PHYSICAL REVIEW A VOLUME 34, NUMBER 5 NOVEMBER 1986

Rydberg states in crossed fields: The gyropendulum

A. R. P. Rau
Department of Physics and Astronomy, Louisiana State University, Baton Rouge, Louisiana 70803
(Received 9 June 1986)

The spectrum of degenerate Rydberg states of an n-manifold of the hydrogen atom when placed in crossed (perpendicular) electric and magnetic fields (the diamagnetic term is disregarded) is shown to conicide with that of a charged-particle harmonic oscillator in a magnetic field. A mechanical counterpart is identified as the gyropendulum. These correspondences give further insight into the nature of Rydberg states in external fields.

I. INTRODUCTION

There has been considerable interest in the study of Rydberg states in external static electric and magnetic fields.[1] Detailed experiments of considerable sophistication are possible and there is intrinsic interest in what we learn about dynamics from the observed phenomena. The varied and conflicting symmetries of electronic motion in the atomic and external fields make for a rich variety of phenomena,[2] some of which still remain to be understood. The regimes of most interest are when the external fields are far from being perturbative, forcing us to confront the full complexity of the nonseparable quantum-mechanical Hamiltonian. This aspect lends further importance to the study of Rydberg states in external fields as prototypes for other nonseparable, nonperturbative, and mixed symmetry problems in atomic physics and other areas of physics.[3]

One particular study which this paper focuses on is of a highly excited state of the hydrogen atom with principal quantum number n in perpendicular electric (**E**) and magnetic (**B**) fields for field strengths where the mixing of states from neighboring n manifolds can still be ignored.[4] We are dealing then with what for either field by itself would be the linear Stark and the linear Zeeman effect. The question is what happens under the superposition of both fields, with say **E** along the x axis and **B** along the z axis. Stated thus, this problem can be fully solved and was in fact solved theoretically long ago,[5] although a full and convincing experimental demonstration of the resulting quantization has only recently been given.[6] Yet, as will be shown here, some interesting aspects of this problem have been insufficiently understood and some perhaps even misunderstood. The aim of this paper is to give a simple and complete accounting of this problem which does not invoke Laplace-Runge-Lenz vectors and the $O(4)$ group[5,6] but works instead with straightforward electromagnetic and mechanical pictures and analogs. Some of the insights gained—most notably, that the system is exactly like a gyropendulum,[7] and that with the **E** field alone, the linear Stark effect in an n manifold may be interpreted as a charged particle harmonic oscillator—should be useful in considering other more complicated regimes when the mixing between n manifolds becomes important and complete solutions are as yet unavailable.

II. ATOM IN CROSSED (E,B) FIELDS

With all energies measured with respect to the unperturbed position of the nth hydrogenic level, the Hamiltonian is

$$H=(e/2\mu c)\mathbf{B}\cdot(\mathbf{L}+2\mathbf{S})+e\mathbf{E}\cdot\mathbf{r}\ . \qquad (1)$$

Here μ is the reduced mass of the electron, **L** and **S** its orbital and spin angular momentum. I have ignored the diamagnetic term which is quadratic in B, but note that in other studies of Rydberg states in a **B** field, particularly for larger n and B, this term leads to the interesting spectrum observed of resonances around the ionization limit $(n=\infty)$.[1,2] This as yet unsolved problem and the suggestion from observations and numerical calculations that there are symmetries in this problem that have not yet been fully understood[8] do not concern this paper.

Following the initial work of Pauli,[5] it has become customary to analyze the degenerate manifold n when subjected to H in Eq. (1) by making the replacement $\mathbf{r}\rightarrow\frac{3}{2}na_0\mathbf{a}/\hbar$, where **a** is the scaled Lenz vector[9] (dimension of angular momentum), and a_0 is the Bohr radius. By next introducing the generators[9] of $O(4)$, $\mathbf{j}_{1,2}=(\mathbf{L}\pm\mathbf{a})/2$, and the Larmor $\boldsymbol{\omega}_B=e\mathbf{B}/2\mu c$ and linear Stark $\boldsymbol{\omega}_E=\frac{3}{2}ne\mathbf{E}a_0/\hbar$ frequencies, H is rewritten as

$$H=2\hbar\omega_B M_s+\boldsymbol{\omega}_1\cdot\mathbf{j}_1+\boldsymbol{\omega}_2\cdot\mathbf{j}_2\ , \qquad (2)$$

where

$$\boldsymbol{\omega}_{1,2}=\boldsymbol{\omega}_B\pm\boldsymbol{\omega}_E,\ \omega_1=\omega_2=\omega=(\omega_B^2+\omega_E^2)^{1/2}\ , \qquad (3)$$

