Adiabatic-invariant change due to separatrix crossing

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A slowly varying Hamiltonian with one degree of freedom and nearly closed orbits has an adiabatic invariant. This adiabatic invariant is conserved to all orders in ϵ, the slowness parameter, except for orbits that cross a separatrix. The present work discusses the change of the adiabatic invariant during this crossing process through order ϵ. First, a calculation of the change of the adiabatic invariant is presented. This calculation is general and, hence, encompasses previous results for specific cases. The change in the adiabatic invariant is shown to depend on a maximum of five parameters, which are functions of the Hamiltonian of interest. Second, the statistics of this process are derived. Finally, these results are applied to the motion of a particle in a wave with changing amplitude and phase velocity.

I. INTRODUCTION

Nearly periodic orbits of a Hamiltonian $H(q,p,\epsilon t)$ with one degree of freedom and slow time dependence $\epsilon \ll 1$ have been extensively studied. The action $I(h,\lambda \equiv \epsilon t)$, the area enclosed by a contour of constant energy (i.e., value of the Hamiltonian) $h = H(q,p,\epsilon t)$, is an adiabatic invariant to lowest order in the slowness parameter ϵ. Furthermore, there exists[1,2] an adiabatic invariant

$$J = I + \epsilon J_1(p,q,\lambda) + \epsilon^2 J_2(q,p,\lambda) + \cdots , \tag{1}$$

which is conserved to all orders in ϵ for a time that is $O(1/\epsilon)$. The corrections $J_1, J_2, J_3, \ldots$ have been calculated in many special cases.[3-5] In Appendix B the calculation for general Hamiltonians with 1 degree of freedom is presented.

For practical application of adiabatic theory one must verify that the terms in Eq. (1) are successively smaller. This allows one to use only the first few terms of the series. This is important since the series is usually asymptotic, not convergent, and, in any case, only the first few terms in the series are ever calculated and used.

The general theory (see Appendix B) shows that the ratio of successive terms in Eq. (1) is of order $\delta(h,\lambda) \equiv \epsilon T_0(h,\lambda)$, where $T_0(h,\lambda)$ is the period of the orbit of energy h for the frozen Hamiltonian. (The frozen Hamiltonian at parameter value λ is the time-independent Hamiltonian obtained by the limit $\epsilon \to 0$ with λ held constant.) The requirement $\delta \ll 1$ is simply that the Hamiltonian must change little in one period for adiabatic theory to apply.

This requirement is not satisfied for trajectories in the vicinity of the separatrix. (A separatrix is a contour of H containing an x point, a saddle point of H. The generic separatrix, a figure eight, is shown in Fig. 1.) The lack of validity arises because the period T_0 is infinite on the separatrix. Hence, no matter how small ϵ is, the parameter δ is arbitrarily large in some neighborhood of the separatrix.

This would be of little consequence were few trajectories ever near the separatrix, but, in fact, adiabatic theory indicates that many trajectories cross separatrices.[6] For example, consider a trajectory initially in region c (referring to Fig. 1). As long as the trajectory is far from the separatrix, the adiabatic invariant J remains at its initial value J_i. However, if the sum of the areas $Y_a(\lambda)$ inside lobe a and $Y_b(\lambda)$ inside lobe b are increasing in time, then there eventually comes a time $\epsilon t_x \equiv \lambda_x$ we call the pseudocrossing time, such that $J_i = Y_c(\lambda_x) \equiv Y_a(\lambda_x) + Y_b(\lambda_x)$. At later times the trajectory has crossed the

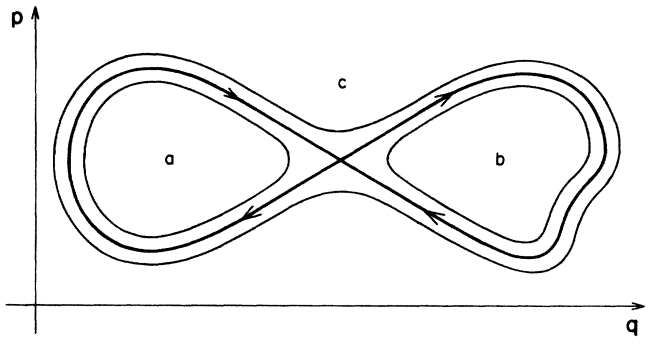

FIG. 1. Contours of Hamiltonian with a typical, figure eight separatrix. Arrows indicate direction of flow when the energy increases upon entering region c.

separatrix and is confined to either lobe a or lobe b. A new adiabatic invariant is well conserved at some final value J_f. However, its definition has changed while crossing the separatrix, during which time adiabatic theory is not valid.

