Stochastic development of an electron avalanche

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(Received 12 November 1985)

The stochastic evolution in space of an electron avalanche has been investigated with the use of Monte Carlo methods. From these investigations we have obtained the energy distribution as a function of distance from the cathode and avalanche size, $f(n,\epsilon,x)$; the probability distribution for avalanche size, $P(n,x)$; and the ionization probability, per unit distance at x, for electrons belonging to an avalanche of size n, $\alpha_s(n,x)$. We believe that this is the first time that any information has been presented on $f(n,\epsilon,x)$ and $\alpha_s(n,x)$ which takes microscopic processes into account. The investigations were done in nitrogen for $E/N \geq 300$ Td. We suggest that pulse-height distribution experiments may be a sensitive technique for determining the E/N conditions at which runaway electrons appear.

I. INTRODUCTION

The evolution of an electron avalanche from a single initiatory electron occupies a pivotal position in the study of electrical breakdown of gases, the resolution of proportional counters, and electron-photon cascades. Two approaches, termed deterministic and stochastic, can be taken in the velocity-averaged, theoretical description of this evolution.[1] The deterministic approach describes the growth in the number of electrons, n, based on the ionization rate coefficient; while the stochastic approach opts for obtaining the probability distribution for the random variable n.

Since electron-molecule collisions, described in terms of probabilities (i.e., cross sections), determine the evolution of the avalanche, a stochastic description seems almost necessary. This is certainly the case in the early stages when the avalanche is small and fluctuations in the number of electrons have large effects on the size of the avalanche at later times.

Traditionally, the stochastic multiplication in the number of electrons in an avalanche has been formulated in terms of the growth in number as a function of distance away from the cathode. The interest has been to determine the total number of electrons collected by the anode. In the context of an avalanche, "distance from the cathode" refers to the centroid of the avalanche. This interpretation is followed throughout the text. The most important issue in this formulation has been the determination of the probability of ionization per unit path length, a_i, for electrons in the avalanche. Models for a_i have been proposed by Snyder,[2] Legler,[3] and Byrne.[4] In these models, the a_i's actually have different physical meanings. In Legler's model, for example, a_i is the ionization probability per unit distance from the point at which the electron was created; whereas in Byrne's model, it is the ionization probability per unit distance away from the cathode. Alkhazov[5] has shown that these models are basically equivalent. In his paper he also investigated a number of models for a_i.

The weakness of these models is that, in essence, no connection has been made between a_i and the single-particle energy distribution for the electrons as they move away from the cathode. Close to the cathode, the distribution is rapidly changing with distance. The nature of this variation must be ascertained before arriving at a proper expression for a_i. From another perspective, the probability for ionization per unit distance (for either interpretation given above) is energy dependent, and, moreover, secondary electrons entering the distribution do not, in general, have the same probability as either the primary, nor the "average" electron. Stochastic kinetic models[6] take the energy distribution of electrons into account, but are significantly more difficult to solve. Similarly, multigroup models[5] based upon the regeneration point method account somewhat for the differences in probabilities, but again, are difficult to solve. Both of these facts must be incorporated into the model for a_i.

Realizing that the mean energy of the distribution is a monotonic decreasing function of avalanche size, and assuming that the mean energy is indicative of this ionization probability, Byrne[4] tried to incorporate "energy dependence" by proposing that a_i is dependent on the avalanche size, n, as follows:

$$a_i = \theta(x)(a_0 + a_1/n) , \tag{1}$$

where a_0 and a_1 are constants, $\theta(x)$ is a spatially varying function to account for the variation of a_i in situations where the fields are nonuniform, and x is the spatial variable (in cylindrical coordinates, x is taken to be the radial distance). With this interpretation of θ, $a_i(n,x)$ would be independent of x for uniform fields, a rather severe assumption. Byrne has been criticized[3] for this conclusion and for an error in the physical interpretation of the parameters $\lambda = a_0 + a_1$ and $b = a_0/(a_0 + a_1)$. Although simplistic and unjustified, this was the first macroscopic

stochastic model of an avalanche which included energy dependence in the ionization coefficients.

Using a Monte Carlo technique,[7] we have been able to investigate the stochastic development in space of an avalanche starting from an initial electron. We have obtained the energy distribution of electrons, at a distance x from the cathode belonging to avalanches from size n; that is, $f(n,\epsilon,x)$. From this, we have determined the ionization probability per unit distance along the field direction for electrons belonging to an avalanche of size n. This coefficient is the same as Byrne's a_i, and will be denoted in the text by α_s, the stochastic ionization coefficient (α_s is a function of n and x). This is the first time that any information on $\alpha_s(n,x)$ has been presented, which takes microscopic electron-molecule processes occurring in the gas into account. From the calculations, we have also obtained the probability distribution for avalanche size at x, i.e., $P(n,x)$. The study was conducted in nitrogen at values of $E/N \geq 300$ Td. One of the reasons for going to such high values of E/N is that we have been interested in using pulse height distribution experiments as a sensitive technique for determining the conditions for the appearance of runaway electrons.

In the following section a description is given of the procedure followed in the calculations. In Sec. III the results are presented and discussed.

II. THE STOCHASTIC VARIABLES AND THE METHOD OF COMPUTATION

A Monte Carlo code has been developed to study the evolution in space of an avalanche starting from a single initiatory electron. The simulation is divided into a number of spatial intervals of width x_i, $i=1,2,3$, etc. An initial electron, with specified energy, is started from the cathode (the origin of our coordinate system), in a random direction toward the anode. The electron is assumed to follow a classical trajectory between collisions, and the distance between collisions is determined by using the "null collision" method.[8] The outcome of a collision event, i.e., type of event and properties of the electron(s) emerging from the event, is determined by a set of random numbers obtained from a uniform distribution in the interval [0,1], probability distribution for the processes, and the differential scattering cross sections for each process. When an ionizing event occurs, a new electron is created.

The primary and all secondaries are tracked up to a time t_1, determined by the condition that the centroid of the avalanche is at the plane $x=x_1$. At this time, the velocity distribution and the size of the avalanche at the time of crossing is recorded.