and $M_s=\pm\frac{1}{2}$ are the two spin projections. Since $j_1/\hbar=j_2/\hbar=\frac{1}{2}(n-1)$ and their projections on any axis run over the values $-\frac{1}{2}(n-1)$ to $\frac{1}{2}(n-1)$ in steps of unity,[9] the Hamiltonian in Eq. (2) describes a set of $2n-1$ equally spaced levels with spacing $\hbar\omega$. Working with $n=34$ states of rubidium, B values of a few hundred gauss, and E values of a few tens V/cm, a completely convincing experimental demonstration[6] has been given of such a spacing which depends on the square root of the sum of terms involving the squared field strengths.

Let us consider Eq. (1), however, more directly in the standard fashion of degenerate perturbation theory in any quantum-mechanical text. For concreteness, let us first look at the $n=2$ states with such a Hamiltonian and also set $M_s=\frac{1}{2}$. Of the four degenerate states the $2p0$ does not couple to the others and remains at the unperturbed energy position save for the contribution $\hbar\omega_B$ due to its spin coupling to B. The remaining three states $2s0$, $2p1$, and $2p-1$, give the matrix

$$\begin{pmatrix} \omega_B & \omega_E/\sqrt{2} & -\omega_E/\sqrt{2} \\ \omega_E/\sqrt{2} & 2\omega_B & 0 \\ -\omega_E/\sqrt{2} & 0 & 0 \end{pmatrix}$$

so that the eigenfrequencies are ω_B and

$$\omega_B \pm (\omega_B^2+\omega_E^2)^{1/2} \ . \tag{4}$$

The spectrum consists then of two essentially unchanged eigenvalues (except for the spin contribution of ω_B) and two nondegenerate eigenvalues on either side with splittings $\pm(\omega_B^2+\omega_E^2)^{1/2}$. In a similar manner, manifolds of higher n can be analyzed to obtain the result that apart from a common spin-dependent shift of $2\omega_B M_s$ for all states, the spectrum is described by equally spaced eigenvalues in steps of ω. The central value at the unperturbed position is n-fold degenerate, with each successive state on either side having a degeneracy less by 1. It is also clear from the structure of the above matrix and of Eq. (4) that the results of pure B field or pure E field are similar to each other and to the crossed field situation.[10] The degenerate manifold in any one of these situations "fans out linearly" with equally spaced states.[11] The result in Eq. (4) should therefore not be regarded[6] as a "completely new type of quantization and organization of the spectrum in crossed (**E**,**B**) fields, different from either the linear Stark or the Zeeman ones." They are all exactly similar problems,[12] as the following sections will further exemplify.

III. THE GYROPENDULUM

The gyropendulum[7] [Fig. 1(a)] is a simple pendulum but with the ordinary bob mass replaced by a rapidly rotating flywheel. We follow here the discussion in Pippard[7] which was, in fact, the trigger for this paper. The original discussion as well as the following section in that book on the gyromagnetic top are highly recommended to all interested in the problem of Rydberg states in external fields. From Fig. 1(a), the torque equation is

$$\frac{d}{dt}\left[\frac{Lr}{l}-iml\dot{r}\right]=imgr \ , \tag{5}$$

where dots will denote differentiation with respect to time, g is the acceleration due to gravity, L the angular momentum of the flywheel, l the length of the pendulum, and the linear approximation for the displacement, $r \ll l$, has been made. Defining two frequencies in a notation that conforms to the previous sections, $\omega_E=(g/l)^{1/2}$, $\omega_B=L/2ml^2$, Eq. (5) may be rewritten as

$$\ddot{r}+2i\omega_B\dot{r}+\omega_E^2 r=0 \ . \tag{6}$$

The characteristic frequencies are

$$\omega_B \pm (\omega_B^2+\omega_E^2)^{1/2} \ , \tag{7}$$

and the gyropendulum performs a looping motion as sketched in Fig. 1(b) under the influence of the two frequencies. For further discussion of the energy exchanges between the two modes, we refer to the original reference.[7]

The significant point for our purpose is, of course, that Eq. (7) coincides with Eq. (4). From this perspective, a rereading of the previous section shows that a Rydberg state in crossed (**E**,**B**) may be viewed as a simple pendulum (harmonic oscillator)—constituted by $H(n\geq 2)$ in an **E** field—which is subjected to a **B** field. This view is reinforced by considering in ordinary electromagnetism the problem of a charged two-dimensional (x,y) harmonic oscillator (frequency ω_E) in an external magnetic field.[13] The equations of motion with $\mathbf{F}=(e/c)\mathbf{v}\times\mathbf{B}$ are

$$\begin{aligned} m\ddot{x}+m\omega_E^2 x &= (eB/c)\dot{y} \ , \\ m\ddot{y}+m\omega_E^2 y &= -(eB/c)\dot{x} \ , \end{aligned} \tag{8}$$

and when written in terms of $r=x+iy$ coincide precisely with Eq. (6).

