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Effect of dissipation on squeezed quantum fluctuations

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The decay of squeezed quantum fluctuations of a system coupled to a dissipative environment is discussed. For a model where the heat bath consists of an infinite set of harmonic oscillators, exact results for the time evolution of the mean-square fluctuations are obtained for arbitrary bath temperature and arbitrary damping strength.

Recently, there has been a great deal of interest in the generation of quantum states with fluctuations of a dynamical variable which are reduced below the size of the vacuum fluctuations at the expense of enhanced noise in the conjugate variable. The theoretical discussion of these so-called squeezed states started over a decade ago.[1] In the last few years various experimental setups to generate and detect them were suggested,[2,3] and recently a first successful experiment using four-wave-mixing techniques[3] has been reported.[4] While the theoretical interest in squeezed states arises primarily from fundamental questions connected with the quantum theory of measurement,[5] their experimental relevance lies in the possibility of improving the signal-to-noise ratio of high-precision measurements in fields such as gravitational wave detection, fiber optics, or optical communication systems.[6]

The purpose of the present work is to discuss how a squeezed state, once it has been generated, evolves in time when coupled to a dissipative environment. The period of time over which the squeezing persists is of considerable interest since even the best experimental setup will be affected by environmental influences causing dissipation. This problem was previously discussed by Milburn and Walls[7] using a Fokker-Planck equation approach. Their results are valid for weak damping (which seems to be the interesting regime for most applications) and also (and more restrictively) for not too low temperatures. The following analysis relies on a functional integral method[8,9] and gives exact results for a model Hamiltonian for arbitrary strength of the damping and arbitrary temperatures.

We consider a quantum mode which we will describe as a particle with mass M, coordinate q, and momentum p (e.g., q may represent the amplitude of an electric field component). This mode is damped through a bilinear coupling to a heat bath which we assume to consist of harmonic oscillators such as phonons or photons. The system is then governed by the Hamiltonian

$$H=\frac{p^2}{2M}+\frac{1}{2}M\omega_0^2q^2+\sum_{n=1}^{N}\left[\frac{1}{2}\frac{p_n^2}{m_n}+\frac{1}{2}m_n\omega_n^2\left(x_n-\frac{c_n}{m_n\omega_n^2}q\right)^2\right] , \quad (1)$$

describing the situation when the squeezed state is propagated away from its source. Although the time evolution of a squeezed state may well be described as the motion in a harmonic potential, the mechanism to generate it is necessarily nonlinear. Without going into details, the preparation of squeezed states always requires a perturbational force acting on the system which has components proportional to $da^2+d^*(a^\dagger)^2$ for squeezing the variances which are frequently combined with components proportional to $ca+c^*a^\dagger$ for displacing the state. Here a and $a^\dagger$ are the usual boson creation and annihilation operators. Such a perturbation leads to terms of the form of the displacement operator $D(\alpha_0)=\exp(\alpha_0a^\dagger-\alpha_0^*a)$ and the squeeze operator $S(z)=\exp[\frac{1}{2}za^2-\frac{1}{2}z^*(a^\dagger)^2]$ in the time-evolution operator.

In previous work[1-3,6] it was assumed that the perturbational force acts on the oscillator vacuum. In the following we will let it act on the equilibrium state ρ_β of a damped harmonic oscillator at temperature $T=1/k_B\beta$ according to

$$\rho_s=D(\alpha_0)S(z)\rho_\beta S^\dagger(z)D^\dagger(\alpha_0) . \quad (2)$$

This modified definition of the initial squeezed state ρ_s is certainly more realistic since the environmental coupling is always present and cannot be switched on and off. In the limit of zero damping and zero temperature the "real" squeezed state (2) reduces to the "ideal" squeezed state discussed in the literature. Using the form

$$\rho_\beta(q,q')=(2\pi\langle q^2\rangle_\beta)^{-1/2}\exp\left[-\frac{1}{2\langle q^2\rangle_\beta}\left(\frac{q+q'}{2}\right)^2-\frac{\langle p^2\rangle_\beta}{2\hbar^2}(q-q')^2\right] \quad (3)$$

for the reduced equilibrium density matrix of a damped harmonic oscillator with variances $\langle q^2\rangle_\beta$ and $\langle p^2\rangle_\beta$, the coordinate representation of the initial state (2) is readily found to read

$$\rho_s(q,q')=[2\pi\sigma_q(0)]^{-1/2}\exp\left[-\frac{1}{2\sigma_q(0)}\left(\frac{q+q'}{2}-q_0\right)^2-\frac{\sigma_p(0)}{2\hbar^2}(q-q')^2+\frac{i}{\hbar}p_0(q-q')\right] . \quad (4)$$

