Time-dependent harmonic oscillator with variable mass under the action of a driving force

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An exact solution is presented for the problem of a harmonic oscillator of time-dependent mass with an external driving force. The wave functions for both pseudostationary and quasicoherent states and the Green's function are calculated. A solution is obtained in the Heisenberg picture. The amplitude for transition from a coherent state to a simple harmonic-oscillator coherent state is calculated.

I. INTRODUCTION

Widespread attention has been paid in the last few decades to the problem of the time-dependent harmonic oscillator.[1–5] The problem of a variable-frequency, or variable-mass harmonic oscillator has many applications in physics, e.g., in quantum chemistry, quantum optics,[6–9] plasma physics,[10,11] and perhaps also in quantum-field theory, see, e.g., Ref. 12. Considerable effort has been devoted to finding invariants of the motion for such a system[13–15] (see also Ref. 16 and references quoted therein). In a previous communication I and my co-workers have considered some special time-dependent mass parameters[17–20] in the absence of a driving force, while in Refs. 21 and 22, respectively, we discussed the case of strongly and weakly pulsating mass in the presence of a periodic driving force. In the present paper we shall develop the work of Ref. 5 by including a driving force in the case of a harmonic oscillator with mass parameter a general function of the time. The variable-mass harmonic oscillator with a driving force is described by the Hamiltonian[1,2,21]

$$H(t)=\frac{p^2}{2M(t)}+[M(t)/m][\tfrac{1}{2}m\omega_0^2q^2-mf(t)q]\ , \tag{1.1}$$

where the mass variable will be taken as

$$M(t)=m\exp\left[2\int\gamma(t)dt\right] \tag{1.2}$$

and $f(t)$ and $\gamma(t)$ are arbitrary functions of the time. The technique we use gives an exact solution for the Schrödinger equation

$$H(t)\,|\,\psi(t)\rangle=i\hbar\frac{\partial}{\partial t}\,|\,\psi(t)\rangle \tag{1.3}$$

and leads to a definition of the Dirac operator which we shall refer to as the "best" operator. Malkin and Man'ko[23] have constructed coherent states for a system described by Eq. (1.3) based on the employment of quantum integrals of the motion. In 1979, Dodonov and Man'ko[2] employed the linear integrals of the motion in order to construct the best systems of coherent states for a general driven time-dependent oscillator, i.e., a system given by Eq. (1.1). However, in the present paper I have used the canonical transformation approach of Ref. 5 to treat the problem of finding the best Dirac operators by solving the Schrödinger equation (1.3). The organization of the paper is as follows. In Sec. II I work in the Schrödinger picture and in Sec. III we derive the solution in the Heisenberg picture and follow this with an expression for the energy operator. The Dirac operator and quasicoherent states are calculated in Sec. IV. In Sec. V the Green's function and the coherent-state transition amplitude has been calculated, followed by a discussion in Sec. VI.

II. THE WAVE FUNCTION IN THE SCHRÖDINGER REPRESENTATION

As a first step one has to solve the time-dependent Schrödinger equation (1.3). By using the canonical scaling transformation introduced in Refs. 17 and 18 the Hamiltonian (1.1) with $M(t)$ given by Eq. (1.2) takes the form

$$H(t)=\frac{P^2}{2m}+\tfrac{1}{2}m\omega_0^2Q^2+\frac{\gamma(t)}{2}(PQ+QP)-F(t)Q \tag{2.1}$$

with $[Q,P]=i\hbar$, and $F(t)$ is

$$F(t)=mf(t)\exp\left[\int\gamma(t)dt\right]. \tag{2.2}$$

From Eqs. (1.3) and (2.1) the Schrödinger equation is

$$\frac{\partial^2\psi}{\partial Q^2}-\frac{m^2\omega_0^2}{\hbar^2}Q^2\psi+\frac{2im}{\hbar}\gamma(t)Q\frac{\partial\psi}{\partial Q}+\frac{2mF(t)}{\hbar}Q\psi=-\frac{2im}{\hbar}\frac{\partial\psi}{\partial t}-\frac{im\gamma(t)}{\hbar}\psi\ . \tag{2.3}$$

In order to obtain a solution of Eq. (2.3) I introduce two transformations. The first one changes the coordinate Q to x through the relation

$$Q=\frac{x}{\sqrt{\omega(t)}}\ , \tag{2.4a}$$

where $\omega(t)$ is explicitly time dependent given by Eq. (2.13) in Ref. 5, i.e.,

$$\ddot{\rho}+[\omega_0^2-\gamma^2(t)-\dot{\gamma}(t)]\rho=\frac{1}{\rho^3},\quad \rho^{-2}=\omega(t)\ . \tag{2.4b}$$