IV. DISCUSSION

The correspondence established above between the three problems (i) the mechanical gyropendulum, (ii) a charged oscillator in a perpendicular magnetic field, and (iii) a $H(n\geq 2)$ state in crossed **E** and **B** fields, leads to fresh insights into all these problems.[14] The most significant one for the study of Rydberg states in external fields is that the degeneracy of excited states of hydrogen has the effect of making the system $H(n\geq 2)$ in **E** behave like an oscillator. Although the linear Stark effect has long been familiar and in the literature on Rydberg states the fanning out of the n manifold of states linearly in E is also well known,[11] what may have been insufficiently appreciated is that this means a harmonic oscillator nature for Rydberg states in an electric field.

When quasibound resonances were first discovered[15] in the vicinity of the ionization limit (in particular, above the limit) for the spectrum of an atom in an electric field, a particular puzzle was the origin of the binding. Above the

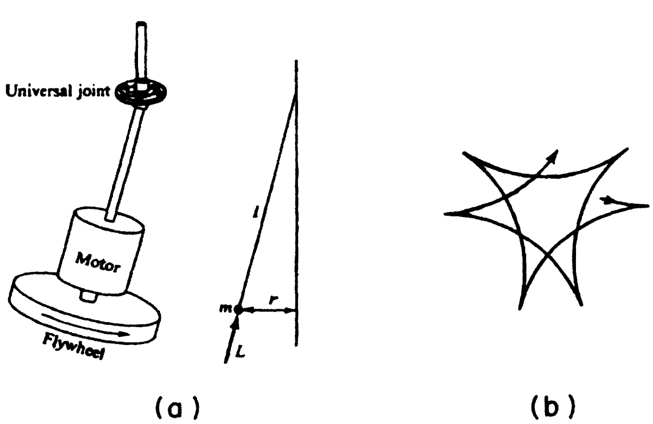

FIG. 1. (a) Gyropendulum; (b) trajectory described by the tip of the pendulum (from Ref. 7).

ionization limit the Coulomb field does not bind and, on the other hand, an electron in an E field also experiences no binding; yet, in the combined fields there is a quasibinding. It was recognized that this phenomenon is related to the same linear fanning out in E exhibited by lower n manifolds and that specific features like the $E^{3/4}$ scaling of the spacing between resonances can thereby be understood.[16] The current discussion adds further insight by showing that whereas a free electron experiences no binding in an E field, the situation is different when there is also present a positive charge center so that the electron is to be viewed as part of a degenerate system, namely, $H(n \geq 2)$. The problem has then become one of a harmonic oscillator. The role of the linear Stark effect or, equivalently, the permanent electric dipole moment of a degenerate manifold, as a restoring force with a spring constant is highlighted by our discussion. Note also that the equations of motion for a free electron in crossed fields, with $\mathbf{E}$ along x and $\mathbf{B}$ along z, differ from Eq. (8) (Ref. 17),

$$m\ddot{x} = (eB/c)\dot{y} + eE , \quad m\ddot{y} = -(eB/c)\dot{x} . \tag{9}$$

When combined as before into a single equation,

$$\ddot{r} + 2i\omega_B \dot{r} - (eE/m) = 0 , \tag{10}$$

there is a crucial difference from Eq. (6) in the last term. The solutions of Eq. (10) are the familiar cyclotron motion with a superposed $\mathbf{E}\times\mathbf{B}$ drift with drift speed (cE/B).[17] The quantization now depends on the frequency ω_B and B alone, the E field contributing only to an overall translational energy. But when we turn from a free electron to an electron in $H(n \geq 2)$ then the two frequencies $\boldsymbol{\omega}_B$ and $\boldsymbol{\omega}_E$ add as two vector frequencies according to Eq. (3). As with a gyropendulum or any of a host of mechanical analogs, Rydberg states when subjected simultaneously to crossed $\mathbf{E}$ and $\mathbf{B}$ fields exhibit oscillations that reflect the sum of two incommensurate perpendicular frequencies.

ACKNOWLEDGMENT

This work has been supported by the National Science Foundation under Contract No. PHY 84-01855.

