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Generation and parametric amplification of solitons in a nonlinear resonator with a Korteweg–de Vries medium

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It is shown that the formation of stationary waveforms in a driven nonlinear microwave resonator with dissipation can be attributed to the generation and parametric amplification of Korteweg–de Vries (KdV) solitons by a pump wave traveling in the same direction. The interaction process and the soliton propagation are described on the basis of a perturbed KdV equation which is solved numerically. The results of an approximate analysis are compared with measurements performed on a transmission line resonator.

In the last years, a growing interest has been given to the evolution of nonlinear dynamical systems to which an external force is applied. The systems are often described by nonlinear partial differential equations with one spatial dimension, for example, the damped and driven sine-Gordon equation,[1,2] the stochastic Korteweg–de Vries (KdV) equation,[3] or the forced nonlinear Schrödinger equation.[4] Besides, discrete systems like the Toda lattice are also under investigation.[5,6] With respect to the solutions of the underlying equations, in all these systems the spatially coherent and stable structures play the most important role. Special emphasis is placed upon the formation of solitons which are well understood in the nonperturbed case. The main questions are concerned with the mechanisms of soliton formation out of dissipation and, as a consequence, with the soliton content of the generated waves.[1,6,7] Furthermore, with respect to the field of self-organization it is important to understand the emergence of such coherent states out of a chaotic one.[1,6]

Taking applications into account, the behavior of such ultrashort pulses in optical devices has additionally aroused much attention.[8,9] For instance, with the help of an external pump signal it seems to be possible to compensate the losses of a long-distance fiber-optic communication system[10] where solitons are the carriers of information.

Quite recently, we have shown that the experimentally observed bistable and multistable behavior of nonlinear microwave ring resonators can be discussed on the basis of characteristic soliton modes[11,12] propagating in a nonlinear medium which has the properties of a lossy KdV system. In this paper, the parametric formation of these stationary waveforms as a result of the influence of an external pump is investigated in detail both experimentally and theoretically.

The experimental observation and theoretical treatment are based upon an iterated high-frequency transmission line where the nonlinearity is introduced by voltage-dependent capacitances. To describe wave propagation in such a medium the equivalent circuit of a single element as shown in Fig. 1 can be used. Here the shunt admittance per section is that of the varactor diode under reverse bias voltage and the series impedance is the lossy inductance of the Lecher-type wire line.[13] Using this nonlinear medium in a ring resonator setup[11] bistability and multistability have been observed[11,12] where the different states of transmission could be characterized by well-defined soliton modes.[12] As an experimental example a typical waveform is illustrated in Fig. 2 which is interpreted as a sinusoidal pump wave with angular frequency ω_p and amplitude V_p and a superimposed pulse with amplitude $\hat{V}$ and phase ϕ with respect to the harmonic wave. Measurements at different positions along the line reveal that the waveform is almost stationary. On the basis of this experimental situation the following theoretical model has been developed in three steps showing that the system can be characterized as a parametric amplifier where a pump wave continuously supplies energy to a dissipative Korteweg–de Vries soliton.

In a first step, wave propagation without external force is studied. The theory is based upon the following difference-differential equations as derived from Fig. 1:

$$V_{n+1}-2V_n+V_{n-1}=\left(L\frac{\partial}{\partial t}+R\right)\frac{\partial}{\partial t}Q_n\ , \qquad (1)$$

$$V_n-V_{1,n}=\frac{1}{G}\frac{\partial}{\partial t}Q_n\ . \qquad (2)$$

Here V_n is the voltage across the nth element and the non-

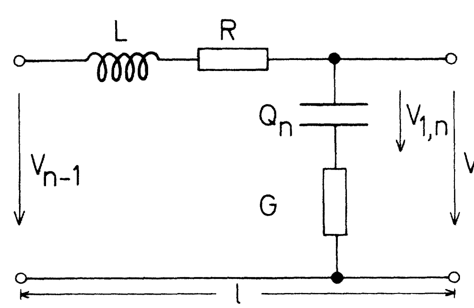

FIG. 1. Equivalent circuit of the nth element of the high-frequency transmission line using varactor diodes under reverse bias as nonlinear elements in a Lecher line (Refs. 11–13). Experimental values are $l=3$ cm; $L=16.4$ nH; $R=0.8$ mΩ; $G=0.7$ S; $C_0=6.6$ pF; $\delta=0.1$ V^{-1} [cf. also Eq. (4)].

