PHYSICAL REVIEW A VOLUME 34, NUMBER 5 NOVEMBER 1986

First-passage times for non-Markovian processes driven by dichotomic Markov noise

M. A. Rodríguez and L. Pesquera
Departamento de Física Teórica, Universidad de Santander, Avenida Los Castros s/n., 39005 Santander, Spain
(Received 11 June 1986)

The first-passage time problem for a general non-Markovian process driven by a dichotomic Markov noise is solved. An equation for the probability to be found in an interval without ever having left this interval is deduced. Exact results for first-passage time moments are obtained.

There has recently been a great deal of interest in the problem of first-passage times for non-Markovian processes.[1-6] The formal theory for this kind of non-Markovian problem has been developed in Ref. 1. The difficulties encountered in obtaining exact first-passage time results for non-Markovian processes have been recently illustrated[2] by an exact study of a non-Markovian, diffusivelike flow $\dot{x}=\xi(t)$, wherein $\xi(t)$ is a dichotomic Markov noise.[7,8] The subtleties arise from the fact that the retarded master operator that characterizes the dynamics of the unrestricted probability must be adjusted so as to prevent backflow of probability into the interval I under consideration.

In this Rapid Communication we study the first-passage time problem for a general non-Markovian flow driven by a dichotomic Markov noise,

$$\dot{x}=f(x)+\xi(t)g(x) \ . \tag{1}$$

An equation for $F_t(x_0)$, the probability that the system is still in interval I at time t, given that it started at $x_0\in I$, is obtained. From this equation exact results for first-passage time moments are obtained.

The process $\xi(t)$ (Refs. 7 and 8) is a discrete two-state Markov process taking the values $a'>0$ and $a<0$ with transition rates μ' and μ, respectively. The stationary mean value of $\xi(t)$ will be assumed equal to zero: i.e., $\mu a'+\mu' a=0$. For the correlation function we find an exponential decay

$$\langle\xi(t)\xi(s)\rangle=\frac{D}{\tau}\exp(-|t-s|/\tau) \ , \tag{2}$$

where $D=a'|a|\tau$, $\tau=(\mu+\mu')^{-1}$. Using the backward equations for the Markov process (x,ξ), one obtains the following equations

$$\dot{F}_t(x_0,a)=[f(x_0)+ag(x_0)]\frac{\partial F_t(x_0,a)}{\partial x_0}+\mu F_t(x_0,a')-\mu F_t(x_0,a) \ , \tag{3a}$$

$$\dot{F}_t(x_0,a')=[f(x_0)+a'g(x_0)]\frac{\partial F_t(x_0,a')}{\partial x_0}+\mu' F_t(x_0,a)-\mu' F_t(x_0,a') \ . \tag{3b}$$

In the following, we consider an interval $I\equiv[A,B]$ such that $f(x_0)+a'g(x_0)>0$ and $f(x_0)+ag(x_0)<0$ for all $x_0\in I$. This situation corresponds to I lying inside the domain bounded by the zeros of $(f+ag)(f+a'g)$.[5] The initial conditions and absorbing boundary conditions are given by

$$F_0(x_0,\Delta)=\Theta(x_0-A)\Theta(B-x_0),\ \ \Delta=a,a' \ , \tag{4a}$$

$$F_t(x_0,\Delta)=0,\ \ \Delta=a,a',\ \ x_0\notin[A,B] \ , \tag{4b}$$

$$F_t(A^+,a)=0,\ \ F_t(B^-,a')=0,\ \ t>0 \ , \tag{4c}$$

where Θ is the Heaviside function. The conditions (4c) account for the fact that a process that begins at $x_0=A$ (B) with initial negative (positive) velocity escapes with certainty.

Now, if we initially prepare the system in state $x_0\in[A,B]$ with $\xi(0)=a$, our goal is to derive an equation for $F_t(x_0,a)$. Taking into account conditions (4a) and (4b), we obtain from (3b)

$$F_t(x_0,a')=O_+(t)\Theta(x_0-A)\Theta(B-x_0)+\mu'\int_0^t dt' O_+(t-t')\Theta(B-x_0)F_{t'}(x_0,a) \ , \tag{5}$$

where we have introduced the operator

$$O_+(t)\equiv\exp\left[-\left(\mu'-[f(x_0)+a'g(x_0)]\frac{\partial}{\partial x_0}\right)t\right] \ . \tag{6}$$

Note that the second condition (4c) is also satisfied. We remark that the use of the Heaviside function in (5) prevents transitions back into the interval $[A,B]$.