The motivation for analyzing this problem comes from studies in many fields. Action-variation calculations have been applied to studies of energy and momentum balance for waves in collisionless plasma,[3,7–10] while slow separatrix crossings have been discussed in studies of mirror containment degeneration due to low-frequency fluctuations[11] and particle transport in strongly turbulent plasmas.[12] Slow separatrix crossings also occur in high-energy accelerators, where coasting particles are trapped and bunched by slowly ramped rf fields, in colliding beam storage rings, where betatron amplitudes can diffuse due to repeated slow crossings of synchrobetatron coupling resonances,[13] and in three-dimensional toroidal plasma-confinement systems.[14] In addition, slow separatrix crossings occur in celestial mechanics[6,15,16] due, e.g., to tidal dissipation effects.

Previous analyses dealt with separatrix crossings for the particular Hamiltonian

$$H=p^2/2+A(t)\cos[q-\varphi(t)] \tag{2}$$

of a particle moving in a wave with a slowly varying amplitude $A=O(\epsilon)$ and phase velocity $\dot{\varphi}=O(\epsilon)$. (The overdot denotes time derivative.) Pocobelli[17] discusses the continuity of the elliptic function solutions across the separatrix. Menyuk[18] numerically analyzed a particular case of (2) obtained by summing two waves of constant amplitude and phase and close in frequency. He found that all particles that cross the separatrix are stochastic in the limit $\epsilon\rightarrow 0$.

Other analyses seek to determine analytically the spreading in the adiabatic invariant due to separatrix crossing. However, these analyses applied only to the cases of fixed amplitude[19] or fixed phase velocity.[20,21] The present results include these previous results as special cases.

The present work is a general analysis of the separatrix crossing process. A detailed calculation of previously reported results[22] is presented. Not surprisingly, it is found that to lowest order the final value of the adiabatic invariant is phase independent and is given by $J_f=Y_\beta(\lambda_x)$ for a particle ultimately in region β ($\beta=a,b$, or c). However, it is also shown that there are phase-dependent deviations $\delta J_{f\beta}\equiv J_f-Y_\beta(\lambda_x)$ of order $\epsilon ln(\epsilon)$. [This is an improvement over the previous result of $\epsilon ln^4(\epsilon)$ of Janicke.[23]] This phase dependence implies that for separatrix crossing orbits, the adiabatic invariant is not conserved to all orders in ϵ, but only to lowest order.

The main calculation of this paper is in Secs. II–V. Section II is an overview of the calculation. Preliminary to calculating the deviation, the adiabatic theory for particles near the separatrix is developed in Sec. III. To correctly calculate the deviation of the adiabatic invariant through order ϵ, the first-order correction to the adiabatic invariant must be obtained. Then, in Sec. IV, the change of the adiabatic invariant for a single oscillation period is obtained. In Sec. V the sum of changes due to all the steps is calculated.

For many problems a statistical analysis is important. For this purpose an ensemble of trajectories with initial conditions uniform in an adiabatic invariant annulus is considered in Sec. VI. Phase mixing causes this to be the asymptotic ($t\rightarrow\infty$) distribution for any initial distribution with a narrow spread in J. For such a distribution the probability distribution for the crossing parameter h_0 and various moments of δJ_f are calculated in certain special cases.

In Sec. VII these results are applied to the motion of a particle in a wave with slowly varying amplitude and phase velocity. Specializing to the cases of constant amplitude ($\ddot{\varphi}\neq 0$, $\dot{A}=0$) and constant phase velocity ($\ddot{\varphi}=0$, $\dot{A}\neq 0$) allows comparison with previously obtained results. The present results agree with those of Timofeev[20] for the constant-phase-velocity case. However, there is a discrepancy between the present results and those found previously for the constant amplitude case in Ref. 19.

II. OVERVIEW OF THE CALCULATION

The main goal of this work is the calculation of the change of the adiabatic invariant due to separatrix crossing. This calculation is performed in Secs. III–VI. In this section, an overview of the calculation is presented to lay out the major issues.