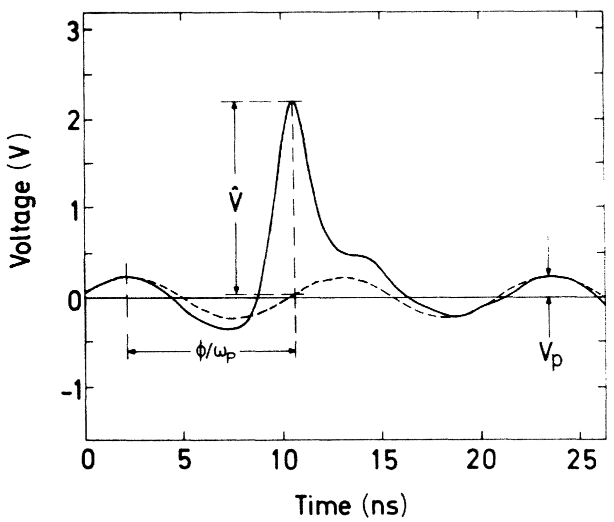

FIG. 2. Experimental waveform as measured in a ring resonator with 90 elements at $f_p = \omega_p/2\pi = 93.2$ MHz. The dashed line is a guide to the eye illustrating the harmonic pump wave of the system.

linearity is introduced by a nonlinear relation between the charge Q_n of the nth capacitor and the applied voltage $V_{1,n}$. In case of small dissipation and by setting

$$Q_n = C_0 V_0 \ln\left(1+\frac{V_{1,n}}{V_0}\right) , \tag{3}$$

where C_0 and V_0 are constants denoting the operating point, Eqs. (1) and (2) describe the wave propagation in the one-dimensional Toda lattice.[14] In the linear regime, the propagation constant can be obtained by applying Floquet's theorem. As a result of the periodic structure a dispersion with low-pass filter characteristic is established, the cut-off frequency being expressed by $\omega_c = 2(LC_0)^{-1/2}$.

For the purpose of a continuum approximation it is now assumed that the guide wavelength is sufficiently large as compared to the length of one section and that the nonlinearity is weak. In this limit the index n can be dropped and the charge voltage relation yields a quadratic nonlinearity

$$Q' = C_0'(1-\tfrac{1}{2}\delta V_1)V_1 , \tag{4}$$

where the dashes now characterize distributed elements as usual, for example, $Q' = Q/1$. Moreover, the following partial differential equation of the form of a lossy KdV equation is derived from Eqs. (1), (2), and (4), cf. Ref. 13:

$$V_\xi = \delta V V_\tau + \kappa V_{\tau\tau\tau} - aV + bV_{\tau\tau} , \tag{5}$$

where $\xi = x/2u_0$ and $\tau = t - x/u_0$ are transformed coordinates and $u_0 = (L'C_0')^{-1/2}$ is the quasistatic small signal phase velocity. Besides, $\delta = V_0^{-1}$, $\kappa = 1/3\omega_c^2$, $a = R'/L'$, and $b = C_0'/G'$ denote the parameters of nonlinearity, dispersion, frequency-independent, and frequency-dependent losses. The solutions of Eq. (5), which is an extended KdV–Burgers equation, have been investigated thoroughly and neglecting dissipative effects the well-known one-soliton solution is given by[13]

$$V(\xi,\tau) = \hat{V}\,\mathrm{sech}^2\left[\left(\frac{\delta\hat{V}}{12\kappa}\right)^{1/2}\left(\tau+\frac{\delta}{3}\hat{V}\xi\right)\right] . \tag{6}$$

In the damped case, the amplitude $\hat{V}$ and velocity $-3/\delta\hat{V}$ of the soliton additionally vary with ξ, cf. Ref. 15.

Now, in a second step, the observed interaction of a soliton with a harmonic pump wave is taken into account. In the parametric model to be considered it is assumed that the pump gives rise to an explicit time and space dependence of the charge now being given by

$$Q' = C_0' V_1(1-\tfrac{1}{2}\delta V_1)[1-f(x,t)] , \tag{7}$$

where the function f is determined by the impressed pump wave. As above, a wave equation is derived which is found to be identical with Eq. (5), with the exception of an additional term describing an inhomogeneity due to the pump. Accordingly,

$$V_\xi = \delta V V_\tau + \kappa V_{\tau\tau\tau} - aV + bV_{\tau\tau} + (fV)_\tau , \tag{8}$$

where in the special case of a harmonic pump wave, f is given by

$$f(\xi,\tau) = \delta V_p \cos[\omega_p(\tau - \Delta\xi)] , \tag{9}$$

and $\Delta < 0$ is a parameter of dispersion describing the phase velocity of the pump within the nonlinear medium.[11,16] Splitting the last term of Eq. (8) into $f_\tau V + fV_\tau$ it becomes evident that on the one hand there is an additional damping or amplification depending on the sign of f_τ. For instance, the case of $f_\tau = \mathrm{const} > 0$ leads to the situation where the static losses "a" can be compensated exactly. On the other hand, the expression fV_τ gives rise to a supplementary contribution to the nonlinearity. A similar equation has been studied by Yagi[17] to describe the behavior of KdV solitons in an inhomogeneous medium.