Then the wave function $\psi(Q,t)$ changes to $\phi(x,t)$ and Eq. (2.3) takes the form

$$\frac{\partial^2\phi}{\partial x^2}-\frac{m^2\omega_0^2}{\hbar^2}\frac{x^2}{\omega^2(t)}\phi+\frac{2im}{\hbar\omega(t)}\left[\gamma(t)+\frac{\dot{\omega}(t)}{2\omega(t)}\right]x\frac{\partial\phi}{\partial x}+\frac{2mF(t)}{\hbar^2}\frac{x}{\omega^{3/2}(t)}\phi=\frac{-2im}{\omega(t)\hbar}\frac{\partial\phi}{\partial t}-\frac{im\gamma(t)}{\hbar\omega(t)}\phi\ . \tag{2.5}$$

The second transformation translates the x axis to a y axis by the equation

$$x=y+\zeta(t)\ , \tag{2.6}$$

where $\zeta(t)$ is to be determined. The wave function must now be changed from $\phi(x,t)$ to $\bar{\phi}(y,t)$ and Eq. (2.5) is recast in the form

$$\frac{\partial^2\bar{\phi}}{\partial y^2}+\frac{2im}{\hbar}\left[R(t)[y+\zeta(t)]-\frac{\dot{\xi}(t)}{\omega(t)}\right]\frac{\partial\bar{\phi}}{\partial y}-\frac{m^2\omega_0^2}{\hbar^2\omega^2(t)}[y^2+2\xi(t)y]\bar{\phi}+\frac{2mF(t)\bar{\phi}}{\hbar^2\omega^{3/2}(t)}y-\frac{im\gamma(t)\bar{\phi}}{\hbar\omega(t)}$$
$$+\frac{2mF(t)}{\hbar^2\omega^{3/2}(t)}\zeta(t)\bar{\phi}-\frac{m^2\omega_0^2}{\hbar^2\omega^2(t)}\zeta^2(t)\bar{\phi}=\frac{-2im}{\hbar\omega(t)}\frac{\partial\bar{\phi}}{\partial t}\ , \tag{2.7}$$

where

$$R(t)=\frac{1}{\omega(t)}\left[\gamma(t)+\frac{\dot{\omega}(t)}{2\omega(t)}\right]\ . \tag{2.8}$$

Let us seek a separation in the form

$$\bar{\phi}(y,t)=G(y)T(t)\exp\left[\frac{-im}{\hbar}\left[R(t)[\tfrac{1}{2}y^2+\zeta(t)y]-\frac{\dot{\zeta}(t)}{\omega(t)}y\right]\right]\ , \tag{2.9}$$

which leads to a partially separated equation in the form

$$\frac{1}{G}\frac{d^2G}{dy^2}-\frac{m^2}{\hbar^2}y^2+\frac{2m^2}{\hbar\omega}\left[\left[\dot{R}(t)+\omega(t)R^2(t)-\frac{\omega_0^2}{\omega(t)}\right]\xi(t)-\frac{d}{dt}\left[\frac{\dot{\xi}(t)}{\omega(t)}\right]+\frac{F(t)}{m\omega^{1/2}(t)}\right]y$$
$$=\frac{-2im}{\hbar\omega(t)}\frac{1}{T}\frac{dT}{dt}+\frac{im}{\hbar}R(t)-\frac{im\gamma(t)}{\hbar\omega(t)}-\frac{2mF(t)}{\hbar^2\omega^{3/2}(t)}\zeta(t)+\frac{\omega_0^2m^2}{\hbar^2\omega^2(t)}\zeta^2(t)-\frac{m^2}{\hbar^2}\left[R(t)\xi(t)-\frac{\dot{\xi}(t)}{\omega(t)}\right]^2\ . \tag{2.10}$$

In order to complete the separation, we must eliminate the coefficient of y on the left-hand side of Eq. (2.10) and choose

$$\ddot{\zeta}(t)-\frac{\dot{\omega}(t)}{\omega(t)}\dot{\zeta}(t)+\omega^2(t)\zeta(t)=\frac{\sqrt{\omega(t)}}{m}F(t)\ . \tag{2.11}$$

