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Uniform convergence to an effective Fokker-Planck equation for weakly colored noise

Ronald Forrest Fox
School of Physics, Georgia Institute of Technology, Atlanta, Georgia 30332
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Uniform convergence to an effective Fokker-Planck equation for weakly colored noise is proved. Corresponding constraints on the correlation time for the colored noise are obtained. Examples of bistable nonlinearities are presented and discussed.

I. INTRODUCTION

In a recent paper,[1] the full functional calculus was used to elucidate the connection between stochastic differential equations and associated Fokker-Planck equations. Of particular interest in that paper was the fact that for weakly colored noise, i.e., a non-Markovian process, an effective Fokker-Planck equation may be derived, both for additive and multiplicative noise. It was also shown that certain numerical simulation results[2] for mean first-passage times in bistable systems could be directly explained by the effective Fokker-Planck equation. However, a key step in the derivation of the Fokker-Planck equation was left implicit. Moreover, the character of the results has also been misinterpreted. In this paper, an explicit proof of uniform convergence to the effective Fokker-Planck equation for weakly colored noise is provided.

Attention is restricted here to stochastic differential equations in one variable x:

$$x = W(x) + g(x)f(t) \ , \tag{1}$$

in which $W(x)$ and $g(x)$ may be nonlinear functions of x, and $f(t)$ is the noise term. When $g(x)=1$, the noise is "additive"; otherwise it is "multiplicative." The noise function $f(t)$ is assumed to be Gaussian, and it may be either "colored" or "white." The white-noise case was discussed in detail in the earlier paper,[1] so that the colored-noise case is emphasized here, although the white noise case may be retrieved in an appropriate limit. It is assumed that $f(t)$ has first and second moments

$$\langle f(t)\rangle = 0 \ , \tag{2}$$

$$\langle f(t)f(s)\rangle = \frac{D}{\tau}\exp\left(-\frac{|t-s|}{\tau}\right) \equiv C(t-s) \ , \tag{3}$$

in which τ is the correlation time. The weakly colored noise regime corresponds to small τ, and white noise results in the limit $\tau \to 0$.

Functional calculus may be used to characterize the noise. The probability distribution functional for $f(t)$ is

$$P[f] = N\exp\left[-\tfrac{1}{2}\int ds \int ds' f(s)f(s')K(s-s')\right], \tag{4}$$

in which the normalization factor N is defined by the functional integral (path integral)

$$N^{-1} = \int\!\!\int \mathcal{D}f \exp\left[-\tfrac{1}{2}\int ds \int ds' f(s) \times f(s')K(s-s')\right] . \tag{5}$$

The kernel $K(s-s')$ is the inverse of the correlation function $C(t-s)$ in (3) and satisfies

$$\int ds\, K(t-s)C(s-s') = \delta(t-s') \ , \tag{6}$$

in which $\delta(t-s')$ is the Dirac δ function.

The functional integral is also used to define the probability distribution functional for $x(t)$, the solution to (1). This functional is

$$P(y,t) = \int\!\!\int \mathcal{D}f P[f]\delta(y-x(t)) \ , \tag{7}$$

in which $x(t)$ is implicitly a functional of $f(t)$.

We show below that for sufficiently small τ, an effective Fokker-Planck equation exists for $P(y,t)$ given by

$$\frac{\partial}{\partial t}P = -\frac{\partial}{\partial y}[W(y)P] + D\frac{\partial}{\partial y}g(y)\frac{\partial}{\partial y}\left\{g(y)\Big/\left[1-\tau\left(W'(y)-\frac{g'(y)}{g(y)}W(y)\right)\right]\right\}P \ , \tag{8}$$

in which W' and g' denote the first y derivatives of W and g, respectively. The diffusion term in (8) differs slightly in form from the diffusion term given in our earlier paper.[1] This is a result of the systematic procedure used here, and in the case of additive noise [for which $g(y)=1$, so that $g'=0$] the results are identical.

The following points deserve emphasis: (1) Colored-noise problems are non-Markovian whereas Fokker-Planck equations are traditionally derived for Markov processes only. The existence of an effective Fokker-Planck equation for weakly colored noise is not guaranteed and should be viewed as a surprise. (2) The effective Fokker-Planck equation in (8) is valid for weakly colored noise only, i.e., small τ, and not for arbitrarily large τ. Explicit conditions for "large" and "small" τ are given below. (3) Uniform convergence to (8) means that this equation is approached irrespective of the value of y. It is this feature of the analysis which has been missing previously, and which was only implicit in an earlier paper. An explicit proof of uniform convergence is given below. (4) The approach used here is *not* a perturbation analysis in τ. Instead, a nonperturbative technique is used to obtain results valid for sufficiently small τ. Below, it will be shown that a corresponding perturbative interpretation violates uniform convergence, and it is this feature which has plagued all earlier approaches.