Equation (8), together with Eq. (9), has numerically been integrated by a finite difference method using periodic boundaries. As an initial condition the soliton solution according to Eq. (6) has been used. As expected, in the purely damping case ($V_p = 0$) the amplitude of the soliton and also its velocity decrease, whereas the width increases with the normalized distance, cf. Fig. 3(a). For the same values of the parameters, but with an additional finite amplitude $V_p \neq 0$ the situation is quite different. Provided that the initial phase difference with respect to the soliton has been chosen appropriately the waveform now remains almost stationary as illustrated in Fig. 3(b). As can be seen, the soliton experiences additional gain and travels with constant velocity. Consequently, the compensation of energy loss together with synchronization between soliton and pump lead to stationary waveforms. Besides, it should, however, be mentioned that in contrast to the experimental observation of Fig. 2, the pump wave in the theoretical model serves only as a parametric energy source and is therefore not visible in Fig. 3.

In a final step, the conditions which allow for parametric amplification and stationary propagation of a soliton are examined by investigation of the mutual dependences between soliton amplitude $\hat{V}$, pump amplitude V_p, and phase difference ϕ (cf. Fig. 2). In a first-order ap-

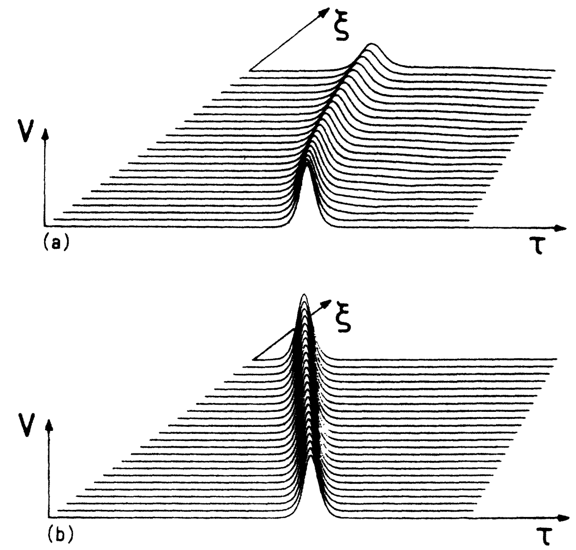

FIG. 3. Numerical results by integration of the inhomogeneous KdV equation given by Eqs. (8) and (9). (a) Damped soliton motion as described by Eq. (5), i.e., $V_p=0$, (b) same as (a) but with finite pump amplitude $V_p=0.29$ V. The length of the τ axis shown corresponds to four temporal periods of the pump wave. Parameters are $\delta=0.1\ \mathrm{V}^{-1}$; $\kappa=3.67\times10^{-20}\ \mathrm{s}^2$; $a=4.8\times10^4\ \mathrm{s}^{-1}$; $b=9.6\times10^{-12}$ s; $\omega_p=5.87\times10^8\ \mathrm{s}^{-1}$; $\Delta=-0.15$; $\hat V(0)=4.6$ V; $\phi(0)=1.5\pi$.

proach it is assumed that under the influence of the pump wave the amplitude $\hat V$ and phase ψ of the soliton are slowly varying functions of time and space. Accordingly, by substituting the modified soliton ansatz

$$V(\xi,\tau)=\hat V(\xi,\tau)\mathrm{sech}^2\left[\left(\frac{\delta\hat V}{12\kappa}\right)^{1/2}\left(\tau+\frac{\delta}{3}\hat V\xi+\psi(\xi,\tau)\right)\right] \tag{10}$$

into Eq. (8) one obtains the following set of differential equations:

$$\hat V_\xi=-\hat V\left(\omega_p\delta V_p\sin\phi+a+\frac{2}{3}\frac{b}{\kappa}\delta\hat V+\frac{b}{\kappa}\delta V_p\cos\phi\right), \tag{11}$$

$$\hat V_\tau=-\frac{b}{\kappa}\hat V, \tag{12}$$

$$\phi_\xi=-\omega_p\left(\delta V_p\cos\phi+\Delta+\frac{1}{3}\delta\hat V-\frac{2b^2}{\kappa}\right), \tag{13}$$

$$\phi_\tau=0, \tag{14}$$

where $\phi=-\omega_p(\psi+\Delta\xi+\delta\hat V\xi/3)$ has been taken into account. In the special case of $b=0$, similar equations have been obtained from energetic considerations[18] and from Eqs. (11) and (13) the following properties can be recognized. The soliton amplitude is amplified by the pump in the angular range from $\pi<\phi<2\pi$ and reduced by the static losses. Aside from the influence of the dispersion as represented by the parameter Δ, the phase ϕ depends on the velocity of the soliton which is a function of its own amplitude $\hat V$ and is again determined by V_p.

