. CONCLUSIONS

The above results show that it is of interest to perform a nonperturbative quasiclassical approach to the stability properties characterizing several potentials involved in the description of quarkonia. The present GSE results have been established just using the virial equation $d\delta H(r)/dr=0$, which has to be combined with the concavity condition of $\delta H(r)$. Now one has the possibility to obtain, in a quite simple manner, useful analytical forms characterizing the GSE's and the corresponding stability thresholds. Such thresholds are byproducts of the virial equation itself and/or of the combination of this equation with the concavity condition. Some general remarks concerning the onset of this approach can still be made. The main point is that the virial theorem relying on ϕ_a can be formulated adequately by replacing the usual variational problem for the energy average $E[\psi]=\langle\hat{H}\rangle_\psi$ with the minimization of the quotient $\delta H(r)=\tilde{H}\phi_a/\phi_a$. Suffice it to say that $E[\psi]$ and $\delta H(r)$ exhibit well-defined minima under the influence of the dilations $\psi(r)\to\psi'(r)\sim\psi(\lambda' r)$ and $r\to r'=\lambda' r$ $(\lambda'=1+\epsilon)$. The minimization of $\tilde{H}\psi_a/\psi_a$ can also be performed, but this relies on Hamiltonians with negative kinetic energy. However, our approach is based on the assumption that d_0 is the same for both ϕ_a and ψ_a. On the other hand, the usual variational procedure ceases to work if ψ is a scale-invariant function, i.e., if $\psi'(r)\sim\psi(r)$. This necessitates the proposal of an appropriate extension of the virial theorem for scale-invariant powerlike probe functions, as we did above.

In this paper we have concerned ourselves with some typical quark potentials. Sensible nonperturbative results concerning the underlying mass spectrum, the upper and/or lower bounds on the GSE, or the relevant domains of admissible couplings have been established. Such results appear to provide a better understanding of the small distance region. Now we are able to distinguish in greater detail between candidate potentials. Spin-dependent forces can also be discussed by averaging separately the spin factors. General $\mathbf{x}$-dependent potentials are also of further interest. Such potentials can be analyzed in a similar manner using the probe function $(\mathbf{x}\cdot\mathbf{n})^{-id_0}$ instead of ϕ_a. Here $\mathbf{n}$ denotes a unit-vector parameter. In this way a consistent quasiclassical picture of the GSE for realistic atomic, molecular, or nuclear potentials is now emerging.

We end with a brief presentation of some corrections relying on power potentials. The static electromagnetic potential energy of two point-charge scalar particles is given to fourth order by a repulsive $n=2$ potential.[27] Fourth-order gravitational potential energies[27] and quantum-electrodynamical corrections to the static gravitational potential energy of the electron[28] have also been analyzed in terms of $n=2$ potentials. The onset of van der Waals forces acting between color-neutral objects,[29] as well as many other similar forces acting between charged and neutral systems, should also be noticed.[30] Vacuum fluctuations relying on $n=-4$ potentials,[31] as well as universal potential curves for quarkonia,[32] should also be mentioned. Now we have the possibility to obtain sensible quasiclassical results and to make further syntheses. In this respect Eqs. (4.8), (4.9), (B2), (B5), and (B7) can be useful. Of course, "exact" d_0 values are still desirable. In this sense we might notice that $d_0=1$ gives the exact Dirac GSE for the superposition of scalar and vectorial Coulomb potentials,[33] as one might expect. The same remains valid for the scalar Coulomb potential.[34] Moreover, $d_0\cong 1.376$ gives the exact nonrelativistic GSE for the linear potential,[35] whereas the "exact" GSE of the zero-mass Dirac Hamiltonian with the scalar linear potential comes from $d_0\cong 1.311$.[36] Further mathematical details concerning the present quasiclassical approach will be presented in a subsequent paper.

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APPENDIX A: THE CONVEXITY BEHAVIOR

Let us parametrize a certain potential energy $V(r)$ as $V(r)=\lambda U(r)$, where λ expresses the coupling. Therefore $U(r)$ is independent of λ. The nonrelativistic GSE is given by

$$E(\lambda)=\lambda U+\hbar^2 d_0^2/2m_0 r^2 , \tag{A1}$$

where $r=r(\lambda)$ is subject to the virial equation

$$\lambda U'=\hbar^2 d_0^2/m_0 r^3>0 , \tag{A2}$$

as well as to the concavity condition

$$F^{(+)}(r)\equiv\lambda(r^2U''+3rU']>0 . \tag{A3}$$

Here $U'=dU/dr$ and $U''=d^2U/dr^2$. In consequence,

$$\frac{dE}{d\lambda}=U+\frac{dr}{d\lambda}\left[\lambda U'-\frac{\hbar^2 d_0^2}{m_0 r^3}\right]=U[r(\lambda)] , \tag{A4}$$