Using (5) in (3a), we have

$$\dot{F}_t(x_0,a)=[f(x_0)+ag(x_0)]\frac{\partial F_t(x_0,a)}{\partial x_0}-\mu F_t(x_0,a)+\mu O_+(t)\Theta(x_0-A)\Theta(B-x_0)+\mu'\mu\int_0^t dt' O_+(t-t')\Theta(B-x_0)F_{t'}(x_0,a);\ \ x_0\in[A,B] \ . \tag{7}$$

This is the main result of the paper. We remark that in contrast to the retarded backward equation derived in Ref. 3, Eq. (7) does not contain transitions back into the interval $[A,B]$.

An integral equation for $F_t(x_0,a)$ can be derived from (7) using conditions (4a) and (4b):

$$F_t(x_0,a)=O_-(t)F_0(x_0,a)+\mu\int_0^t dt' O_-(t-t')\Theta(x_0-A)O_+(t')F_0(x_0,a)$$
$$+\mu\mu'\int_0^t dt'\int_0^{t'} dt'' O_-(t-t')\Theta(x_0-A)O_+(t'-t'')\Theta(B-x_0)F_{t''}(x_0,a)\ , \quad (8)$$

where we have introduced the operator

$$O_-(t)=\exp\left[-\left[\mu-[f(x_0)+ag(x_0)]\frac{\partial}{\partial x_0}\right]t\right]\ . \quad (9)$$

Note that again the Heaviside functions in (8) suppress transitions back into the interval.

In order to obtain first-passage time moments, we perform the Laplace transform of (7):

$$s\tilde{F}_s(x_0,a)-1=[f(x_0)+ag(x_0)]\frac{\partial\tilde{F}_s}{\partial x_0}-\mu\tilde{F}_s+\mu\int_{x_0}^B dx\,\frac{\exp\left[-\left[(\mu'+s)\int_{x_0}^x dy\,[f(y)+a'g(y)]^{-1}\right]\right]}{f(x)+a'g(x)}$$
$$+\mu\mu'\int_{x_0}^B dx\,\frac{\exp\left[-\left[(\mu'+s)\int_{x_0}^x dy\,[f(y)+a'g(y)]^{-1}\right]\right]}{f(x)+ag(x)}\tilde{F}_s(x,a)\ . \quad (10)$$

An equation for the first-passage time moments, given by

$$T_n(x_0,a)=(-1)^n n\left.\frac{\partial^{n-1}\tilde{F}_s(x_0,a)}{\partial s^{n-1}}\right|_{s=0}\ , \quad (11)$$

can be derived from (10). The equation satisfied by T_n is

$$-\tau[f(x_0)+a'g(x_0)][f(x_0)+ag(x_0)]\frac{\partial^2 T_n}{\partial x_0^2}-\{-f(x_0)+\tau[f(x_0)+a'g(x_0)][f'(x_0)+ag'(x_0)]\}\frac{\partial T_n}{\partial x_0}=S_n\ , \quad (12)$$

where

$$S_n=-nT_{n-1}+n(n-1)(1-2\delta_{n,2})\tau T_{n-2}$$
$$+\tau[2f(x_0)+(a+a')g(x_0)]n\frac{\partial T_{n-1}}{\partial x_0},\quad (T_0=1) \quad (13)$$

with the boundary conditions

$$T_n(A,a)=0\ , \quad (14a)$$

$$[f(B)+ag(B)]\left.\frac{\partial T_n}{\partial x_0}\right|_B=\mu T_n(B,a)-nT_{n-1}(B,a)\ . \quad (14b)$$

Conditions (14a) and (14b) are obtained from (4c) and (10), respectively. These boundary conditions and Eq. (12) have been obtained for $n=1$ in Ref. 4(b).