II. UNIFORM CONVERGENCE TO AN EFFECTIVE FOKKER-PLANCK EQUATION

In order not to reproduce here the analysis contained in an earlier paper,[1] I refer the reader to that paper for important preliminary details. Ultimately, a rigorously exact equation is obtained [Eq. (53) in Ref. 1]:

$$\frac{\partial}{\partial t}P=-\frac{\partial}{\partial y}[W(y)P]+\frac{\partial}{\partial y}g(y)\frac{\partial}{\partial y}\int_0^t ds'C(t-s')\iint\mathcal{D}fP[f]\delta(y-x(t))$$
$$\times\exp\left[\int_{s'}^t ds\,[W'(x(s))+g'(x(s))f(s)]\right]g(x(s'))\ . \qquad (9)$$

This is *not* yet a Fokker-Planck equation because the diffusion term does not contain $P(y,t)$ because the functional integral contains non-Markovian dependence on $x(s)$ [and $x(s')$] for $s<t$. It is now shown that for sufficiently small τ, an effective Fokker-Planck equation may be obtained *uniformly in y.*

We begin by observing that

$$\frac{d}{dt}g(x(t))=g'(x(t))\frac{d}{dt}x(t)=g'(x(t))[W(x(t))+g(x(t))f(t)]$$
$$=\frac{g'(x(t))}{g(xIt))}[W(x(t))+g(x(t))f(t)]g(x(t))\ . \qquad (10)$$

This yields the formal expression

$$g(x(s'))=\exp\left[\int_t^{s'}ds\frac{g'(x(s))}{g(x(s))}[W(x(s))+g(x(s))f(s)]\right]g(x(t))\ . \qquad (11)$$

When this expression, together with the explicit exponential form for $C(t-s')$ given by (3), is inserted into (9), we obtain

$$\frac{\partial}{\partial t}P=-\frac{\partial}{\partial y}[W(y)P]+\frac{\partial}{\partial y}g(y)\frac{\partial}{\partial y}\int_0^t ds'\frac{D}{\tau}\exp\left[-\frac{t-s'}{\tau}\right]\iint DfP[f]\delta(y-x(t))$$
$$\times\exp\left[\int_{s'}^t ds\left[W'(x(s))-\frac{g'(x(s))}{g(x(s))}W(x(s))\right]\right]g(x(t))\ . \qquad (12)$$

Notice the cancellation of the gf terms in the exponential. Now, change integration variable from s' to $(t-s')/\tau\equiv\theta$. The integral in the diffusion term now becomes

$$D\int_0^{t/\tau}d\theta e^{-\theta}\iint\mathcal{D}fP[f]\delta(y-x(t))\exp\left[\int_{t-\tau\theta}^t ds\left[W'(x(s))-\frac{g'(x(s))}{g(x(s))}W(x(s))\right]\right]g(x(t))$$
$$\underset{\tau\to 0}{\to}D\int_0^\infty d\theta e^{-\theta}\iint\mathcal{D}fP[f]\delta(y-x(t))\exp\left[\tau\theta\left[W'(x(t))-\frac{g'(x(t))}{g(x(t))}W(x(t))\right]\right]g\left[x(t)\right]$$
$$=D\int_0^\infty d\theta e^{-\theta}\exp\left[\tau\theta\left[W'(y)-\frac{g'(y)}{g(y)}W(y)\right]\right]\iint\mathcal{D}fP[f]\delta(y-x(t))g(y)$$
$$=\left\{Dg(y)\Big/\left[1-\tau\left[W'(y)-\frac{g'(y)}{g(y)}W(y)\right]\right]\right\}P(y,t)\ . \qquad (13)$$

In the limit $\tau\to 0$, we have replaced the integral argument of the exponential by the first term of its Taylor expansion about time t. In the second step, each $x(t)$ has been replaced by y because of $\delta(y-x(t))$. The third step is valid, *uniformly in y,* provided

$$1-\tau\left[W'(y)-\frac{g'(y)}{g(y)}W(y)\right]>0 \text{ for all } y\ . \qquad (14)$$

This provides the constraint on τ. This is the same condition for which the first step is valid. The resulting effective Fokker-Planck equation is clearly (8).