Now in the general case $b\neq0$ and with respect to stationary propagation effects, a transformation into the reference system of the pump wave yields the following steady-state solutions:

$$\delta V_p=\left[\frac{2b}{\kappa}\left(\frac{2b^2}{\kappa}-\frac{1}{2}\Delta\right)+a\right]\left(\frac{b}{\kappa}\cos\phi-\omega_p\sin\phi\right)^{-1}, \tag{15}$$

$$\delta\hat V=3\left(\frac{2b^2}{\kappa}-\Delta-\delta V_p\cos\phi\right). \tag{16}$$

According to Eqs. (15) and (16), the relations between soliton amplitude $\hat V$, phase ϕ, and pump amplitude V_p are illustrated in Fig. 4. Furthermore these analytical results are compared with measurements where V_p has been changed experimentally. It is obvious that a soliton can only be generated if the amplitude of the pump wave exceeds a clear threshold value. From Eqs. (15) and (16) this threshold value $V_{p\,\mathrm{th}}$ of the pump amplitude can be obtained yielding

$$\delta V_{p\,\mathrm{th}}=\omega_p^{-1}\left(a-\frac{\Delta b}{\kappa}+\frac{4b^3}{\kappa^2}\right)\left[1+\left(\frac{b}{\kappa\omega_p}\right)^2\right]^{-1/2}. \tag{17}$$

As expected, the threshold value is enhanced for increasing losses and vanishes in the nondamping case. Moreover, there is an interval of V_p where two values of the soliton amplitude exist. Here a stability analysis of Eqs. (11)–(14) yields that in agreement with the experiment merely the larger solitons are stable. As a result, in the $(\hat V,\phi)$-phase plane the stable solutions correspond to stable foci or nodes, while the unstable ones are revealed to be saddle points.

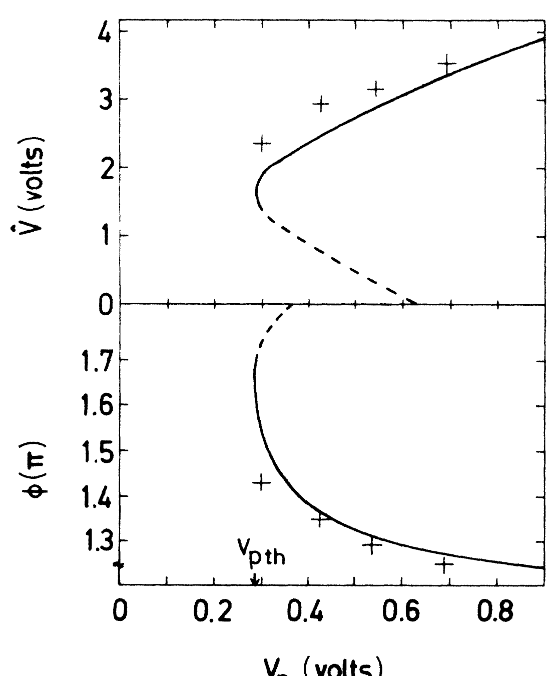

FIG. 4. Parametrically generated solitons; experimental results (+) and steady-state values (solid lines) according to Eqs. (15) and (16) with $\Delta=-0.06$; remaining parameters as in Fig. 3. Dashed lines denote unstable soliton solutions.

In conclusion, it is shown that spatially coherent structures in the form of solitons do play an important role in a driven damped Toda chain or KdV system. The analytical model which has been used elucidates the underlying mechanism on the basis of a distributed parametric interaction between solitons and a pump wave. Starting from the experimentally observed soliton modes in a high-frequency ring resonator, an extended KdV equation is approximately derived. The theoretical and numerical results which are confirmed by experiments yield stationary propagation of solitons above well-defined threshold levels of the pump wave. This model is quite similar to that describing the behavior of a charge carrier in a particle accelerator or in a microwave traveling wave amplifier where, for example, electromagnetic energy is delivered to the electron or vice versa, respectively. From a technical viewpoint the soliton concept is that of a special modelocking of the involved Fourier components leading to a pulse compression by converting a cw input signal into short video pulses of high power. Very recently a similar idea has been discussed in connection with the soliton laser.[8] Additionally, such soliton modes are shown to be relevant to the explanation of wave oscillations on anharmonic molecular chains like the DNA.[19] Finally, our further investigations have to deal with instabilities like chaos, which has already been observed in the resonator,[20,21] and their relation to soliton propagation.

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