The electrons are then tracked to time t_2, chosen again by the requirement that the centroid is at the plane $x=x_2$. The properties of the electrons and the size of the avalanche at this time are again recorded. This procedure is carried out up to a plane x_n for which information is desired. A new electron is started from the cathode and the whole procedure described above is repeated. A total of N avalanches are investigated (up to 8000). For avalanches with $n \leq 4$, a different procedure is followed. N electrons are started at the cathode and tracked concurrently. The properties of these electrons are used for obtaining the desired information for avalanches of size 1. When these electrons ionize, they are subsequently counted in avalanches of size 2 and so on. With this procedure, the problem of negative mean velocity and multiple passing of the avalanche centroid through the plane x_i, is avoided. Both of these situations can occur when the avalanche size is small, i.e., one or two electrons.

From the sampled information we can obtained the energy distribution at the plane x_i, for electrons belonging to avalanches of size n; and the avalanche size distribution, $P(n,x_i)$, that is the probability for having an avalanche of size n at position x_i. The probability distribution obtained as described above differs from one obtained with a real anode. This is due to nonequilibrium effects near electrodes with absorbing boundaries. If the electrode is far from the cathode, such that the size of the arriving avalanches are large, the effect on the distribution will be small. In any event, the calculations do not take anode boundary effects into account.

The ionization probability per unit distance at x_i, for an electron belonging to an avalanche of size n, i.e., the stochastic ionization coefficient, was obtained in the calculations making use of the equation,

$$\alpha_s(n,x_i)=\frac{\nu_i(n,x_i)}{V_d(n,x_i)}\ , \tag{2}$$

where $\nu_i(n,x_i)$ is the ionization rate for an electron in an avalanche of size n at x_i, and $V_d(n,x_i)$ is the drift velocity of the center of mass of such an avalanche. $\nu_i(n,x_i)$ is obtained from the distribution functions as follows:

$$\nu_i(n,x_i)=\int \sigma_i(\mathbf{v})f(n,\mathbf{v},x_i)v\,d\mathbf{v} \tag{3a}$$

and $V_d(n,x_i)$ is calculated from

$$V_d(n,x_i)=\left\langle \frac{dX_c}{dt} \right\rangle \Bigg|_{x_i}\ , \tag{3b}$$

where X_c is the centroid of the avalanche of size n, averaged over all avalanches of size n. Note that $\nu_i(n,x_i)$ and $V_d(n,x_i)$ are properties of the average avalanche of size n. Also computed in the simulations is the mean electron energy for avalanches with the same size n, i.e., $\epsilon(n,x_i)$.

Given the stochastic ionization coefficient, $\alpha_s(n,x)$, the transition probability q that an avalanche of size n will increase its population by one electron in the interval $(x, x+\Delta x)$ is $q=n\alpha_s(n,x)dx$. The probability of an increase by more than one electron in this interval is assumed to be negligibly small. The avalanche size distribution $P(n,x)$ may then be determined from the equation,[3]

$$\partial_x P(n,x)=-n\alpha_s(n,x)P(n,x) + (n-1)\alpha_s(n-1,x)P(n-1,x) \tag{4}$$

with initial conditions $P(1,0)=1$, $P(n,0)=0$ for $n\neq 0$. Multiplying Eq. (1) by n and summing over all n, an equation for the mean avalanche size, $\bar{n}$ is obtained; namely,