Equation (2.11) is a second-order nonhomogeneous differential equation and in the absence of a driving force $\zeta(t)$ is zero. When the driving force is applied, the solution of this equation may be taken for convenience in the form

$$\zeta(t)=\{[I_c(t)-I_c(0)]\sin[g(t)]-[I_s(t)-I_s(0)]\cos[g(t)]\}\ , \tag{2.12}$$

where

$$I_c(t)=\int\frac{f(t)\exp\left[\int\gamma(t)dt\right]}{\sqrt{\omega(t)}}\cos[g(t)]dt\ , \tag{2.13a}$$

$$I_s(t)=\int\frac{f(t)\exp\left[\int\gamma(t)dt\right]}{\sqrt{\omega(t)}}\sin[g(t)]dt\ , \tag{2.13b}$$

and

$$g(t)=\int_0^t\omega(\tau)d\tau\ . \tag{2.13c}$$

The separation is completed and the general solution is of the form

$$\psi_n(Q,t)=\left[\frac{m\omega(t)}{\hbar\pi}\right]^{1/4}2^{-n/2}(n!)^{-1/2}H_n\left[\left[\frac{m\omega(t)}{\hbar}\right]^{1/2}[Q-\zeta_1(t)]\right]$$
$$\times\exp\left\{\frac{-m}{2\hbar}\left[\omega(t)+i\left[\gamma(t)+\frac{\dot\omega(t)}{2\omega(t)}\right]\right][Q-\zeta_1(t)]^2\right\}$$
$$\times\exp\left[\frac{-im}{\hbar}\{[\gamma(t)\zeta_1(t)-\dot\zeta_1(t)][Q-\zeta_1(t)]\}\right]\exp\left[-i(n+\tfrac{1}{2})g(t)+\frac{i\phi(t)}{\hbar}\right], \quad (2.14)$$

where

$$\zeta_1(t)=\zeta(t)/\sqrt{\omega(t)} \quad (2.15a)$$

and

$$\phi(t)=\int_0^t F(t')\zeta_1(t')dt' -(m/2)\int_0^t\{\omega_0^2\zeta_1^2(t') -[\gamma(t')\zeta_1(t')-\dot\zeta_1(t')]^2\}dt' . \quad (2.15b)$$

When we consider the replacement $\gamma\to-\nu\tan(\nu t)$ it is interesting to compare the solution given in Eq. (2.14) and the result given by Eq. (54) of Ref. 21. In the absence of a driving force, so that $F(t)=0$, Eq. (2.14) agrees with Eq. (3.10) of Ref. 5, and Eq. (3.10) of Ref. 24.

III. THE HEISENBERG EQUATIONS OF MOTION

Having solved the problem in the Schrödinger picture, we shall turn our attention to the solution in the Heisenberg picture. The Hamiltonian (2.1) gives

$$\frac{dQ}{dt}=\frac{P}{m}+\gamma(t)Q , \quad (3.1a)$$

$$\frac{dP}{dt}=-m\omega_0^2Q-\gamma(t)P+F(t) . \quad (3.1b)$$

The elimination of P leads to

$$\frac{d^2Q}{dt^2}+(\omega_0^2-\dot\gamma-\gamma^2)Q=\frac{F(t)}{m} . \quad (3.2)$$

By introducing the transformation

$$Z(t)=\sqrt{\omega(t)}Q(t) \quad (3.3)$$

Eq. (3.2) may be cast in the form

$$\frac{d^2Z}{dt^2}-\frac{\dot\omega(t)}{\omega(t)}\frac{dZ}{dt}+\left[\omega_0^2-\gamma^2-\dot\gamma+\frac{3}{4}(\dot\omega/\omega)^2-\frac{1}{2}\frac{\ddot\omega}{\omega}\right]Z =\frac{\sqrt{\omega(t)}}{m}F(t) . \quad (3.4)$$

With the aid of Eq. (2.4b) one may write the general solution in the following form:

$$Q(t)=\left[\left[\frac{\omega(0)}{\omega(t)}\right]^{1/2}\cos[g(t)]+\frac{J(0)}{\sqrt{\omega(t)\omega(0)}}\sin[g(t)]\right]Q(0)+\frac{\sin[g(t)]}{m\sqrt{\omega(t)\omega(0)}}P(0)$$
$$+\frac{1}{\sqrt{\omega(t)}}\{I_s(0)-I_s(t)\cos[g(t)]+[I_c(t)-I_c(0)]\sin[g(t)]\} \quad (3.5)$$