A trajectory that crosses the separatrix consists of three parts. Long before the actual separatrix crossing, adiabatic theory is valid. For those oscillations in the vicinity of the separatrix, adiabatic theory breaks down. Long after the crossing, adiabatic theory again is valid. The change of the adiabatic invariant occurs during the critical oscillations in the vicinity of the separatrix.

This scenario suggests that the analysis of the trajectory can be split into three portions. The portions of the trajectory long before and long after the crossing can be treated by standard adiabatic-invariant theory. According to standard adiabatic theory, the adiabatic invariant is conserved to all orders. This implies that the first-order corrected adiabatic invariant $J^1\equiv J_0+\epsilon J_1$ deviates from being constant by the magnitude of the next correction $\epsilon^2 J_2$ during these two portions of the orbit.

To connect these two parts of the trajectory the portion of the trajectory near the separatrix must be analyzed. This analysis cannot rely on the validity of the adiabatic invariant, because the series for the adiabatic invariant rapidly diverges near the separatrix, as is shown in Sec. III. Instead, the variation of only the first two terms, $J_0+\epsilon J_1$, of the series is calculated for the portion of the trajectory near the separatrix.

This part of the analysis is greatly simplified by assuming the Hamiltonian has the form

$$H=\omega(p^2-q^2)/2+\delta H(q,p,\lambda)\ , \tag{3a}$$

where

$$\delta H=O(q^3,q^2p,qp^2,p^3)\ , \tag{3b}$$

and ω is a constant. In Appendix A it is shown that an arbitrary slowly varying Hamiltonian can be put in this form to arbitrarily high order in ϵ. With this choice one

can use the approximation of small energy h in the calculation of the variation of $J_0+\epsilon J_1$ for the portion of the trajectory near the separatrix.

For the figures and purposes of discussion, the choice is made that the energy inside the lobes is negative, so the flow proceeds clockwise around the lobes. This choice helps prevent sign confusion. However, provided one calculates the various line integrals, such as in Eq. (17), by integrating along the flow, all of the formulas are valid with other conventions for the energy.

The first part of the calculation is the development of adiabatic theory for particles near the separatrix. This is carried out using standard techniques in Sec. III.

There are three types of separatrix crossings to be discussed (see Fig. 2). A double crossing occurs when a trajectory begins in one lobe and ends in the other. The trajectory crosses the separatrix upon both leaving the initial lobe and entering the final. The two types of single crossings occur for trajectories that begin or end outside both lobes. Without loss of generality the choice is made that either the particle is initially in lobe a, and/or it is ultimately in lobe b.

As in the work of Timofeev,[20] the small-h analysis proceeds by breaking up the crossing into a sequence of steps. Each step is a single- or double-lobe encirclement from one vertex, a point on the trajectory close to the x point, to the next vertex. A vertex is defined to be an intersection of an orbit with the line $q=0$ or the line $p=0$. The four possible types of vertices are labeled as shown in Fig. 3. The vertices are not uniquely defined because the canonical transformation to q and p is not unique. In Appendix F we will show that the final answer is nevertheless independent of the coordinate system.

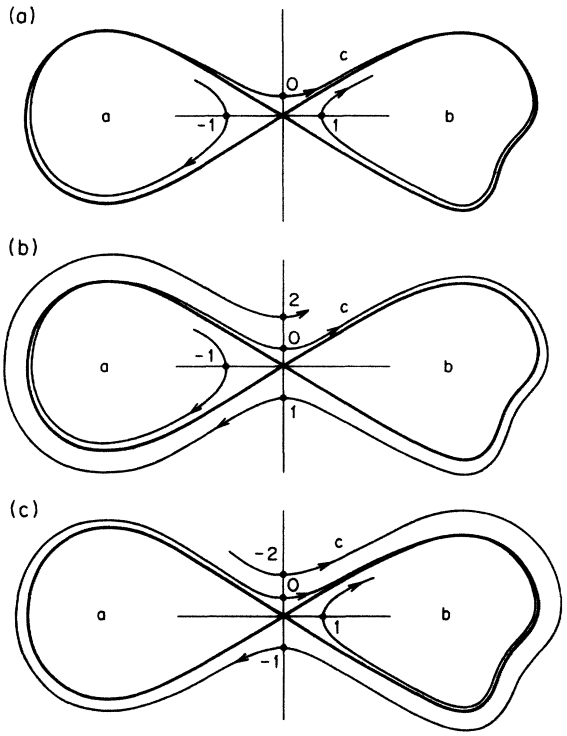

FIG. 2. Illustration of possible types of motion. A double crossing for a particle beginning in lobe a and ending in lobe b is shown in (a). Single crossings are shown for a particle beginning in a lobe (b) and ending in a lobe (c). These illustrations are only schematic. Actually, the motion loops stay roughly constant in size while the separatrix-enclosed areas change.