$$\frac{d\bar{n}}{dx}(x)=\bar{\alpha}(x)n(x)\ , \tag{5}$$

where

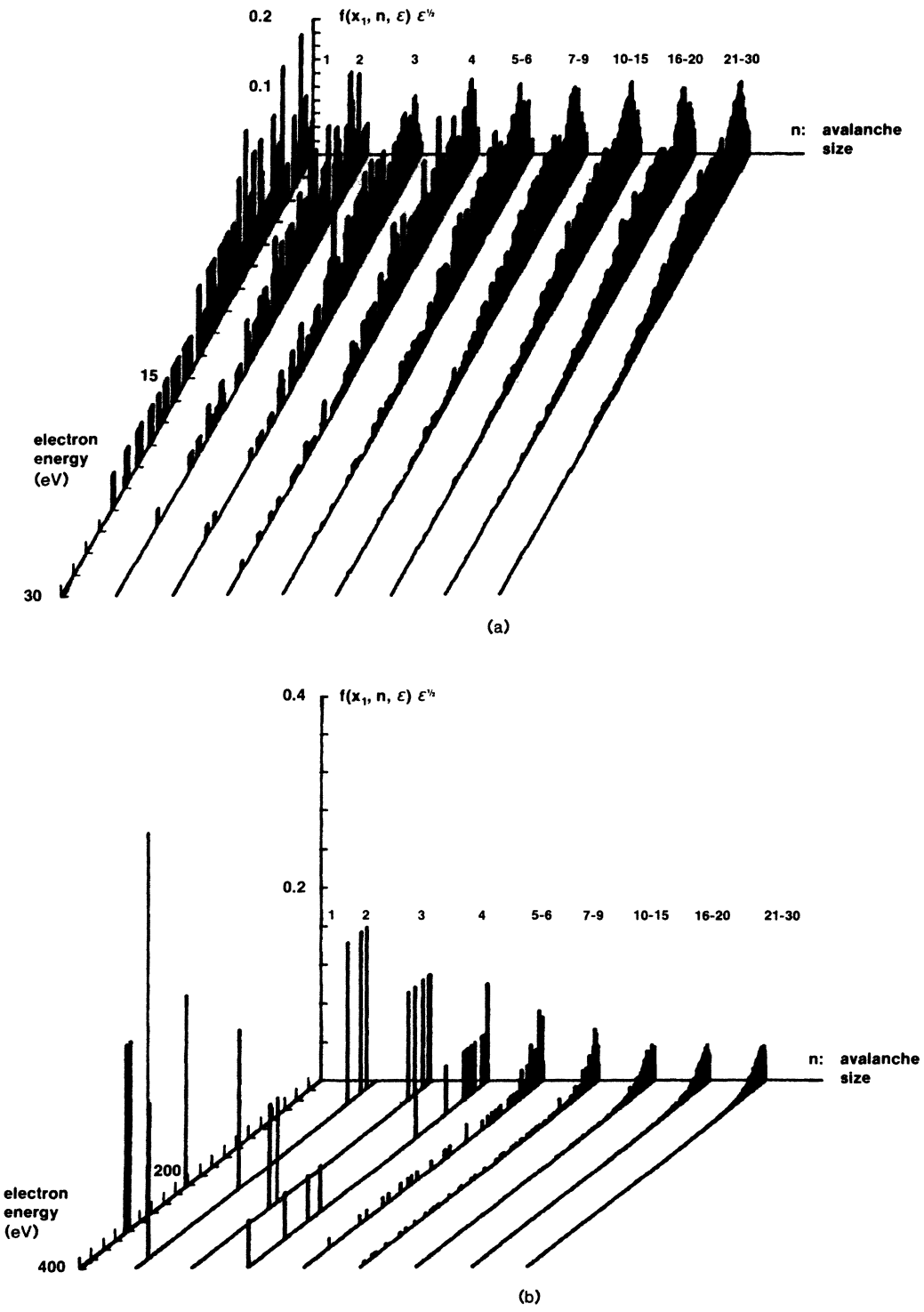

FIG. 1. Electron energy distribution for avalanches of size n in nitrogen for (a) $E/N=300$ Td and $x=0.03$ cm; (b) $E/N=1500$ Td and $x=2.5\times10^{-3}$ cm; and (c) $E/N=3000$ Td and $x=1.25\times10^{-3}$. $N=10^{19}$ cm^{-3} in all the figures.

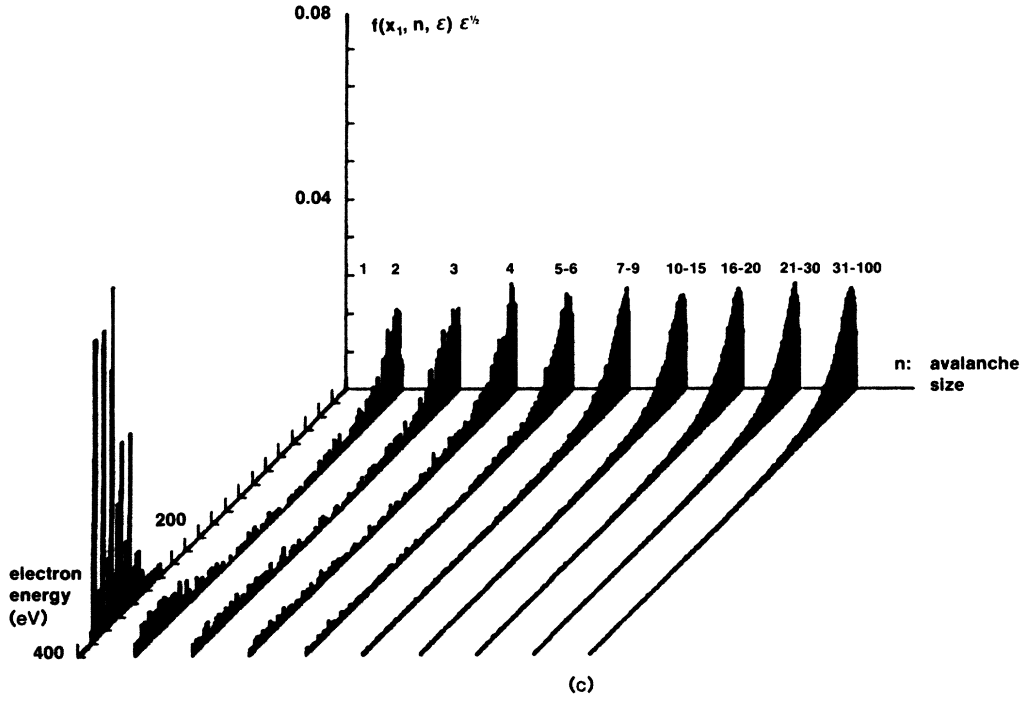

FIG. 1. (*Continued*).

$$\bar{\alpha}(x)=\frac{\sum_n n\alpha_s(n,x)P(n,x)}{\sum_n nP(n,x)} \tag{6}$$

and $\bar{n}=\sum_n nP(n,x)$. From Eq. (2), $\bar{\alpha}(x)$ plays the role of a mean local ionization coefficient. For large x, $\bar{\alpha}(x)$ approaches the bulk ionization coefficient, α_i,

$$\lim\bar{\alpha}(x)=\alpha_i \quad \text{as } x\to\infty \ ,$$

and Eq. (5) reduces to the usual equation for avalanche growth in a constant field.[1-6,9]

The average drift velocity $\bar{V}_d(x)$ and average energy $\bar{\epsilon}(x)$ are calculated as follows:

$$\bar{V}_d(x)=\frac{\sum_n nP(n,x)V_d(n,x)}{\sum_n nP(n,x)} \tag{7}$$

and

$$\bar{\epsilon}(x)=\frac{\sum_n nP(n,x)\epsilon(n,x)}{\sum_n nP(n,x)} \ . \tag{8}$$

At large x, these averaged quantities approach constant values. In this domain, the average avalanche is said to have reached dynamic equilibrium.

III. RESULTS AND DISCUSSION

The fluctuation in the size of an avalanche collected by an anode far from the cathode, is primarily determined from the fluctuation in the multiplication of electrons at the very early stages of development, i.e., the first few ionizing collisions. For this reason, it is not necessary to carry the simulation to large numbers of electrons to determine the statistical behavior of the avalanche. The critical region extends from the cathode to the plane where the mean avalanche size is ≈ 100 electrons. With the Monte Carlo code described in the preceding section, we have calculated the stochastic energy distribution function $f(n,\epsilon,x)$, ionization coefficient $\alpha_s(n,x)$, ionization rate

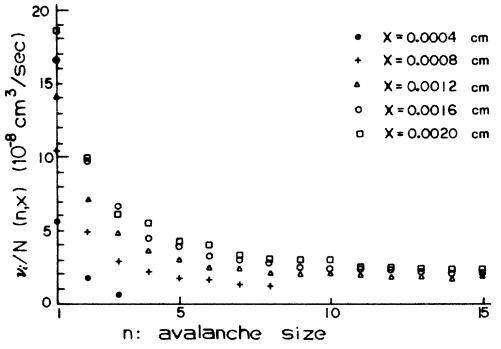

FIG. 2. Stochastic ionization rate, $\nu_i(n,x)$, for avalanches of size n at a distance x from the cathode for $E/N=1500$ Td.