and

$$P(t)=\left[\left[\frac{\omega(t)}{\omega(0)}\right]^{1/2}\cos[g(t)]-\frac{J(t)}{\sqrt{\omega(t)\omega(0)}}\sin[g(t)]\right]P(0)$$
$$+mQ(0)\left[\frac{\omega(t)J(0)-J(t)\omega(0)}{\sqrt{\omega(0)\omega(t)}}\cos[g(t)]-\sqrt{\omega(0)\omega(t)}\left[1+\frac{J(0)J(t)}{\omega(0)\omega(t)}\right]\sin[g(t)]\right]$$
$$+\frac{m}{\sqrt{\omega(t)}}(\{[I_c(t)-I_c(0)]\omega(t)+[I_s(t)-I_s(0)]J(t)\}\cos[g(t)]$$
$$+\{[I_s(t)-I_s(0)]\omega(t)-[I_c(t)-I_c(0)]J(t)\}\sin[g(t)]) , \quad (3.6)$$

where $I_s(t)$ and $I_c(t)$ are given by Eq. (2.13) and

$$J(t)=\left[\gamma(t)+\frac{1}{2}\frac{\dot\omega(t)}{\omega(t)}\right] . \quad (3.7)$$

From Eqs. (3.5) and (3.6) we can prove that

$$[Q(t),P(t)]=[Q(0),P(0)]=i\hbar . \quad (3.8)$$

Now one is in a position to extend the expression for the energy given in Refs. 5 and 21 to include the work done by the driving force. From Eqs. (1.1), (3.5), and (3.6) one finds the following expression for $E(t)$:

$$E(t)=E_0(t)+mQ(0)\{U_Q(t)\cos[g(t)]+W_Q(t)\sin[g(t)]\}$$
$$+P(0)\{U_p(t)\cos[g(t)]+W_p(t)\sin[g(t)]\}+\tfrac{1}{2}m\left[\omega_0^2\zeta_1^2+[\dot{\zeta}_1-\gamma(t)\zeta_1(t)]^2-\frac{2}{m}F(t)\xi_1(t)\right], \tag{3.9}$$

where

$$U_Q(t)=\left\{\left[\left[\frac{\omega(0)}{\omega(t)}\right]^{1/2}J(t)-\left[\frac{\omega(t)}{\omega(0)}\right]^{1/2}J(0)\right][\gamma(t)\zeta_1(t)-\dot{\zeta}_1(t)]+\left[\frac{\omega(0)}{\omega(t)}\right]^{1/2}\left[\omega_0^2\xi_1(t)-\frac{F(t)}{m}\right]\right\}, \tag{3.10a}$$

$$W_Q(t)=\frac{J(0)}{\sqrt{\omega(0)\omega(t)}}\left[\omega_0^2\xi(t)-\frac{F(t)}{m}\right]+[\gamma(t)\zeta_1(t)-\dot{\zeta}_1(t)]\left[\sqrt{\omega(0)\omega(t)}+\frac{J(0)J(t)}{\sqrt{\omega(0)\omega(t)}}\right], \tag{3.10b}$$

$$U_p(t)=\sqrt{\omega(t)/\omega(0)}[\dot{\zeta}_1(t)-\gamma(t)\zeta_1(t)], \tag{3.10c}$$

$$W_p(t)=\{[\omega_0^2+\gamma(t)J(t)]\zeta_1(t)-[F(t)+J(t)\dot{\zeta}_1(t)]\}/\sqrt{\omega(t)\omega(0)}, \tag{3.10d}$$

where $\zeta_1(t)$ is given by Eq. (2.15a) and $E_0(t)$ denotes the expression for the energy in the absence of a driving force,

$$E_0(t)=\omega^{-1}(0)\left[\omega(t)\cos^2[g(t)]+\left[\frac{J^2(t)+\omega_0^2}{\omega(t)}\right]\sin^2[g(t)]-J(t)\sin[2g(t)]\right]T(0)$$
$$+\omega^{-1}(t)\left\{\left[\omega(0)+\frac{[J(0)\omega(t)-J(t)\omega(0)]^2}{\omega_0^2\omega(0)}\right]\cos^2[g(t)]\right.$$
$$+\left[\frac{J^2(0)}{\omega(0)}+\frac{[\omega(0)\omega(t)+J(t)J(0)]^2}{\omega_0^2\omega(0)}\right]\sin^2[g(t)]$$
$$\left.+\left[J(0)-[J(0)\omega(t)-J(t)\omega(0)]\left[\omega(t)+\frac{J(0)}{\omega(0)}J(t)\right]\right]\sin[2g(t)]\right\}V(0)$$
$$+\frac{1}{4}\left[\left[\frac{\omega_0^2+J^2(t)-\omega^2(t)}{\omega(t)}-\frac{2J(0)J(t)}{\omega(0)}\right]\sin[2g(t)]+\frac{2\omega_0^2J(0)}{\omega(t)\omega(0)}\sin^2[g(t)]\right][Q(0),P(0)]_+ . \tag{3.11}$$