so that

$$\frac{d^2E}{d\lambda^2}=U'\frac{dr}{d\lambda} . \tag{A5}$$

Differentiating (A2) with respect to λ gives

$$U'=-\frac{dr}{d\lambda}\frac{F^{(+)}}{r^2}\sim-\frac{dr}{d\lambda} , \tag{A6}$$

insofar as $dd_0/d\lambda=0$, where (A3) has been used. Then (A5) and (A6) yield $d^2E/d\lambda^2<0$, which shows that the GSE is a convex function of λ. In other words, the convexity of the GSE with respect to the coupling is a consequence of the concavity of the energy dispersion $\delta H(r)$, provided that d_0 is independent of the coupling.

In the relativistic case one proceeds quite similarly. Now,

$$\mathcal{E}(\lambda)=\lambda U+f_0(r) , \tag{A7}$$

where $f_0(r)=(m_0^2+\hbar^2 d_0^2/r^2)^{1/2}$. Above λU is a vectorial potential. Then

$$\frac{d^2\mathcal{E}}{d\lambda^2}=U'\frac{dr}{d\lambda} , \tag{A8}$$

where the virial equation

$$\lambda U'=\hbar^2 d_0^2/r^3 f_0>0 \tag{A9}$$

has been used. Now the concavity condition is

$$G^{(+)}(r)\equiv\lambda U''+3\hbar^2 d_0^2/r^4 f_0-\hbar^4 d_0^4/r^6 f_0^3>0 . \tag{A10}$$

Thus the differentiation of (A9) with respect to λ yields $U'\sim-(dr/d\lambda)G^{(+)}$, provided again that d_0 is independent of λ. In consequence $d^2\mathcal{E}/d\lambda^2<0$, which exhibits the convexity of the GSE for relativistic Hamiltonians with vectorial potentials. In addition one has $\lambda(dr/d\lambda)<0$, both in relativistic and nonrelativistic cases. We notice that for scalar potentials, or for equally mixed scalar and vectorial potentials, the convexity of the GSE is able to be preserved for suitable λ values only.

APPENDIX B: SINGULAR POTENTIALS

In this appendix we will present GSE results concerning the superposition $V(r)=V_2(r)+V_n(r)$ of the singular $V_2(r)$ potential with other power potentials $V_n(r)$, where $n\neq 2$. First let us consider that $V_2(r)$ is a repulsive potential so that $V_2(r)=|C_2|/r^2$. Using the parametrization $C_n=m_0\gamma_n l_1^n$, the dispersion of the nonrelativistic (two-body) Hamiltonian reads

$$\delta H(x)=m_0\left[\frac{1+\gamma_2^0}{x^2}+\frac{\gamma_n}{x^n}\right] , \tag{B1}$$

where $\gamma_n^0=|\gamma_n|$ and $x=r/l_1$. Next one realizes that the minimization (2.1) works only for attractive $n<2$ potentials ($n\gamma_n<0$). This yields the GSE

$$E(n<2;\gamma_2^0)=m_0(1+\gamma_2^0)\frac{n-2}{n}\left[\frac{|n|\gamma_n^0}{2(1+\gamma_2^0)}\right]^{2/(2-n)} . \tag{B2}$$

On the other hand, Eq. (2.4) reads

$$\begin{aligned}P_2(d)&\equiv d^2+d-l(l+1)-m_0|C_2|/\hbar^2\\&=m_0C_n r^{2-n}/\hbar^2 .\end{aligned} \tag{B3}$$

This means, in general, that $P_2(d)<0$ if $0<n<2$, whereas $P_2(d)>0$ for $n<0$. However r^{2-n} vanishes near the origin, so that the dominant behavior of $P_2(d)$ is given by $P_2(d)=0$. Then we find

$$|d(l=0;\gamma_2^0)|=\frac{1}{2}+\left[\frac{1}{4}+\frac{m_0C_2}{\hbar^2}\right]^{1/2} , \tag{B4}$$

where the irregular choice of the power exponent has been considered. Now we have the possibility to make the following two identifications:[2] $d_0^2=|d(l=0;C_2)|$ or $d_0=|d(l=0;C_2)|$. Using the first alternative yields $d_0^2=1+\gamma_2^0$. So the GSE becomes

$$\begin{aligned}E(n<2;\gamma_2^0)&=m_0\frac{n-2}{n}(d_0^2)^{-n/(2-n)}\\&\times\left[\frac{|n|\gamma_n^0}{2}\right]^{2/(2-n)} ,\end{aligned} \tag{B5}$$

in which d_0^2 is given by (B4). For the Kratzer potential[10] we have to set $n=1$, so that

$$E(1;\gamma_2^0)=-m_0\frac{\gamma_1^{02}}{4d_0^2}=-m_0\frac{C_1^2}{4\hbar^2 d_0^4} . \tag{B6}$$