$\nu_i(n,x)$, drift velocity $V_d(n,x)$, and probability distribution for the random variable n at positions x, $P(n,x)$. There has been no quantitative information on any of these functions, except $P(n,x)$.[10–12] We have calculated them for nitrogen at $E/N \geq 300$ Td to provide insight into the stochastic development in space of an electron avalanche.

In Figs. 1(a), 1(b), and 1(c), the stochastic distribution function $f(n,\epsilon,x)$ is displayed for $x_i = 3\times10^{-3}$ cm, 2.5×10^{-3} cm, and 1.25×10^{-3} cm, at $E/N = 300$, 1500, and 3000 Td, respectively. The energy distribution for avalanches of size one, i.e., electrons which have made no ionizing collisions, has a relatively high mean energy ($\sim$300 eV at 3000 Td, 200 eV at 1500 Td, and 10 eV at 300 Td). As the electrons multiply, the distribution changes rapidly. Since there is a high probability that the secondary electron produced in an ionization event emerges with low energy,[13] the population of electrons at low energies increases with n due to ionization. The ionizing primaries enter the distribution near the interval $\epsilon-\epsilon_i$, where ϵ is the energy of the primary and ϵ_i is the ionization potential. In the transition from an avalanche of size one to another of a size two, at $E/N \geq 1500$ Td, a low-energy electron is produced and the primary stays in the high-energy region (recall that ϵ is near 300 eV at these values of E/N). Thus, the distribution for $n=2$ consists of two isolated groups of electrons. This is clearly seen in Fig. 1(c). As n increases, the region in energy between these two groups gets populated due to the broadening in energy of both groups caused by ionization and scattering. At low E/N, this behavior is less pronounced since the population of electrons for avalanches of size one is relatively broad and the mean energy of the electrons is not high. The ionizing primaries have energies relatively near the ionization potential so that after undergoing an ionizing collision they too enter the distribution at low energies. The two distinct groups are not observed at 300 Td, but rather a broad distribution is. For avalanches with 100 electrons or more, the distribution reaches a quasisteady equilibrium. That is, $f(n,\epsilon,x) \simeq f(\epsilon)$ for all n greater than (approximately) 100.

For values of E/N above 1500 Td (and for the cross-section set used in these simulations), runaway electrons are observed so that an avalanche never reaches complete equilibrium. An electron is a runaway if it does not circulate through all the energy states available to it at a given E/N, but on average moves towards high-energy states. These electrons are clearly visible in Figs. 1(b) and 1(c).

From distributions such as those described above, the ionization rate ν_i for electrons belonging to an avalanche of size n at position x_i is computed making use of Eq. (3a). The dependence of ν_i on n for various distance from the cathode is shown in Fig. 2, at E/N of 1500 Td. The stochastic drift velocity $V_d(n,x)$ is shown in Fig. 3. This function has been obtained from the local slope of the x_i vs $\bar{t}$ data, where x_i is the position at which avalanches of size n have a mean arrival time $\bar{t}$ [see Eq. (3b)]. The stochastic ionization coefficient $\alpha_s(n,x)$ is obtained from Eq. (2). A plot of $\alpha_s(n,x)$ vs n, at E/N of 1500 TD with x as a parameter is shown in Fig. 4. The mean energy as a function of n is shown in Figs. 5(a)–5(c) (for E/N of 300, 1500, and 3000 Td, respectively). For $E/N < 1500$ Td, α_s, ν_i, V_d, and ϵ_s are monotonically decreasing functions of n, at a fixed position. This is in agreement with Byrne's[1] hypothesis that since there are more electrons to share the energy, the macroscopic parameters should decrease with n, for a fixed position. At high E/N, this is no longer true since there are electrons with energies well above the maximum in the cross section. This can be seen in Fig. 6. The ionization coefficient for avalanches of size 1 decrease with E/N. For avalanches of size 3, α is seen to go through a maximum at approximately 1750 Td. These avalanches basically constitute a beam of electrons.

For a fixed avalanche size, the ionization rate and the drift velocity are monotonically increasing functions of x, throughout the domain of the simulation. However, the rate of increase of the ionization rate decreases with x, while the rate of change drift velocity stays relatively constant. This gives rise to a maximum in ionization coefficient, α_s, as a function of x, for a fixed n (see Fig. 4). It is possible that ν_i and V_d behave similarly at large x, for a fixed n. This is certainly the case at high E/N for avalanches with low n.

An attempt has been made to fit the curves in Fig. 4 for $\alpha_s(n,x)$ with the function suggested by Byrne [i.e., Eq.

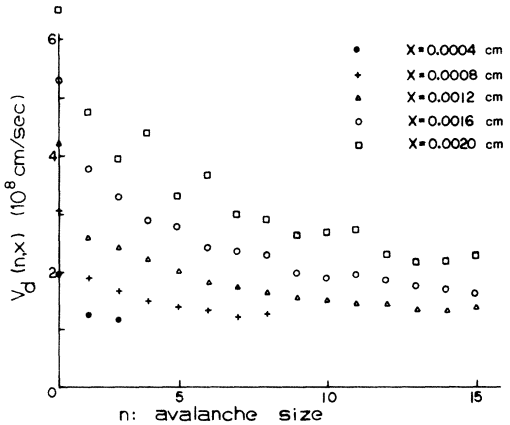

FIG. 3. Stochastic drift velocity, $V_i(n,x)$, for avalanches of size n at a distance x from cathode for $E/N = 1500$ Td.