Here $\sigma_q(0)=\langle q^2\rangle_\beta \exp(-2z)$ and $\sigma_p(0)=\langle p^2\rangle_\beta \exp(2z)$ are the initial variances of coordinate and momentum, while the initial first moments q_0 and p_0 are connected to the displacement parameter α_0 by

$$\alpha_0=(M\omega_0/2\hbar)^{1/2}q_0+i(2\hbar M\omega_0)^{-1/2}p_0 .$$

In writing the form (4) for the initial density matrix we have assumed the squeezing parameter z to be real, which means that the nondiagonal variance

$$\sigma_{pq}(0)=\tfrac{1}{2}\langle pq+qp\rangle_0-p_0q_0$$

vanishes. Then the principal axes of the uncertainty ellipse initially point in the p and q direction which is the case of interest, generally.

Now, the equilibrium variance of the coordinate is given by[10]

$$\langle q^2\rangle_\beta=\frac{1}{\beta M}\sum_{n=-\infty}^{\infty}[\omega_0^2+\nu_n^2+|\nu_n|\hat{\gamma}(|\nu_n|)]^{-1} , \tag{5}$$

where $\nu_n=2\pi n/\hbar\beta$ are the Matsubara frequencies, and where

$$\hat{\gamma}(\omega)=\frac{1}{M}\sum_{n=1}^{N}\frac{c_n^2}{m_n\omega_n^2}\frac{\omega}{\omega^2+\omega_n^2} \tag{6}$$

is the frequency-dependent damping coefficient describing the influence of the heat bath. Note that $\langle q^2\rangle_\beta$ is always reduced by dissipation. For low temperatures and not too weak damping we even find that the equilibrium fluctuations in q can be considerably smaller than the vacuum value $\langle q^2\rangle_{\rm vac}=\hbar/2M\omega_0$. This static reduction of coordinate fluctuations must be distinguished from the dynamical squeezing arising from the squeeze operators in Eq. (2). In a dynamically squeezed state the variance periodically dips below the level set by the equilibrium state. On the other hand, the static squeezing of the equilibrium variance below its vacuum value does not arise from the initial nonequilibrium state. Nevertheless, this effect of the dissipative environment may also be of relevance in connection with high-precision measurements.

To determine the time evolution of the initial state (5) we use a functional integral method based on the influence functional theory of Feynman and Vernon.[8] Since the initial state does not factorize into separate contributions from the quantum mode and the reservoir, the method has to be generalized to allow for a description of the initial correlations between environment and oscillator. Details of our approach will be given elsewhere.[9] The basic equation for the density matrix at time t is

$$\rho(q_f,q_f',t)=\int dq_i\,dq_i'\,d\bar{q}\,d\bar{q}'\lambda(q_i,\bar{q},q_i',\bar{q}')Z^{-1}\int D[\tilde{q}]\exp\left[\frac{i}{\hbar}(S[q]-S[q'])-\frac{1}{\hbar}S^E[\bar{q}]\right]F[\tilde{q}] . \tag{7}$$

Here, the functional integral runs over all paths

$$\tilde{q}(z)=\begin{cases} q'(s) \text{ for } z=s,\ 0\le s\le t, \\ \bar{q}(\tau) \text{ for } z=-i\tau,\ 0\le\tau\le\hbar\beta, \\ q(s) \text{ for } z=-i\hbar\beta+s,\ 0\le s\le t \end{cases} \tag{8}$$

on a contour C in complex time $z=s-i\tau$ (Fig. 1) satisfying the boundary conditions $q(0)=q_i$, $q(t)=q_f$, $q'(0)=q_i'$, $q'(t)=q_f'$, and $\bar{q}(0)=\bar{q}'$, $\bar{q}(\hbar\beta)=\bar{q}$. The functions

$$S[q]=\int_0^t ds\frac{M}{2}(\dot{q}^2-\omega_0^2q^2),$$
$$S^E[q]=\int_0^t ds\frac{M}{2}(\dot{q}^2+\omega_0^2q^2) \tag{9}$$