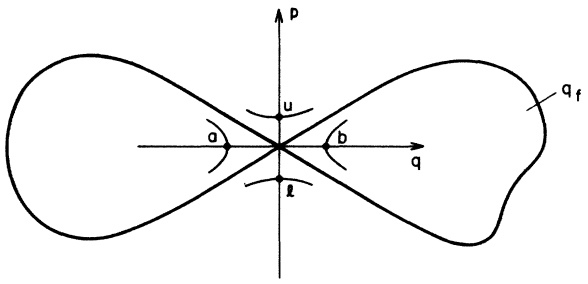

FIG. 3. Orientation of phase space for the calculation. The figure defines the labeling of the distinct vertices u and l of region c.

Next, the change of the adiabatic invariant in a single step is calculated in Sec. IV. The single-step period and energy change are calculated using time-dependent perturbation theory. These results allow one to calculate the change in the adiabatic invariant for each step.

The final value J_f of the adiabatic invariant is calculated in Sec. V by summing the effects of the steps before and after the crossing. This gives the result in terms of h_0 and t_0, the values of h and t at the zero vertex, which is the vertex in region c which is closest to the separatrix crossing(s) (see Fig. 2). To complete the calculation, the relation between t_0 and the pseudocrossing time t_x must be found. Use of this relation gives the formula for J_f in terms of the initial value J_i of the adiabatic invariant and the crossing parameter h_0.

III. ADIABATIC THEORY NEAR THE SEPARATRIX

To cross the separatrix, a trajectory must be near the separatrix. For this reason it is necessary to first develop adiabatic theory for particles near the separatrix. That is, the various quantities of adiabatic-invariant theory must be calculated for small h. The reason for this calculation is that it underlies the calculation of the effects of separatrix crossing. As will be shown later, the values of h relevant to the crossing process are of order ϵ.

In the most familiar form of Hamilton's equations, t is the independent variable, while q and p are dependent variables. In the present calculation it is more convenient to use q as the independent variable and t and h as the dependent variables. Given q, t, and h, one can solve

$$H(q,P(q,h,\lambda),\lambda)=h\ , \tag{4}$$

for the function $P(q,h,\lambda)$. Application of the chain rule to Hamilton's equations yields the equations of motion

$$\frac{dt}{dq}=\frac{\partial P(q,h,\lambda)}{\partial h} \tag{5a}$$

and

$$\frac{dh}{dq}=-\frac{\partial P(q,h,\lambda)}{\partial t}=-\epsilon\frac{\partial P(q,h,\lambda)}{\partial\lambda}\ . \tag{5b}$$

(These equations can also be derived from the principle of least action expressed in canonical coordinates.[24]) An inessential difficulty of this formalism arises in the analysis of trajectories on which $\partial H/\partial p$ vanishes.[25]

In the analysis that follows, phase space is divided into two regions. In the region far from the x point, the various functions can be evaluated by Taylor expansion in h. Near the x point this approximation is invalid. Instead, the solution of Eq. (4) may be found by perturbation theory based on the Taylor expansion (3). For exterior orbits ($h>0$) the lowest-order solution

$$P_0\equiv(2h/\omega+q^2)^{1/2} \tag{6}$$

is found by neglecting δH. The first correction is obtained by Taylor expansion and resubstitution,

$$P\approx P_0-(\partial P_0/\partial h)\delta H(q,P_0(q,h),\lambda)\ . \tag{7}$$

For later reference the derivatives

$$\begin{aligned}\frac{\partial P}{\partial h}\approx&(2\omega h+\omega^2q^2)^{-1/2}\\&+\omega(2\omega h+\omega^2q^2)^{-3/2}\delta H(q,(2\omega h+\omega^2q^2)^{1/2},\lambda)\\&-(2\omega h+\omega^2q^2)^{-1}\frac{\partial\delta H(q,(2\omega h+\omega^2q^2)^{1/2},\lambda)}{\partial p}+\cdots,\end{aligned} \tag{8}$$

and

$$\partial P/\partial\lambda\approx(\omega P_0)^{-1}\partial\delta H/\partial\lambda+\cdots\ , \tag{9}$$

are given.