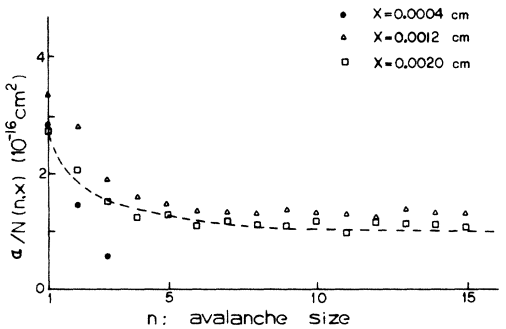

FIG. 4. Stochastic ionization coefficient, $\alpha_s(n,x)$ for $E/N = 1500$ Td, as a function of n, for various values of x.

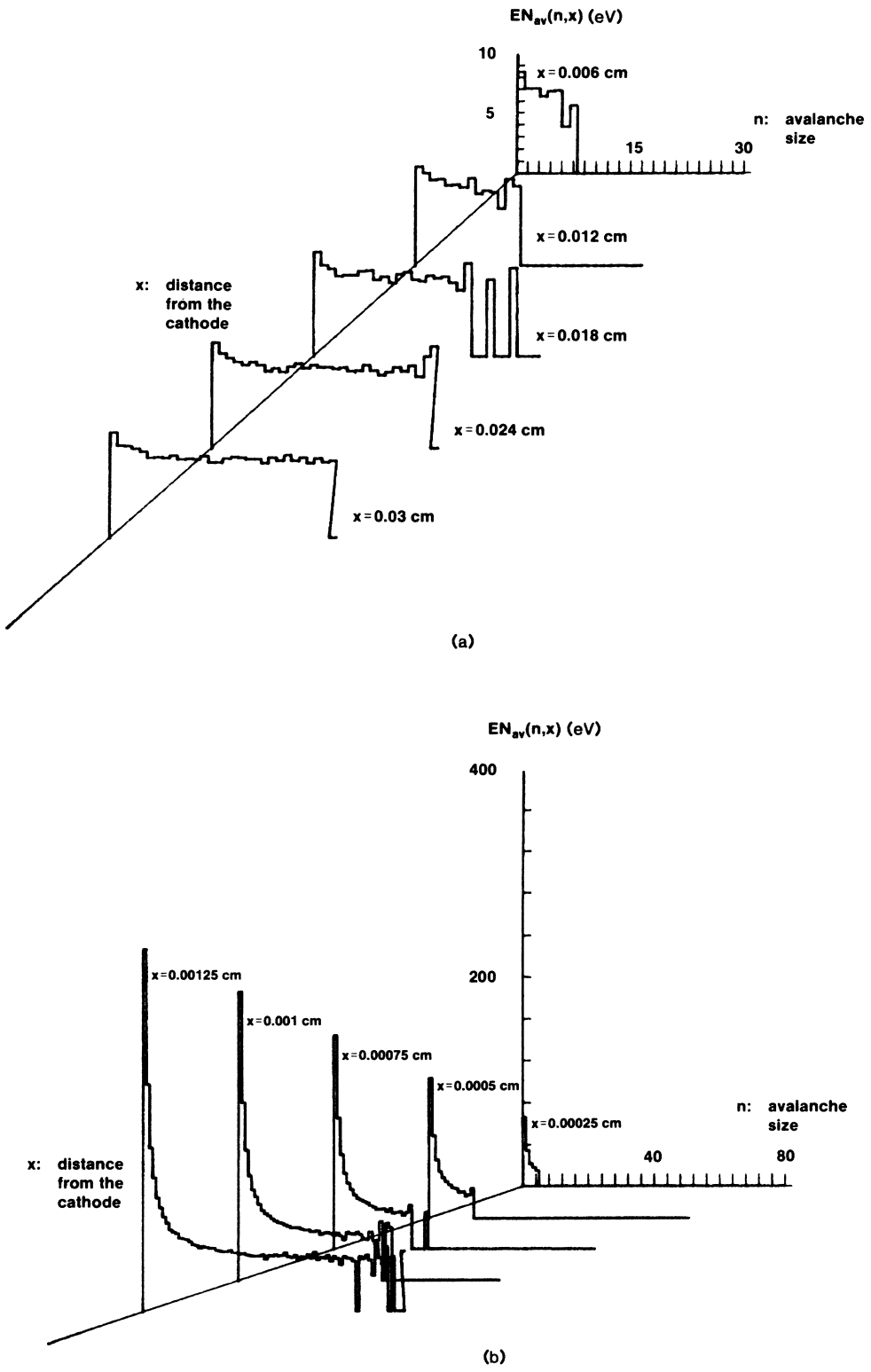

FIG. 5. Average electron energy, $\epsilon(n,x)$, for avalanches of size n at a distance x from the cathode at E/N equal to 300 Td, (b) 1500 Td, and (c) 3000 Td.

(1)]. It is clear from the discussion above and the figures that the coefficients a_0 and a_1 would have to be space dependent. The dashed curve in Fig. 4 is a plot of a two-parameter fit to the date for $x=2\times10^{-3}$ cm, using Eq. (1); i.e.,