IV. THE DIRAC OPERATOR, QUASICOHERENT STATES, AND NUMBER STATES

In this section I shall derive the best Dirac operator in the presence of a driving force. Since the quasicoherent state $\psi_\alpha(Q,t)$ can be expanded in a power series of α, i.e.,

$$\psi_\alpha(Q,t)=\exp(-\tfrac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\psi_n(Q,t), \tag{4.1}$$

where $\psi_n(Q,t)$ is given by Eq. (2.14). Therefore with aid of the identity

$$\exp(2v\theta-\theta^2)=\sum_{n=0}^{\infty}H_n(v)\frac{\theta^n}{n!}, \tag{4.2}$$

where $H_n(v)$ is the Hermite polynomial one can easily obtain $\psi_\alpha(Q,t)$ in the form

$$\psi_\alpha(Q,t)=\left[\frac{m\omega(t)}{\hbar\pi}\right]^{1/4}\exp(-\tfrac{1}{2}|\alpha|^2)\exp\left[\left[\frac{2m\omega(t)}{\hbar}\right]^{1/2}[Q-\zeta_1(t)]\alpha(t)-\frac{\alpha^2(t)}{2}\right]$$
$$\times\exp\left[\frac{-m}{2\hbar}\{[\omega(t)+iJ(t)][Q-\zeta_1(t)]^2+2i[\gamma(t)\zeta_1(t)-\dot{\zeta}_1(t)][Q-\zeta_1(t)]\}\right]\exp\left[\frac{i}{\hbar}\left[\phi(t)-\frac{\hbar}{2}g(t)\right]\right], \tag{4.3}$$

where

$$\alpha(t)=\alpha(0)\exp\left[-i\int_0^t\omega(t')dt'\right] \tag{4.4}$$

and $\phi(t)$ is given by Eq. (2.15b).

In order to construct the best operator in the presence of a driving force, we shall differentiate Eq. (4.3) partially with respect to Q. Thus

$$\frac{\partial\psi_\alpha}{\partial Q}=\left[\left[\frac{2m\omega(t)}{\hbar}\right]^{1/2}\alpha(t)-\frac{m}{\hbar}[\omega(t)+iJ(t)][Q-\zeta_1(t)]\right.$$
$$\left.-\frac{im}{\hbar}[\gamma(t)\zeta_1(t)-\dot{\zeta}_1(t)]\right]\psi_\alpha(Q,t)\ . \tag{4.5}$$

Since α is the eigenvalue of any operator $A(t)$ satisfying the relation

$$[A,A^\dagger]=1 \tag{4.6}$$

such that

$$A(t)\,|\,\alpha(t)\rangle=\alpha(t)\,|\,\alpha(t)\rangle \tag{4.7}$$

one uses the fact that

$$p=\frac{\hbar}{i}\frac{\partial}{\partial Q} \tag{4.8}$$

to find the best Dirac operator in the form

$$A(t)=[2m\omega(t)\hbar]^{-1/2}\{m[\omega(t)+iJ(t)]Q(t)$$
$$+iP(t)-mK(t)\}\ , \tag{4.9}$$

where

$$K(t)=\zeta_1(t)\left[\omega(t)+i\frac{d}{dt}[\ln\zeta(t)]\right]\ . \tag{4.10}$$

Alternatively, this operator can be found from the linear combination of the coordinate and momentum in Eqs. (3.5) and (3.6). The scaled coordinate Q and momentum P may be expressed as

$$Q(t)=\{[\hbar/2m\omega(t)]^{1/2}(A+A^\dagger)+\zeta_1(t)\}\ , \tag{4.11}$$

$$P(t)=\frac{\sqrt{m\omega(t)\hbar}}{2}A^\dagger\left[i-\frac{J(t)}{\omega(t)}\right]-A\left[i+\frac{J(t)}{\omega(t)}\right]$$
$$+m[\dot{\zeta}_1(t)-\gamma(t)\zeta_1(t)]\ . \tag{4.12}$$