For interior orbits ($h<0$) this perturbation theory breaks down because $\partial P_0/\partial h$ vanishes at the vertex. Instead, such orbits are analyzed by using p as the independent variable. In analogy with Eqs. (4) and (5), the function $Q(p,h,\lambda)$ satisfying

$$H(Q(p,h,\lambda),p,\lambda)=h\ , \tag{10a}$$

must be found. The corresponding equations of motion are

$$\frac{dt}{dp}=-\frac{\partial Q(p,h,\lambda)}{\partial h} \tag{10b}$$

and

$$\frac{dh}{dp}=\frac{\partial Q(p,h,\lambda)}{\partial h}=-\epsilon\frac{\partial Q(p,h,\lambda)}{\partial\lambda}\ . \tag{10c}$$

The perturbation expansions and the calculations of the interior orbits are entirely analogous to those of the exterior orbits. Hence, in the interest of brevity, they are not presented.

A. The instantaneous period

The instantaneous period $T_0(h,\lambda)$ is defined to be the period of an orbit of energy h at the parameter value λ for fixed ϵ. That is, it is the period for the Hamiltonian frozen at λ. The instantaneous period is given by the loop integral

$$T_0(h,\lambda)=\oint dq\frac{\partial P}{\partial h}(q,h,\lambda)\ .$$

The contribution from encircling lobe b to the period of a particle in region c is given by

$$T_{0b}(h,\lambda)=\int_{q_u}^{q_l}dq\frac{\partial P(q,h,\lambda)}{\partial h}\ .$$

Of course, $q_u=q_l=0$. This notation indicates that the integral proceeds along the flow from the upper vertex to the lower vertex. This integral is evaluated by dividing it into the contributions of the upper and lower halves. The contribution

$$T_{0bu}(h,\lambda)=\int_0^{q_B}dq\frac{\partial P(q,h,\lambda)}{\partial h}\ , \tag{11}$$

of the upper half is defined to be that part of the integral from the vertex to some final curve given by $q_B(h,\lambda)$.

From the expansion (8), it is seen that the integral (11) is singular near $q=0$. Therefore, the singular portion is subtracted off,

$$\begin{aligned}T_{0bu}(h,\lambda)=&\int_0^{q_B}dq(2\omega h+\omega^2q^2)^{-1/2}\\&+\int_0^{q_B}dq\,R(q,h,\lambda)\ ,\end{aligned} \tag{12}$$

where

$$R\equiv\partial P/\partial h-(2\omega h+\omega^2q^2)^{-1/2}\ . \tag{13}$$

The first integral is found by standard methods to be

$$\int_0^{q_B}dq(2\omega h+\omega^2q^2)^{-1/2}=(2\omega)^{-1}\ln|2\omega q_B^2/h|+O(h)\ . \tag{14}$$

The second integral of Eq. (12) is evaluated by dividing the integration region at q_I, a value which has order less than unity,

$$T_{0bu}(h,\lambda)=\int_0^{q_I}dq\,R(q,h,\lambda)+\int_{q_I}^{q_B}dq\,R(q,h,\lambda)\ . \tag{15}$$

To evaluate the first integral, the perturbation expansion (10) is used. This gives

$$\int_0^{q_I}dq\,R(q,h,\lambda)=O(q_I)\ .$$

To evaluate the second integral of Eq. (15) the Taylor expansion

$$R(q,h,\lambda)\approx R(q,0,\lambda)+h\frac{\partial R(q,0,\lambda)}{\partial h}+\cdots\ ,$$

is used. When integrating the first term in this expansion, the lower limit of the integral may be set to zero. This introduces an error that is $O(q_I)$. The term $\partial R/\partial h$ in this equation is bounded above by an order-unity constant divided by q^2. Hence, upon integration, it yields a contribution of order h/q_I. Therefore, the second integral of Eq. (15) is given by

$$\int_{q_I}^{q_B}dq\,R(q,h,\lambda)=\int_0^{q_B}dq\,R(q,0,\lambda)+O(h/q_I)\ .$$

When the two integrals of Eq. (12) are combined, the various constants of order unity may be collected into the logarithm. The final result is

$$T_{0bu}=(2\omega)^{-1}\ln|h_{bu}/h|+O(h/q_I)+O(q_I)\ . \quad (16)$$

The constant h_{bu} is given by

$$\ln|h_{bu}|=\ln|2\omega q_B^2|+2\omega\int_0^{q_B}dq\,R(q,0,\lambda)\ . \quad (17)$$

From Eq. (16) it is seen that the error is minimized by the choice, $q_I=O(h^{1/2})$, in which case the error of Eq. (16) is $O(h^{1/2})$.