$$\alpha_s(n,x=2\times10^{-3}\text{ cm})=(0.985+1.71/n)\,.$$

The discrepancies are not large and it seems that α_s may

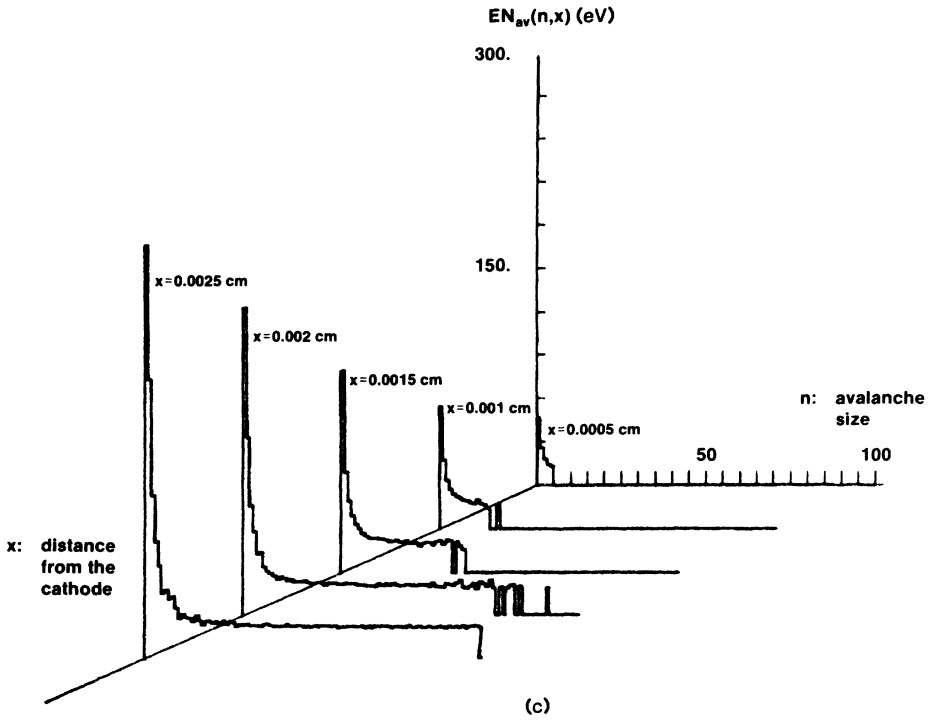

FIG. 5. (*Continued*).

be described by Eq. (1) with space-dependent coefficients. It is also possible to obtain empirical equations for $\alpha_s(n,x)$. However, at this time it makes no sense to do so before making a physical connection between the behavior of α_s and the microscopic processes. This is presently sought.

By counting the number of avalanches with size n at position x_i, the probability distribution $P(n,x_i)$ can be determined. The evolution in space of $P(n,x_i)$ is shown in Figs. 7(a), 7(b), and 7(c), for $E/N=300$, 1500, and 3000 Td, respectively. The distributions have been calculated for distances close to the cathode. It is this region which governs the statistics of an avalanche. At low E/N, the relative variance Δ ($\Delta^2=\sigma/\bar{n}^2$, where σ is the variance and $\bar{n}$ is the mean avalanche size), decreases with increasing E/N.[5] For $E/N \gtrsim 1500$ Td, in nitrogen, it begins to increase due to the appearance of runaway electrons. Two features in the electron-molecule collision cross section make this phenomenon possible. First, the total cross section decreases with energy at high energies, and second, the angular scattering becomes predominantly forward at these energies. Since the ionization probability for these electrons decreases with energy, the flow of avalanches of size n into $n+1$, for low values of n, decreases.

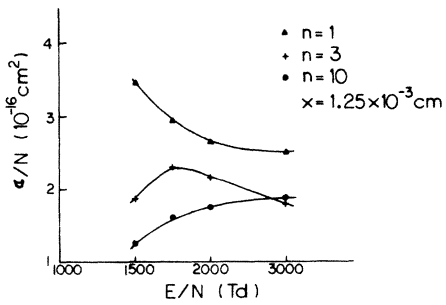

FIG. 6. E/N dependence of the stochastic ionization coefficient, $\alpha_s(N,x)$, at $x=1.25\times10^{-3}$ cm.

This effect can be clearly seen in the shape of $P(n,x)$ for $E/N=3000$ Td, Fig. 7. The probability distribution for low n at $E/N=1500$ Td is nearly constant, and at 3000 Td the distribution has two maxima [see Fig. 7(c)]. We suggest that from measurements of the relative variance versus E/N (using pulse height distribution experiments), a "threshold" value of E/N can be determined above which runaway electrons can occur (in a given spatial domain).

From Eqs. (6)–(8), the evolution in space of the mean avalanche parameters, averaged over the avalanche size distribution, can be determined. The dependence with x of the mean energy, drift velocity, and ionization coefficient are displayed in Figs. 8(a) and 8(b), for $E/N=300$ and 1500 Td, respectively. After a distance d_i equal to ϵ_i/E (where ϵ_i is the ionization potential), the probability for an electron to ionize is greater than zero. For $x>d_i$, the ionization coefficient $\bar{\alpha}$ is greater than zero [see Figs. 8(a) and 8(b)]. A maximum is observed in $\bar{\alpha}$ and in the mean energy $\bar{\epsilon}$ as functions of x. The overshoot follows from the gross behavior of the distribution with x. The spatial evolution of the energy distribution, integrated