From Eqs. (4.11) and (4.12) Eq. (2.1) can be written as follows:

$$H(t)=\hbar\left[\omega(t)+\frac{1}{2}\frac{\dot{J}(t)}{\omega(t)}\right](A^\dagger A+\tfrac{1}{2})+\frac{\hbar}{4\omega(t)}\left[\frac{d}{dt}[J(t)-i\omega(t)]\right](A^\dagger)^2$$
$$+\frac{\hbar}{4\omega(t)}\left[\frac{d}{dt}[J(t)+i\omega(t)]\right]A^2$$
$$+\left[\frac{m\hbar}{2\omega(t)}\right]^{1/2}\left[[\omega_0^2-\gamma^2(t)]\zeta_1(t)+i\left[\omega(t)+i\frac{\dot{\omega}(t)}{2\omega(t)}\right]\dot{\zeta}_1(t)-f(t)\exp\left[\int\gamma(t)dt\right]\right]A^\dagger$$
$$+\left[\frac{m\hbar}{2\omega(t)}\right]^{1/2}\left[[\omega_0^2-\gamma^2(t)]\zeta_1(t)-i\left[\omega(t)-i\frac{\dot{\omega}(t)}{2\omega(t)}\right]\dot{\zeta}_1(t)-f(t)\exp\left[\int\gamma(t)dt\right]\right]A$$
$$+\tfrac{1}{2}m\left[[\omega_0^2-\gamma^2(t)]\zeta_1^2(t)+\dot{\zeta}_1^2(t)-2f(t)\zeta_1(t)\exp\left[\int\gamma(t)dt\right]\right]\ . \tag{4.13}$$

Since the time-dependent number states of any operator $A,A^\dagger$ are given by

$$A\,|\,n(t)\rangle=\sqrt{n}\,|\,n-1\rangle\ , \tag{4.14a}$$

$$A^\dagger\,|\,n(t)\rangle=\sqrt{n+1}\,|\,n+1\rangle\ , \tag{4.14b}$$

$$A^\dagger A\,|\,n(t)\rangle=n\,|\,n(t)\rangle,\quad n=0,1,2,\ldots \tag{4.14c}$$

the expectation values of $E(t)$ and $H(t)$ can be found in state $|\,n\rangle$. Thus

$$\langle n\,|E(t)|\,n\rangle=\langle n\,|E_0(t)|\,n\rangle$$
$$+\tfrac{1}{2}m\{\omega_0^2\zeta_1^2+[\dot{\zeta}_1(t)-\gamma_1(t)\xi_1]^2\}$$
$$-F(t)\xi_1(t) \tag{4.15}$$

and

$$\langle n\,|H(t)|\,n\rangle=\langle n\,|H_0(t)|\,n\rangle$$
$$+\tfrac{1}{2}m\left[[\omega_0^2-\gamma^2(t)]\xi_1^2+\dot{\zeta}_1^2(t)\right.$$
$$\left.-2f(t)\zeta_1(t)\exp\left[\int\gamma(t)dt\right]\right]\ , \tag{4.16}$$

where $\langle n\,|E_0(t)|\,n\rangle$ and $\langle n\,|H_0(t)|\,n\rangle$ are the expectation values for the operators $E(t)$ and $H(t)$ in the absence of a driving force respectively.

V. GREEN'S FUNCTION

Since the solution in the Heisenberg picture is given in Sec. III, one is in a position to construct the Green's function.

The definition of the Green's function $G(Q,Q_0,t)$ is

$$G(Q,Q_0,t)=\exp\left[-\frac{i}{\hbar}\int_0^t H(\tau)d\tau\right]\delta(Q-Q_0)\ , \quad (5.1)$$

therefore for any later time t, we have

$$Q(-t)G(Q,Q_0,t)=Q_0G(Q,Q_0,t),\quad t>0\ . \quad (5.2)$$

Then from Eqs. (3.5) and (5.2) together with Eq. (1.3),

$$G(Q,Q_0,t) =G_0(Q,t)\exp\left[\frac{i}{2\hbar B}[AQ_0^2-2QQ_0+2\zeta_1(t)Q_0]\right]\ , \quad (5.3)$$

where A, B, and ζ_1 are the corresponding time-dependent coefficients in the right-hand side of Eq. (3.5), respectively. Similarly, from Eq. (3.6) one has

$$\frac{1}{G}\frac{\partial G}{\partial G}=\frac{i}{B\hbar}[DQ-Q_0+(B\eta-\zeta_1D)]\ , \quad (5.4)$$

where

$$D(t)=\left[\left[\frac{\omega(t)}{\omega(0)}\right]^{1/2}\cos[g(t)] -\frac{J(t)}{\sqrt{\omega(t)\omega(0)}}\sin[g(t)]\right] \quad (5.5a)$$

and

$$\eta(t)=m[\dot{\zeta}_1(t)-\gamma(t)\zeta_1(t)]\ . \quad (5.5b)$$

By differentiating Eq. (5.3) partially with respect to Q and equating the result with Eq. (5.4),

$$G_0(Q,t)=N(t)\exp\left[\frac{i}{2\hbar B}[DQ^2+2(B\eta-\zeta_1D)Q]\right]\ , \quad (5.6)$$

where $N(t)$ is a constant of integration.