The period for the lower half of the orbit can be obtained in the same way. The only difference in the answer is a change in the constant, $h_{bu}\to h_{bl}$. However, even these may be made equal ($h_b\equiv h_{bu}=h_{bl}$) by a choice of $q_B(h,\lambda)$, since modifying q_B changes T_{0bu} and T_{0bl} by opposite amounts of order unity. This choice amounts to requiring

$$\int_0^{q_B}dq\,R(q,0,\lambda)=\int_{q_B}^{0}dq\,R(q,0,\lambda)\ .$$

These integrals are defined to proceed along the flow. This choice of q_B gives the results

$$T_{0bu}=(2\omega)^{-1}\ln|h_b/h|+O(h^{1/2})$$

and

$$T_{0bl}=(2\omega)^{-1}\ln|h_b/h|+O(h^{1/2})\ .$$

The total instantaneous period is found by adding these two results,

$$T_{0b}(h,\lambda)=\omega^{-1}\ln|h_b/h|+\mu_b(\lambda)|h|^{1/2}+o(|h|^{1/2})\ . \quad (18a)$$

The quantity $\mu_b(\lambda)$ need not be known for this analysis. However, the fact that it is a function of λ alone is needed to determine the error in the change of the adiabatic invariant in a single step. The period for an interior orbit, which is found by using the alternative formalism (10), is also given by (18a). The analysis of lobe a is identical. Therefore,

$$T_{0a}(h,\lambda)=\omega^{-1}\ln|h_a/h|+\mu_a(\lambda)|h|^{1/2}+o(|h|^{1/2})\ . \quad (18b)$$

For a particle encircling both lobes the total instantaneous period is simply the sum of the separate periods,

$$T_{0c}(h,\lambda)=(2/\omega)\ln|h_c/h|+\mu_c(\lambda)|h|^{1/2}+o(|h|^{1/2})\ , \quad (18c)$$

where $h_c\equiv(h_ah_b)^{1/2}$.

B. The action

The action is defined to be the phase-space area enclosed by a contour of H. In terms of the momentum function P, the action is given by

$$I(h,\lambda)=\oint dq\,P(q,h,\lambda)\ . \quad (19)$$

Considered now to be a phase function by substitution, $I(H(q,p,\lambda),\lambda)$, the action has the following property. The value of I on a contour of constant I equals the phase-space area enclosed by that contour. This implies that the action cannot be a continuous function across the separatrix.

The action can be easily calculated from its energy derivative,

$$\frac{\partial I(h,\lambda)}{\partial h}=\oint dq\frac{\partial P(q,h,\lambda)}{\partial h}+T_0(h,\lambda)\ ,$$

and the results of Sec. II. Straightforward integration gives

$$I_a(h,\lambda)=Y_a+(h/\omega)(1+\ln|h_a/h|)+\mu_a(\lambda)|h|^{3/2}+o(|h|^{3/2}) \quad (20a)$$

and

$$I_b(h,\lambda)=Y_b+(h/\omega)(1+\ln|h_b/h|)+\mu_b(\lambda)|h|^{3/2}+o(|h|^{3/2})\ , \quad (20b)$$

where Y_α is the area enclosed by lobe α. For a particle encircling both lobes, the action is simply the sum

$$I_c(h,\lambda)=Y_c+(2h/\omega)(1+\ln|h_c/h|)+\mu_c(\lambda)|h|^{3/2}+o(|h|^{3/2})\ . \quad (20c)$$

C. First correction to the adiabatic invariant

The change in the adiabatic invariant is a quantity of order ϵ. To be properly calculated the adiabatic invariant must itself be computed through order ϵ. In this section the first correction to the adiabatic invariant is calculated using the theory of Appendix B.

The correction to the adiabatic invariant depends on q as well as h and t. Therefore, one must specify the location on the energy contour at which the action is being calculated. For the calculation the adiabatic invariant at the vertices is needed. There are four such vertices, as can be seen in Fig. 3. The four corrections to the adiabatic invariant will be labeled accordingly.