III. BISTABILITY EXAMPLE

A popular example is $W(x)=ax-bx^3$ with either $g(x)=1$ or $g(x)=x$. The first g choice corresponds with decay of a metastable state according to the Kramers theory for chemical reactions.[3] The second g choice corresponds with the theory of dye-laser fluctuations which include pump noise.[4] Consider the multiplicative case $(g(x)=x)$ first. The τ condition (14) requires

$$1-\tau[a-3by^2-(a-by^2)]=1+\tau 2by^2>0 \ . \quad (15)$$

This is clearly valid for all $y\in(-\infty,\infty)$. Thus, there exists no restriction on τ in this case.

In the purely additive case $[g(x)=1 => g'=0]$, this condition is instead

$$1-\tau(a-3by^2)=1-a\tau+\tau 3by^2>0 \ . \quad (16)$$

This is true for all y iff $a\tau<1$, i.e.,

$$\tau<\frac{1}{a} \ . \quad (17)$$

Thus, in the additive-noise case the condition is more restrictive.

IV. PERTURBATION EXPANSION OF THE DIFFUSION TERM

The diffusion term in (8) yields a perturbative expansion in τ previously obtained by the perturbative method called the τ expansion:[5,6]

$$Dg\Big/\left[1-\tau\left(W'-\frac{g'}{g}W\right)\right]\to Dg\left[1+\tau\left(W'-\frac{g'}{g}W\right)\right] \ . \quad (18)$$

Now we notice that when condition (14) is satisfied, the left-hand side of (18) is positive, *uniformly in y*, whereas the right-hand side becomes negative for sufficiently large y [for either $g(x)=x$ or $g(x)=1$]. [In each side of (18), ignore the sign of the overall factor of g because the complete diffusion term in (8) contains a second overall factor of g which compensates this particular sign.] Thus, the perturbative result is not uniform in y. The substitution in (18) not only requires that τ is small but also that y^2 is small! The step in (13) which yielded the left-hand side of (18) was not perturbative but resulted instead from the integral of an exponential (this reminds one of comparable "nonperturbative" results obtained in field theory).

V. COMPARISON WITH EXACTLY COMPUTABLE CASES

Two cases exist which can be treated exactly in closed form. They are $W(x)=-\lambda x$ with either $g(x)=1$ or $g(x)=x$. In the first case $[g(x)=1]$, one gets exactly

$$\frac{\partial}{\partial t}P=\lambda\frac{\partial}{\partial x}(xP)+\frac{D}{1+\lambda\tau}\left[1-\exp\left(-\lambda t-\frac{t}{\tau}\right)\right]\frac{\partial^2}{\partial x^2}P \ , \quad (19)$$

whereas in the second case one gets exactly

$$\frac{\partial}{\partial t}P=\lambda\frac{\partial}{\partial x}(xP)+D\left[1-\exp\left(-\frac{t}{\tau}\right)\right]\frac{\partial}{\partial x}x\frac{\partial}{\partial x}xP \ . \quad (20)$$

Each of these cases corresponds with a simplification in our general analysis, which is seen to occur in Eq. (12). The last exponential there becomes independent of $x(s)$ and may be factored out of the functional integral, thereby yielding (19) and (20) directly. Indeed, it was examination of these special cases which led to the derivation of uniform convergence above. [Clearly, we can treat exactly the case: $W(x)=-\lambda x^n$ with $g(x)=x^n$.]

VI. HIGHER-ORDER τ RESULTS

Any attempt to include higher-order τ terms in the Taylor expansion of the argument of the exponential in (13) will engender $(d/ds)x(s)$ terms, which in turn contain $f(s)$ factors. These no longer cancel out as they have to first order in τ [because of their cancellation already in (12)]. The methods used in the earlier paper indicate that such factors will effectively produce higher order y derivatives for the $P(y,t)$ equation, thereby destroying its Fokker-Planck structure. This simply shows that the process is non-Markovian and that higher-order τ terms will force this on the $P(y,t)$ equation.

VII. BISTABILITY AND MEAN FIRST PASSAGE TIMES

In the earlier paper it was shown that for $W(x)=ax-bx^3$ and $g(x)=1$, the mean first passage time T is given by the formula

$$T=\frac{\pi}{a\sqrt{2}}\left(\frac{1+2a\tau}{1-a\tau}\right)\exp\left(\frac{a^2}{4bD}(1+2a\tau)\right) \ . \quad (21)$$

This formula exhibits an exponential dependence on τ which was first observed in computer simulations.[2] The criticism has been raised that this expression becomes negative for large τ. This result, however, was obtained for weakly colored noise only, and my procedure requires the constraint $\tau<1/a$ as was observed above in (17). Thus, within its domain of validity, formula (21) is well behaved.

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