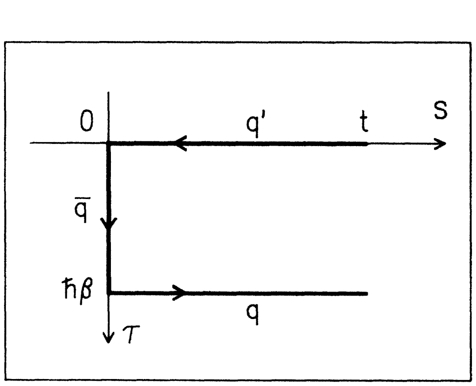

FIG. 1. The integration contour C in the complex time plane $z=s-i\tau$ along which the functional integral (8) is defined.

are the action of the particle in real and imaginary (Euclidean) time, while

$$F[\tilde{q}]=\exp\left[-\frac{1}{\hbar}\left(\int dz\int_{z>z'}dz'K(z-z')\tilde{q}(z)\tilde{q}(z')+\frac{i}{2}\mu\int dz\,\tilde{q}^2(z)\right)\right] \tag{10}$$

is the influence functional describing the frictional influence of the environment. Here the integrals over z and z' are along the contour C and $z>z'$ means that z follows z' in the direction of the arrows in Fig. 1. The functions $K(z)$ and μ are given by

$$K(z)=\sum_{n=1}^{N}\frac{c_n^2}{2m_n\omega_n}\frac{\cosh[\omega_n(\frac{1}{2}\hbar\beta-iz)]}{\sinh(\frac{1}{2}\omega_n\hbar\beta)} \tag{11}$$

and

$$\mu=\sum_{n=1}^{N}\frac{c_n^2}{m_n\omega_n^2} . \tag{12}$$

Finally, the function

$$\lambda_s(q,\bar{q},q',\bar{q}')=\exp\left[z+\frac{i}{\hbar}p_0(q-q')\right]\times\delta(\bar{q}-(q-q_0)\exp(z))\times\delta(\bar{q}'+(q-q_0)\exp(z)) \tag{13}$$

carries the information about the initial state (3).

The main advantage of our approach is that the functional integral (7) describing the decay of a squeezed state can be evaluated exactly in terms of equilibrium properties of the system. For the expectation value of the coordinate at time t we find

$$\langle q\rangle_t = M\dot{\chi}(t)q_0 + \chi(t)p_0 , \qquad (14)$$

while the average momentum follows from $\langle p\rangle_t = Md\langle q\rangle_t/dt$. Here, $\chi(t)$ is the response function of the oscillator, which for arbitrary damping coefficient $\hat{\gamma}(\omega)$ can be written in terms of its Laplace transform

$$\hat{\chi}(\omega) = \frac{1}{M}[\omega^2 + \omega\hat{\gamma}(\omega) + \omega_0^2]^{-1} . \qquad (15)$$

For the special case of frequency-independent damping, $\hat{\gamma}(\omega) = \gamma$, this gives

$$\chi(t) = \frac{1}{M\xi}\sin(\xi t)\exp\left[-\frac{\gamma}{2}t\right] , \qquad (16)$$

where $\xi = (\omega_0^2 - \gamma^2/4)^{1/2}$ is the frequency of damped oscillations. The relaxation of an initial displacement according to (16) follows the classical trajectory and is temperature independent.