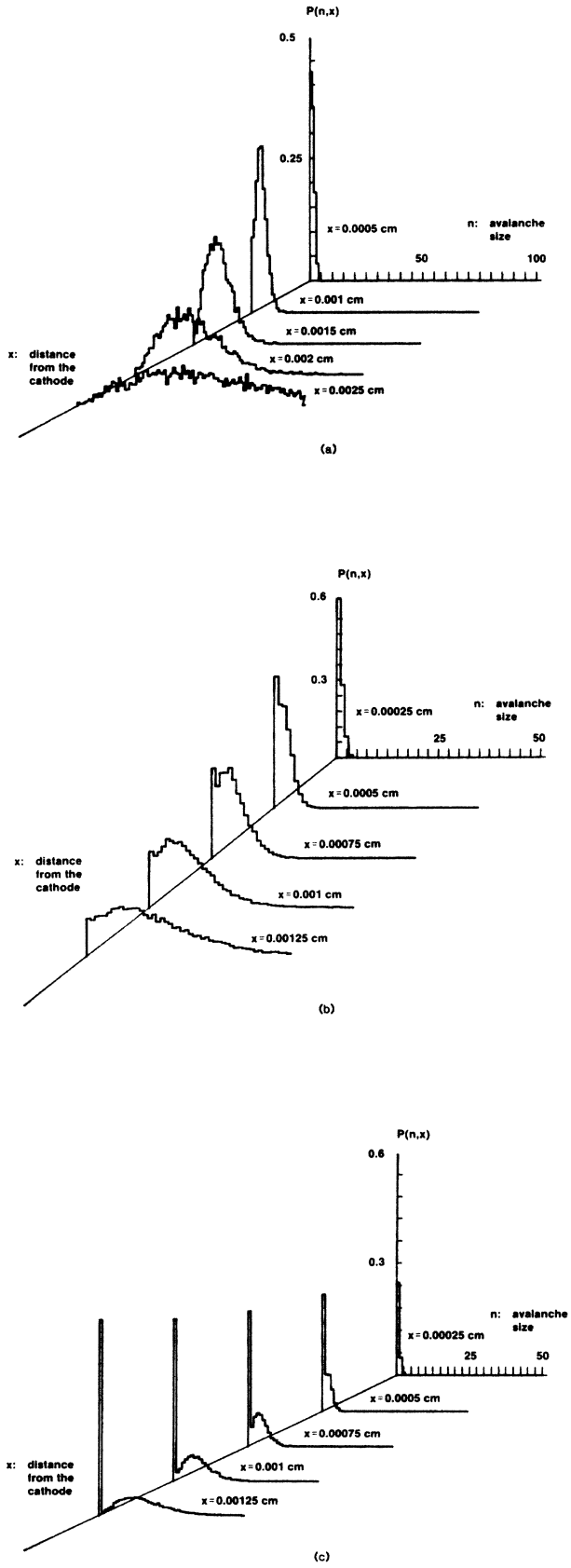

FIG. 7. Avalanche size distribution at E/N equal to (a) 300 Td, (b) 1500 Td, and (c) 3000 Td.

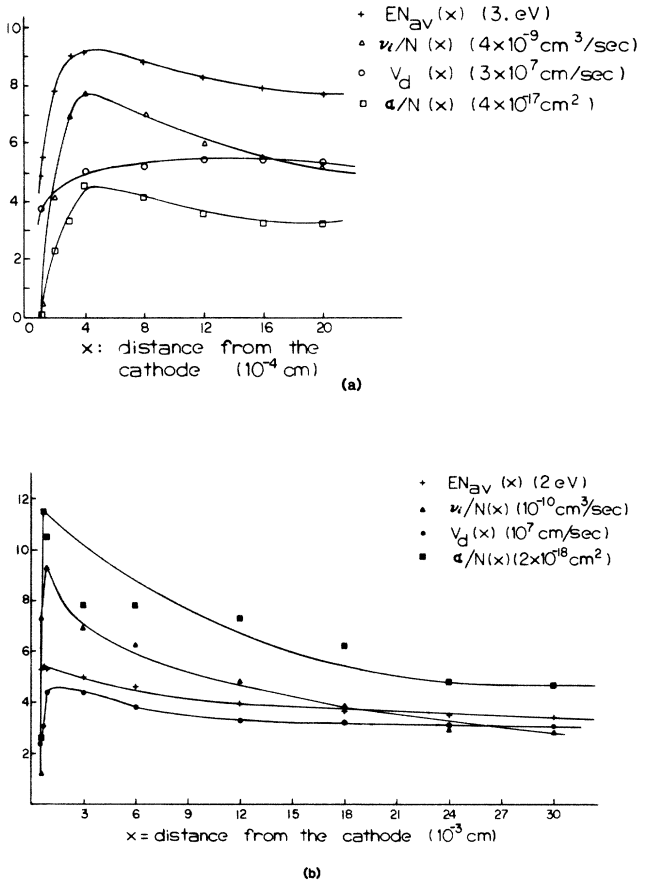

FIG. 8. Spatial evolution of macroscopic avalanche parameters for E/N equal to (a) 300 Td and (b) 1500 Td.

over avalanche size, is shown in Figs. 9(a)—9(c), for $E/N = 300$, 1500, and 3000 Td, respectively. The distribution is observed to "overshoot" the equilibrium state. That is, upon leaving the cathode, the flux of electrons to high energies due to the acceleration of the field is much greater than the return flux (initially there are no electrons at high energies). This results in an overpopulation of the high-energy states relative to their equilibrium value. As the assembly progresses, electrons begin to scatter back to low energies until the fluxes equilibrate. The scattering is highest for electrons near the vibrational peak ($\sim$2.5 eV) and for those with energies above 8 eV. The large number of inelastic processes at lower energies in N_2 prevent the distribution from oscillating about equilibrium. The temporal evolution of the distribution for the average avalanche is to be presented in a future paper. At large x, the macroscopic variables reach their equilibrium value. The behavior with x of these variables is as expected from the discussion given above of the way the distribution (integrated over avalanche size) reaches equilibrium. At 300 Td, the drift velocity also exhibits a maximum as a function of x [see Fig. 8(a)]; however, at 1500 Td [Fig. 8(b)], it monotonically increases with x to its equilibrium value. Cookson and Lewis[14] may have overlooked the differences