The equation obtained in Ref. 3 for the mean first-passage time T_1 coincides with (12) for $n=1$. However, the absorbing boundary conditions $T_1(B,a)=0$ used there disagree with (14b). For moments of order $n>1$, the term S_n in Eq. (12) differs from the corresponding Markovian one. This is in agreement with previous results for $n=2$.[1(a)]

Using (14) in (12) we obtain

$$T_n(x_0,a)=k_n\int_A^{x_0} dx\,G(x,A)$$
$$+\int_A^{x_0} dx\int_A^x dx'\,G(x,x')S_n(x')\ , \quad (15)$$

where

$$G(x,x')=-\frac{\Theta(x-x')}{\tau[f(x)+ag(x)][f(x')+a'g(x')]}\exp\left[\int_{x'}^x\left[\frac{\mu}{f(y)+ag(y)}+\frac{\mu'}{f(y)+a'g(y)}\right]dy\right]\ , \quad (16)$$

$$k_n=\frac{[f(B)+ag(B)]\int_A^B G(B,x)S_n(x)dx-\mu\int_A^B dx\int_A^x dx'\,G(x,x')S_n(x')+nT_{n-1}(B,a)}{\mu\int_A^B dx\,G(x,A)-[f(B)+ag(B)]G(B,A)}\ . \quad (17)$$

It is easy to see from (12)–(14) that in the white Gaussian limit, $|a|=a'\to\infty$, $\tau\to 0$, $D=|a|^2\tau=$const, we recover the well-known results for one-dimensional Markov process.[9]

We finally consider in connection with recent work[4] the first-passage time probability density, $W_t(x_0,\Delta)$

$=-(\partial/\partial t)F_t(x_0,\Delta)$ $(\Delta=a,a')$, that obeys Eqs. (3), but with the following conditions:

$$W_0(x_0,\Delta)=-[f(x_0)+\Delta g(x_0)][\delta(x_0-A)-\delta(B-x_0)],\ \Delta=a,a' ,\tag{18a}$$

$$W_t(A,a)=W_t(B,a')=\delta(t) .\tag{18b}$$

Then, $W_t(x_0,a)$ satisfies Eq. (8) with the initial condition (18a). If we Laplace transform this integral equation, we get

$$\begin{aligned}\tilde{W}_s(x_0,a)=&\exp\left[-\left(\int_{x_0}^{A}\frac{\mu'+s}{f(y)+ag(y)}dy\right)\right]-\mu\int_A^{x_0}dx\,\frac{\exp\left(-\int_{x_0}^{x}[(\mu+s)/f(y)+ag(y)]dy\right)}{f(x)+ag(x)}\\&\times\exp\left(-\int_x^B[(\mu'+s)/f(y)+a'g(y)]\,dy\right)\\&-\mu\mu'\int_A^{x_0}dx\int_x^B dx'\frac{\exp\left(-\int_{x_0}^{x}[(\mu+s)/f(y)+ag(y)]\,dy\right)}{f(x)+ag(x)}\\&\times\frac{\exp\left(-\int_x^{x'}[(\mu'+s)/f(y)+a'g(y)]\,dy\right)}{f(x')+a'g(x')}\tilde{W}_s(x',a) .\end{aligned}\tag{19}$$

This integral equation was also obtained in Ref. 4(a) for the particular case, $f=0$ and $g=1$, and in Ref. 4(b) for the general case using an entirely different procedure. We finally note that Eqs. (13) and (14) can be also derived from (19).

We acknowledge financial support from Comisión Asesora de Investigación Címtífica y Técnica (Spain) Project No. 361/84.

[1]P. Hanggi and P. Talkner, (a) Phys. Rev. Lett. **51**, 2242 (1983); (b) Z. Phys. B **45**, 79 (1981).

[2]P. Hanggi and P. Talkner, Phys. Rev. A **32**, 1934 (1985).

[3]J. M. Sancho, Phys. Rev. A **31**, 3523 (1985).

[4]J. Masoliver, K. Lindenberg, and B. J. West, (a) Phys. Rev. A **33**, 2177 (1986); (b) (to be published).

[5]P. Hanggi, T. J. Mroczkowski, F. Moss, and P. V. E. McClintock, Phys. Rev. A **32**, 695 (1985).

[6]J. M. Sancho, F. Sagués, and M. San Miguel, Phys. Rev. A **33**, 3399 (1986).

[7]W. Horsthemke and R. Lefever, *Noise-Induced Transitions*, Springer Series in Synergetics, Vol. 15 (Springer, Berlin, 1983).

[8]C. Van den Broeck and P. Hanggi, Phys. Rev. A **30**, 2730 (1984).

[9]R. L. Stratonovich, *Topics in the Theory of Random Noise* (Gordon and Breach, New York, 1963), Vol. I.