Let us now discuss the more interesting question of how the squeezed fluctuations decay toward their equilibrium values. We find $[\zeta = \exp(z)]$

$$\sigma_q(t) = 2M\chi(t)\dot{S}(t)(1-\zeta) - 2M\dot{\chi}(t)S(t)(1-\zeta^{-1}) + \langle q^2\rangle_\beta[1 + M^2\dot{\chi}^2(t)(1-\zeta^{-1})^2] + \langle p^2\rangle_\beta\chi^2(t)(1-\zeta)^2, \qquad (17)$$

$$\sigma_p(t) = 2M^3\dot{\chi}(t)\ddot{S}(t)(1-\zeta) - 2M^3\ddot{\chi}(t)\dot{S}(t)(1-\zeta^{-1}) + \langle q^2\rangle_\beta M^4\ddot{\chi}^2(t)(1-\zeta^{-1})^2 + \langle p^2\rangle_\beta[1 + M^2\dot{\chi}^2(t)(1-\zeta)^2] , \qquad (18)$$

while the cross variance is given by $\sigma_{pq}(t) = \frac{1}{2}M\dot{\sigma}_q(t)$. Hence, the entire dynamics of the variances can be expressed in terms of two functions; namely, the response function $\chi(t)$ and the symmetrized coordinate autocorrelation function $S(t) = \frac{1}{2}\langle q(t)q + qq(t)\rangle$. Again, for arbitrary damping mechanism $S(t)$ can be given in terms of its Laplace transform

$$\hat{S}(\omega) = \frac{1}{\beta}\sum_{n=-\infty}^{\infty}\frac{\omega}{\nu_n^2 - \omega^2}[\hat{\chi}(\omega) - \hat{\chi}(|\nu_n|)] , \qquad (19)$$

which for frequency-independent damping can be evaluated in closed form to yield[10]

$$S(t) = \frac{\hbar}{4M\xi}\left[\exp(-\lambda_2 t)\coth\left(\frac{i}{2}\hbar\beta\lambda_2\right) - \exp(-\lambda_1 t)\coth\left(\frac{i}{2}\hbar\beta\lambda_1\right)\right] - \frac{\gamma}{M\beta}\sum_{n=-\infty}^{\infty}\frac{|\nu_n|\exp(-|\nu_n|t)}{(\omega_0^2+\nu_n^2)^2 - \gamma^2\nu_n^2} , \qquad (20)$$

where $\lambda_{1,2} = \gamma/2 \pm i\zeta$. Since the system under consideration is linear, the initial state (5) remains Gaussian for all times so that the density matrix at time t is determined uniquely by the expectation values (14), (17), and (18). In order to discuss the decay of the squeezed fluctuations in more detail it is convenient to introduce the Wigner representation $W(p,q)$ of the density matrix. We readily find

$$W(p,q,t) = \frac{1}{2\pi}[\sigma_q(t)\sigma_p(t)k(t)]^{-1/2}\exp\left[-\frac{1}{2}\left(\frac{(q-\langle q\rangle_t)^2}{\sigma_q(t)k(t)} + \frac{(p-\langle p\rangle_t)^2}{\sigma_p(t)k(t)} - \frac{2\sigma_{pq}(t)}{\sigma_q(t)\sigma_p(t)k(t)}(q-\langle q\rangle_t)(p-\langle p\rangle_t)\right)\right], \qquad (21)$$

where

$$k(t) = 1 - \sigma_{pq}^2(t)/[\sigma_p(t)\sigma_q(t)] .$$

Let us further introduce the dimensionless variables $\tilde{q} = (2M\omega_0/\hbar)^{1/2}q$ and $\tilde{p} = (2/\hbar M\omega_0)^{1/2}p$ and the variances $\tilde{\sigma}_q(t)$, $\tilde{\sigma}_p(t)$, and $\tilde{\sigma}_{pq}(t)$ scaled accordingly. Setting the expression in large parentheses in (21) equal to 1 defines an uncertainty ellipse in the $(\tilde{p},\tilde{q})$ plane which is centered at $\tilde{p} = \langle\tilde{p}\rangle_t$ and $\tilde{q} = \langle\tilde{q}\rangle_t$ and characterizes the width of the fluctuations. At $t = 0$ the principal axes of this ellipse point in the $\tilde{p}$ and $\tilde{q}$ direction and they have the lengths $\tilde{\sigma}_p^{1/2}(0)$ and $\tilde{\sigma}_q^{1/2}(0)$. For $t > 0$ the ellipse rotates and the lengths of the axes oscillate. Note that we have scaled the variables such that the vacuum state is described by a unit circle, while the equilibrium state is ellipsoidal due to the dissipation.