According to Eq. (B14) the correction to the action at vertex b is given by

$$J_{1b}=\frac{1}{2}\frac{\partial\not{\jmath}_{0b}}{\partial h}\frac{\partial\not{\jmath}_{0b}}{\partial\lambda}-\int_{q_B}^{q_B}dq\frac{\partial P(q,h,\lambda)}{\partial h}\int_{q_b}^{q}dq'\frac{\partial P(q',h,\lambda)}{\partial\lambda}\ , \quad (21)$$

where q_b is the location of vertex b, i.e., $P(q_b,h,\lambda)=0$. The loop integral over dq may begin and end at any point. The choice here is to use q_B, which was defined such that the upper and lower half-periods of Sec. III A are equal. As discussed in Appendix B, the branch of the multivalued integral over dq' is specified by requiring the integration to proceed along the flow.

This expression simplifies if the replacement

$$\int_{q_b}^{q}dq'\frac{\partial P(q',h,\lambda)}{\partial\lambda}=\frac{\partial\not{\jmath}_{0b}}{\partial\lambda}-\int_{q}^{q_b}dq'\frac{\partial P(q',h,\lambda)}{\partial\lambda}$$

is made for that part of the dq integral from q_B to q_b. With this the expression (21) for J_{1b} can be rewritten in the form

$$J_{1b}=\frac{1}{2}\frac{\partial \mathcal{J}_{0b}}{\partial h}\frac{\partial \mathcal{J}_{0b}}{\partial \lambda}-\frac{\partial \mathcal{J}_{0b}}{\partial \lambda}\int_{q_B}^{q_b}dq'\frac{\partial P(q',h,\lambda)}{\partial h}+J_{1b+}+J_{1b-}\ , \tag{22}$$

where

$$J_{1b+}\equiv-\int_{q_b}^{q_B}dq\frac{\partial P(q,h,\lambda)}{\partial h}\int_{q_b}^{q}dq'\frac{\partial P(q',h,\lambda)}{\partial \lambda}\ , \tag{23a}$$

and

$$J_{1b-}\equiv\int_{q_B}^{q_b}dq\frac{\partial P(q,h,\lambda)}{\partial h}\int_{q}^{q_b}dq'\frac{\partial P(q',h,\lambda)}{\partial \lambda}\ . \tag{23b}$$

The first two terms of expression (22) for J_{1b} cancel since q_B was chosen to be the halfway point of the loop integral of $\partial P/\partial h$. Thus the expression for J_{1b} is

$$J_{1b}=J_{1b+}+J_{1b-}\ . \tag{24}$$

The value of J_{1b} has a finite value in the limit $h\to 0$. In this limit $q_b=0$, and $\partial P/\partial\lambda=O(q^2)$. [This last fact can be seen in the perturbation expansion (9), upon using the separatrix condition $P_0\approx(\omega q)^{-1}$.] Hence the interior integral of Eqs. (23) is $O(q^3)$. This cancels the $O(1/q)$ singularity of $\partial P/\partial h$. The correction to J_{1b} due to nonzero h is estimated in Appendix C and given by Eq. (C3). Therefore, near the separatrix,

$$J_{1b}=g_b+O(h)\ , \tag{25}$$

where

$$g_b(\lambda)\equiv-\int_0^{q_B}dq\frac{\partial P(q,0,\lambda)}{\partial h}\int_0^{q}dq'\frac{\partial P(q',0,\lambda)}{\partial \lambda}+\int_{q_B}^{0}dq\frac{\partial P(q,0,\lambda)}{\partial h}\int_q^{0}dq'\frac{\partial P(q',0,\lambda)}{\partial \lambda}\ . \tag{26}$$

Identical results are valid for J_{1a} with the substitutions $b\to a$ and $B\to A$.

In all previously studied cases[19–21] the quantities g_α vanish. This vanishing occurred because of a special symmetry between the upper and lower halves of the lobes.