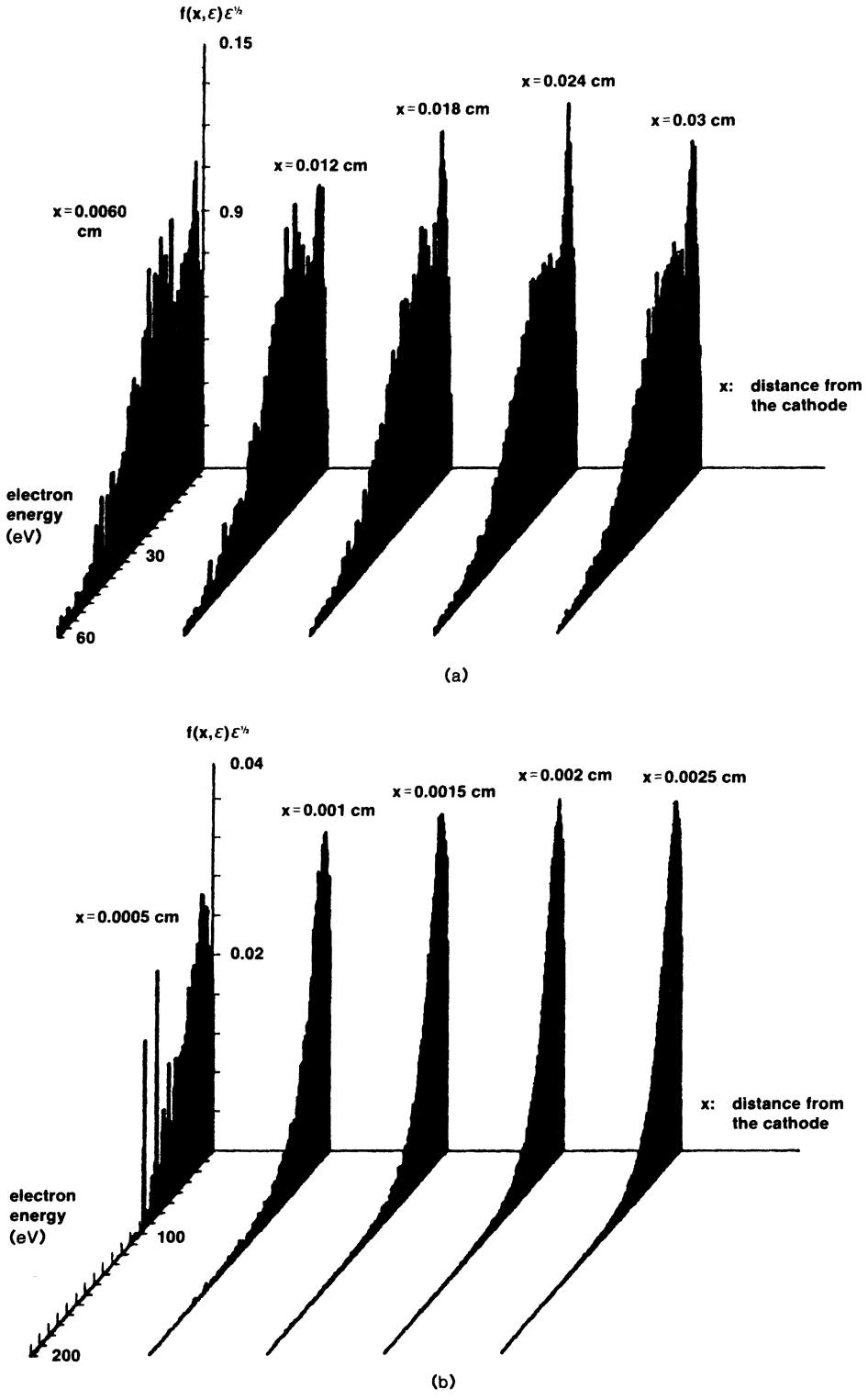

FIG. 9. Spatial evolution of the electron energy distribution integrated over avalanche size for E/N equal to (a) 300 Td, (b) 1500 Td, and (c) 3000 Td.

in the definition of $\alpha_s(n,x)$ and $\bar{\alpha}(x)$ and concluded that $\bar{\alpha}(x)$ is a decreasing function of x. In the stochastic formulation, n and x are independent variables, with n being a random variable distributed according to $P(n,x)$; so that letting $n=n(x)$ in the expression for $\alpha_s(n,x)$ to obtain $\bar{\alpha}(x)$, is not in general correct.

To compare the results that have been presented with experimental values, the calculated probability distribu-

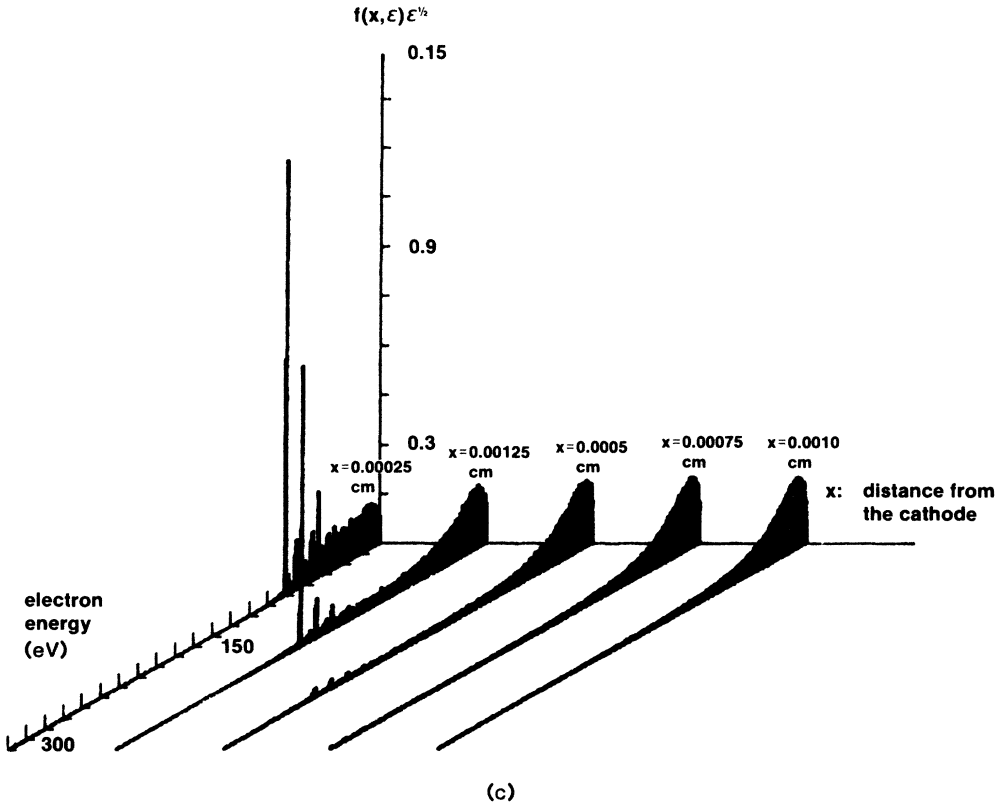

FIG. 9. (*Continued*).

tions need to be extrapolated to values of n, the avalanche size, that can be detected; this is typically above 10^4 electrons. To do so, the further development of avalanches with more than 100 electrons would have to be modeled using average procedures. It would be computationally prohibitive at this time using Monte Carlo techniques. Since the fluctuations are small for avalanches with $n > 100$, this average procedure seems plausible. This is presently being investigated.

ACKNOWLEDGMENT

This work was supported by the U. S. Office of Naval Research.

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