The system is in a squeezed state as long as the minor axis of the uncertainty ellipse is shorter than 1. The principal axes of the rotated ellipse are given by

$$x_\pm(t) = [\tfrac{1}{2}(\tilde{\sigma}_q(t) + \tilde{\sigma}_p(t) \pm \{[\tilde{\sigma}_q(t) - \tilde{\sigma}_p(t)]^2 + 4\tilde{\sigma}_{pq}^2(t)\}^{1/2})]^{1/2} , \qquad (22)$$

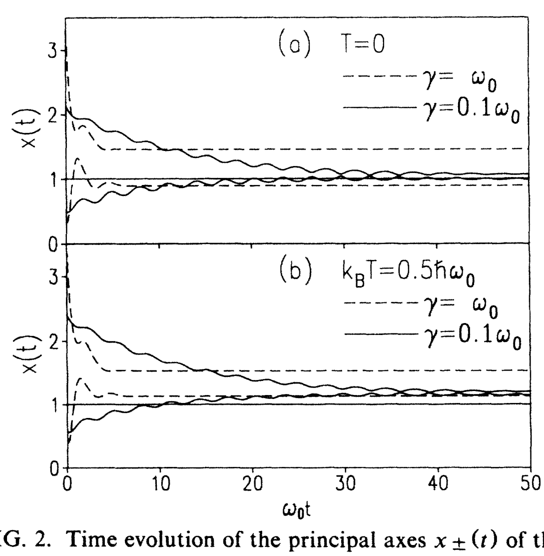

FIG. 2. Time evolution of the principal axes $x_\pm(t)$ of the uncertainty ellipse for a Drude model with $\omega_D = 10\omega_0$, $\exp(z) = 2$. The horizontal line marks the size of the vacuum fluctuations.

where the plus and minus sign hold for the major and minor axis, respectively. Figure 2 shows the time evolution of these axes for a Drude model with $\hat{\gamma}(\omega)=\gamma\omega_D/(\omega+\omega_D)$ for two temperatures. Dissipation influences the squeezing in two ways. It leads to a decay of the squeezed fluctuations and changes their absolute values.

Let us first discuss the lifetime of the squeezing. It is seen from Fig. 2 that the decay rate is roughly temperature independent. This can be understood by using the explicit time dependence of the response and correlation function for frequency-independent damping [Eqs. (16) and (20)]. For high temperatures $T>\hbar\gamma/4\pi k_B$ all terms in Eqs. (17) and (18) for the variances decay as $\exp(-\gamma t)$ or faster so that the damping constant is the only relevant parameter for the lifetime of the squeezing. For lower temperatures we still have terms proportional to $\exp(-\gamma t)$, but some of the additional terms proportional to $\exp[-(\frac{1}{2}\gamma+\nu_n)t]$ decay slower. For moderate to strong damping these latter terms determine the lifetime of the squeezed fluctuations which is of the order of $2\gamma^{-1}$. For weak damping ($\gamma\ll\omega_0$) one has to note that all terms which decay slower than $\exp(-\gamma t)$ are by a factor of γ/ω_0 smaller, so that they become important only after a period of time when most of the squeezing has already died out. Hence, for weakly damped systems the lifetime of the squeezing is of order γ^{-1} for all temperatures, and the result of the weak coupling theory[7] is valid even at $T=0$.

Figure 2 also shows that the initial squeezing below the vacuum fluctuations is increased by dissipation. While stronger damping leads to a faster decay of the dynamical squeezing, the static squeezing of the equilibrium variance below the vacuum level is increased by the damping. This second effect of dissipation is not described by a weak coupling theory and it is more pronounced for lower temperatures and stronger damping. However, even for the weakly damped system shown in Fig. 2 it has the consequence that the $T=0$ curve for the minor axis takes about twice as long to reach the vacuum line than the finite-temperature curve. The condition of low temperatures is not very stringent in the optical regime where even at room temperature $k_BT<0.1\hbar\omega_0$, but it requires millikelvin temperatures if one works with microwaves.

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