From Eqs. (5.3) and (5.6) one has

$$G(Q,Q_0,t)=N(t)\exp\left[\frac{i}{2\hbar B}[AQ_0^2+DQ^2-2QQ_0 +2\zeta_1Q_0+2(B\eta-\zeta_1D)Q]\right]\ . \quad (5.7)$$

To calculate $N(t)$ use the relation

$$\int_{-\infty}^{\infty}G^*(Q,Q_0,t)G(Q,Q_1,t)dQ=\delta(Q_0-Q_1)\ . \quad (5.8)$$

Then we find

$$N(t)=[2\pi\hbar|B(t)|]^{-1/2} \quad (5.9)$$

since the quasicoherent state at any time $t>0$ may be calculated using the Green's function (5.7)

$$\psi_\alpha(Q,t)=\int_{-\infty}^{\infty}G(Q,Q_0,t)\psi_\alpha(Q_0,0)dQ_0\ . \quad (5.10)$$

The quasicoherent state at time $t=0$ is

$$\psi_\alpha(Q_0,0)=\left[\frac{m\omega(0)}{\hbar\pi}\right]^{1/4}\exp\{-\tfrac{1}{2}[|\alpha|^2+\alpha^2(0)]\} \times\exp\left[\frac{-m}{2\hbar}[\omega(0)+iJ(0)]Q_0^2 +\left[\frac{2m\omega(0)}{\hbar}\right]^{1/2}\alpha(0)Q_0\right] \quad (5.11)$$

substitute Eqs. (5.7) and (5.11) into Eq. (5.10) and after straightforward calculations one gets

$$\psi_\alpha(Q,t)=\left[\frac{m\omega(t)}{\hbar\pi}\right]^{1/4}\exp\left[-\tfrac{1}{2}[|\alpha|^2+\alpha^2(t)]+\left[\frac{2m\omega(t)}{\hbar}\right]^{1/2}\alpha(t)[Q-\zeta_1(t)]\right]$$
$$\times\exp\left[\frac{-m}{2\hbar}[\omega(t)+iJ(t)][Q-\zeta_1(t)]^2-\frac{im}{\hbar}[\gamma(t)\zeta_1(t)-\dot{\zeta}_1(t)][Q-\xi_1(t)]\right]$$
$$\times\exp\left\{\frac{-im}{2\hbar}\zeta_1(t)\left[\left[\gamma(t)-\frac{\dot{\omega}(t)}{2\omega(t)}+\omega(t)\cot[g(t)]\right]\zeta_1(t)-2\dot{\zeta}_1(t)\right]-i\frac{g(t)}{2}\right\}\ . \quad (5.12)$$

It is easy to check that Eq. (5.12) is in agreement with Eq. (4.3) if one calculates the integral in Eq. (2.15b) and this can be done easily if one uses Eq. (2.11).

Now let us calculate the transition amplitude $\langle\beta|\alpha\rangle$ between the coherent state $|\alpha\rangle$ given by Eq. (4.3) of the variable-mass oscillator and the coherent state $|\beta\rangle$ of the usual time-independent harmonic oscillator with mass m_0.

The coherent state for time-independent oscillator is in the form

$$\langle Q|\beta\rangle=\left[\frac{m_0\omega_0}{\hbar\pi}\right]^{1/4}\exp\left[\frac{m_0\omega_0}{2\hbar}Q^2+\left[\frac{2m_0\omega_0}{\hbar}\right]^{1/2}\beta Q-\tfrac{1}{2}(|\beta|^2+\beta^2)\right] \quad (5.13)$$

and the transition amplitude is given by

$$\langle\beta|\alpha\rangle=\int_{-\infty}^{\infty}\psi_\alpha(Q,t)\psi_\beta^*(Q,t)dQ\ . \quad (5.14)$$