---

[1]See, for instance, D. Kleppner, M. G. Littman, and M. L. Zimmerman, in *Rydberg States of Atoms and Molecules,* edited by R. F. Stebbings and R. B. Dunning (Cambridge Univ. Press, Cambridge, England, 1983); C. W. Clark, K. T. Lu, and A. F. Starace, in *Progress in Atomic Spectroscopy,* edited by H. J. Beyer and H. Kleinpoppen (Plenum, New York , 1984), Pt. C; J. C. Gay, in *Photophysics and Photochemistry in the Vacuum Ultraviolet,* edited by S. P. McGlynn, G. L. Findley, and R. H. Huebner (Reidel, New York, 1985); A. R. P. Rau, Comments At. Mol. Phys. **10**, 19 (1980); in *Atomic Excitation and Recombination in External Fields,* edited by C. W. Clark and M H. Nayfeh (Gordon and Breach, New York, 1985).

[2]A. R. P. Rau, J. Phys. B **12**, L193 (1979).

[3]U. Fano, in *Atomic Physics,* edited by I. Lindgren and S. Svanberg (Plenum, New York, 1983), Vol. 8; A. R. P. Rau, J. Phys. (Paris) Colloq. **43**, Suppl. 11, C2-211 (1982).

[4]Although we consider the hydrogen atom, substantially the same effect is exhibited by high Rydberg states of any atom because the crucial aspect that there is a degenerate set of l states applies to highly excited states of all atoms: Only a few of the very low values of l have appreciable quantum defects and separate from the rest of the manifold, but the large majority are unaffected and the discussion in this paper remains valid. This is particularly true for high m values.

[5]W. Pauli, Z. Phys. **36**, 336 (1926); Y. Demkov, B. S. Monozon, and V. Ostrovskii, Zh. Eksp. Teor. Fiz. **11**, 691 (1970) [Sov. Phys. JETP **30**, 775 (1970)].

[6]F. Penent, D. Delande, F. Biraben, and J. C. Gay, Opt. Commun. **49**, 184 (1984). See also, E. Korevaar and M. G. Littman, J. Phys. B **16**, L437 (1983); P. A. Braun, *ibid.* **18**, 4187 (1985).

[7]A. B. Pippard, *The Physics of Vibration* (Cambridge Univ. Press, Cambridge, England, 1983), Vol. I, p. 216.

[8]M. L. Zimmerman, M. M. Kash, and D. Kleppner, Phys. Rev. Lett. **45**, 1092 (1980); C. W. Clark and K. T. Taylor, J. Phys. B **13**, L737 (1980); Nature **292**, 437 (1981); E. A. Soloviev, Pis'ma Zh. Eksp. Teor. Fiz. **32**, 971 (1981) [JETP Lett. **34**, 265 (1981)]; D. R. Herrick, Phys. Rev. A **26**, 323 (1982); J. C. Gay, D. Delande, F. Biraben, and F. Penent, J. Phys. B **16**, L693 (1983).

[9]L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory,* 3rd ed. (Pergamon, New York, 1977), p. 124.

[10]When $E = 0$, the above matrix, or its counterparts for higher n, is obviously diagonal and has equally spaced eigenvalues. When $B = 0$ it is off diagonal with precisely the structure which again gives equally spaced eigenvalues. This can most readily be seen by redefining $\mathbf{E}$ to lie along the z axis when, as in many a standard treatment of the linear Stark effect, for each m one has a matrix with nonzero entries only along the two subdiagonals on either side of the main diagonal.

[11]M. L. Zimmerman, M. G. Littman, M. M. Kash, and D. Kleppner, Phys. Rev. A **20**, 2251 (1979).

[12]This point of view is also in keeping with the relativistic invariance of electromagnetism, that because the two invariants are $E^2 - B^2$ and $\mathbf{E}\cdot\mathbf{B}$, the problem of crossed fields can be reduced to pure E or pure B alone.

[13]L. D. Landau and E. M. Lifshitz, in *The Classical Theory of Fields* (Pergamon, New York, 1962), p. 61.

[14]Another concerns the observation in Ref. 7 that the normal modes of the gyropendulum in Eq. (7) are not time-reversed copies of each other. That electromagnetic problems involving a magnetic field break time reversal invariance is well known. In the mechanical problem, it is the angular momentum L, which is odd under time reversal, which plays the role of B.

[15]R. R. Freeman, N. P. Economou, G. C. Bjorklund, and K. T. Lu, Phys. Rev. Lett. **41**, 1463 (1978).

[16]A. R. P. Rau and K. T. Lu, Phys. Rev. A **21**, 1057 (1980).

[17]Reference 13, pp. 62 and 63.