This produces the exact GSE characterizing the single-particle Hamiltonian via the rescaling $m_0\rightarrow 2m_0$. In general, $d_0^2>|d(l=0;\gamma_2^0)|$ for $n<0$, whereas $d_0^2<|d(l=0;\gamma_2^0)|$ if $0<n<2$. This means that $d_0^2=|d(l=0;\gamma_2^0)$ can also be interpreted as the dominant d_0^2 choice characterizing the whole $n<2$ region.

We now consider the attractive $V_2(r)$ potential. This time the GSE reads

$$E(n;-\gamma_2^0)=m_0\frac{n-2}{n}(1-\gamma_2^0)\left[\frac{n\gamma_n}{2(\gamma_2^0-1)}\right]^{2/(2-n)} , \tag{B7}$$

insofar as

$$n\gamma_n > 0, \quad n > 2, \quad \gamma_2^0 > 1 \; ,$$
$$n\gamma_n < 0, \quad n < 2, \quad \gamma_2^0 < 1 \; . \tag{B8}$$

In general, the present power exponent reads

$$d_0^{(\pm)}(\gamma_2^0) \equiv |\, d^{(\pm)}(l=0; -\gamma_2^0)\, | = \frac{1}{2} \pm \left[\frac{1}{4} - \frac{m_0 |C_2|}{\hbar^2} \right]^{1/2} , \tag{B9}$$

provided that $|C_2| \leq \hbar^2/4m_0$. This inequality can be rewritten equivalently as $1 \geq 4\gamma_2^0 d_0^2$. For the attractive $n<2$ potentials ($\gamma_2^0<1$) the discussion is analogous to the one performed above. Indeed, $d_0^2 = d_0^{(\pm)}(\gamma_2^0)$ leads to the dominant choice $d_0^2 = 1-\gamma_2^0$. Then one would have $0 < d_0^2 \leq \frac{1}{2}$ for $1 > \gamma_2^0 \geq \frac{1}{2}$ insofar as the sign "$-$" would be considered, whereas $\frac{1}{2} \leq d_0^2 < 1$ and $0 < \gamma_2^0 \leq \frac{1}{2}$ act for the sign "$+$". In this respect $\mathrm{sgn}(\frac{1}{2} - \gamma_0^2) \equiv \pm$. Note that the inequality $|C_2| \leq \hbar^2/4m_0$ is fulfilled self-consistently in terms of $(2\gamma_2^0 - 1)^2 \geq 0$. Alternatively, one would also have the possibility of probing the choice $d_0 = d_0^{(+)}(\gamma_2^0)$. Then $d_0 = 1/(1+\gamma_2^0)$, so that $1/2 \leq d_0 < 1$ for $0 < \gamma_2^0 \leq 1$.

Considering the repulsive $n>2$ potentials ($\gamma_2^0>1$), one realizes immediately that the right-hand side of (B3) cannot be ignored. Now $P_2(d) > 0$, whereas the d_0-fixing condition $d_0^2 = d_0^{(\pm)}(\gamma_2^0)$ ceases to be meaningful. Under such conditions the only "admissible" choice is $d_0 = d_0^{(-)}(\gamma_2^0)$. This yields $d_0 = 1/(1+\gamma_2^0)$, so that $0 < d_0^2 \leq \frac{1}{4}$ for $\gamma_2^0 \geq 1$. The condition $|C_2| \leq \hbar^2/4m_0$ is also fulfilled self-consistently, now in terms of $(1-\gamma_2^0)^2 \geq 0$. The above results also show that d_0 is subject to complex values as soon as $|C_2| > \hbar^2/4m_0$, thereby involving quasiclassical predictions of corresponding resonances. We might also notice that Eq. (B9) comes from the generalization of Eq. (4.21) of Ref. 2.

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$$(\alpha/r)u_0(r) = (2/\pi)\int_0^\infty dr' \int_0^\infty dk\, k \sin(kr)\sin(kr')u_0(r') \; ,$$

where $u_0(r) = rR_0(r) \sim r^{d'}$. In consequence $\alpha = d' \cot(\pi d'/2)$. This equation has d' roots confined within the interval $d' \in (-1,1)$ if $0 < \alpha < 2/\pi$ only.

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