Thus from Eqs. (4.3) and (5.13) substitute into Eq. (5.14) and then evaluating the integral one had

$$\langle\beta|\alpha\rangle=\sqrt{2}\left[\frac{m_0\omega_0\omega(t)}{m\mu^2(t)}\right]^{1/4}\exp\{-\tfrac{1}{2}[|\alpha|^2+|\beta|^2+\alpha^2+(\beta^*)^2]\}$$

$$\times\exp\left\{\frac{\hbar}{2m\mu(t)}\left[\left[\frac{2m\omega(t)}{\hbar}\right]^{1/2}\alpha(t)+\frac{m}{\hbar}[\omega(t)+iJ(t)]\zeta_1(t)-\frac{im}{\hbar}[\gamma(t)\zeta_1(t)-\dot{\zeta}_1(t)]+\left[\frac{2\omega_0 m_0}{\hbar}\right]^{1/2}\beta^*\right]^2\right\}$$

$$\times\exp\left\{-\zeta_1(t)\left[\left[\frac{2m\omega(t)}{\hbar}\right]^{1/2}\alpha(t)+\frac{m}{2\hbar}[\omega(t)+iJ(t)]\zeta_1(t)+\frac{im}{\hbar}[\gamma(t)\zeta_1(t)-\dot{\zeta}_1(t)]\right]\right\}\exp\left[\frac{i}{\hbar}\phi(t)\right], \quad (5.15)$$

where $\phi(t)$ is given by Eq. (2.15b) and $\mu(t)$ is

$$\mu(t)=[\omega(t)+\omega_0]+iJ(t)\,. \quad (5.16)$$

Expanding β^* in a power series one can obtain an expression for $\langle n_0|\alpha\rangle$, where $|n_0\rangle$ is the nth eigenstate of the time-independent oscillator, that is,

$$\langle n_0|\alpha\rangle=\left[\frac{\sqrt{2}}{\sqrt{\mu(t)(n_0!)}}\right]\left[\frac{m_0\omega_0\omega(t)}{m}\right]^{1/4}\exp\{-\tfrac{1}{2}[\alpha^2+|\alpha|^2+ig(t)]\}$$

$$\times H_{n_0}\left[\left[\frac{m_0\omega_0\hbar}{m\mu(t)}\right]^{1/2}S(t)\right]\left[\frac{m\mu(t)-2\omega_0 m_0}{2m\mu(t)}\right]^{n_0/2}\exp\left[\frac{i}{\hbar}\phi(t)\right]$$

$$\times\exp\left[\frac{\hbar}{2m\mu(t)}\left\{\left[\frac{2m\omega(t)}{\hbar}\right]^{1/2}\alpha(t)+\frac{m}{\hbar}\left[\left[\omega(t)+i\frac{\dot{\omega}(t)}{2\omega(t)}\right]\zeta_1(t)+i\dot{\zeta}_1(t)\right]\right\}^2\right]$$

$$\times\exp\left\{-\zeta_1(t)\left[\left[\frac{2m\omega(t)}{\hbar}\right]^{1/2}\alpha(t)+\frac{m}{2\hbar}[\omega(t)+iJ(t)]\zeta_1(t)+\frac{im}{\hbar}[\gamma(t)\zeta_1(t)-\dot{\zeta}_1(t)]\right]\right\}, \quad (5.17)$$

where

$$S(t)=[m\mu(t)-2\omega_0 m_0]^{-1/2}\left\{\left[\frac{2m\omega(t)}{\hbar}\right]^{1/2}\alpha(t)+\frac{m}{\hbar}\left[\left[\omega(t)+i\frac{\dot{\omega}(t)}{2\omega(t)}\right]\zeta_1+i\dot{\zeta}_1(t)\right]\right\}. \quad (5.18)$$

This result can be compared with Eq. (53) in Ref. 2.

VI. DISCUSSION

I have given a full description of the motion of a harmonic oscillator under the combined action of a time-dependent mass parameter and a variable driving force in both the Schrödinger and Heisenberg pictures of quantum mechanics. I have overcome the difficulty of passing from one picture to the other through two channels (i) either to construct the Dirac operator from the solution in the Heisenberg picture, and then use the definition of the coherent states to find the solution in the Schrödinger picture; or (ii) to use the solution in the Schrödinger picture to find the quasicoherent states which lead to the best Dirac operator. By using the relation $A(t)=A(0)\exp[-ig(t)]$ one obtains the solution in the Heisenberg picture. This is, in fact, due to success in making a complete separation in the Schrödinger picture.

I have calculated the Green's function and the connection with the wave function for quasicoherent states, which has been used to calculate the transition amplitude between the state $|\alpha\rangle$ in our model and the state $|\beta\rangle$ in the ordinary simple harmonic oscillator in the absence of a driving force. Since I have obtained the complete solution for the most general case, I am able to deduce all the other special cases which have been considered earlier. In particular, this work is an extension of that presented in Refs. 5 and 24. I feel that all the results in the present paper could be of paramount importance in quantum optics and perhaps in other branches of physics.

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