It remains to calculate the correction J_1 at the upper vertex. (The value at the lower vertex, while not needed here, follows from the interchange of a and b.) According to Appendix B this correction is given by

$$J_{1u}=\frac{1}{2}\left[\frac{\partial \mathcal{J}_{0a}}{\partial h}+\frac{\partial \mathcal{J}_{0b}}{\partial h}\right]\left[\frac{\partial \mathcal{J}_{0a}}{\partial \lambda}+\frac{\partial \mathcal{J}_{0b}}{\partial \lambda}\right]-\int_{q_u}^{q_u}dq\frac{dP(q,h,\lambda)}{\partial h}\int_{q_u}^{q}dq'\frac{\partial P(q',h,\lambda)}{\partial \lambda}\ .$$

(Of course, $q_u=0$. The notation of this equation is intended to convey that the lower endpoints of the integrals are at $q=0$ and $p>0$.) The integral in this expression can be divided into contributions from the separate loops. With the results (23)–(25), this equation can be put in the form

$$J_{1u}=g_c+\frac{1}{2}\left[\frac{\partial \mathcal{J}_{0a}}{\partial \lambda}\frac{\partial \mathcal{J}_{0b}}{\partial h}-\frac{\partial \mathcal{J}_{0a}}{\partial h}\frac{\partial \mathcal{J}_{0b}}{\partial \lambda}\right]+O(h)\ , \tag{27}$$

where $g_c\equiv g_a+g_b$.

Equations (20) can be used to evaluate this expression. In this evaluation, $\partial I_a/\partial\lambda=dY_a/d\lambda$ to relevant order. Differentiating the remainder term $(h/\omega)(1+ln\,|h_a/h\,|)$ with respect to λ gives a term of order h which may be neglected. The result of these manipulations yields

$$J_{1u}=g_c+[(dY_a/d\lambda)\ln|h_b/h\,|-(dY_b/d\lambda)\ln|h_a/h\,|]/(2\omega)+O(h)\ .$$

A summary of the expressions for the first-order corrected adiabatic invariant,

$$J^1\equiv J_0+\epsilon J_1\ , \tag{28}$$

calculated in this section, follows.

$$J_\alpha^1=Y_\alpha(\lambda)+\delta J_\alpha+\mu_\alpha|h\,|^{3/2}+o(|h\,|^{3/2})+O(\epsilon h)\ , \tag{29}$$

where $\alpha=a$, b, or u, and

$$\delta J_\alpha=(h/\omega)(1+\ln|h_\alpha/h\,|)+\epsilon g_\alpha \tag{30}$$

for $\alpha=a$ or b, while

$$\delta J_u=\delta J_a+\delta J_b+(\dot Y_a\ln|h_b/h\,|-\dot Y_b\ln|h_a/h\,|)/(2\omega)\ . \tag{31}$$

D. Higher-order corrections to the adiabatic invariant

Some estimate of the higher-order corrections to the adiabatic invariant is needed for this calculation. This estimate can be obtained by noting that in the calculation of subsequent orders, an additional derivative with respect to h is introduced in Eq. (B8). This gives the dominant term because it is singular as $h\to 0$. This phenomenon is manifest in the lowest two orders in that J_0 has a finite limit as $h\to 0$, while J_1 [see Eq. (27)] diverges logarithmically. In the next order, another derivative is introduced. Therefore,

$$J_2=O(1/h)\ . \tag{32}$$

IV. ADIABATIC INVARIANT CHANGE DUE TO A SINGLE STEP

In this section the change of the adiabatic invariant in a single step is calculated. In Sec. V the effects of individual steps will be summed to obtain the total change of the adiabatic invariant. The present calculation relies on the two basic calculations of the time required to make a step and the energy lost in making a step. Given these quantities, the change of the adiabatic invariant, which was found as a function of h and t in Sec. III, is easily computed.

Some of the error calculations in this section are particularly tedious. For continuity of development, these error estimates have been relegated to Appendix D.

A step is a single lobe encirclement. The beginning of the step occurs at a vertex, when either $p=0$ or $q=0$. This occurs at the initial time $t=t_i=\lambda_i/\epsilon$. At this time the particle is assumed to have energy h_i. At the final time $t_f=\lambda_f/\epsilon$ the particle is again at a vertex and has energy h_f. The goal is to find t_f and h_f given t_i and h_i. To be specific we consider a step between points 0 and 1 of Fig. 2(b), which begins and ends at $q=0$ and encircles lobe b. The generalization to other types of steps is straightforward.

The values of t_f and h_f are determined from the equations (5) of motion. An approximate solution to these equations for t near t_j and h near h_j ($j=i$ or f) is obtained by perturbation theory using

$$t(q)-t_j=\Delta t_{j0}(q)+\epsilon\,\Delta t_{j1}(q)+\cdots \tag{33}$$

and

$$h(q)-h_j=\epsilon\,\Delta h_{j0}(q)+\epsilon^2\,\Delta h_{j1}(q)+\cdots\ . \tag{34}$$

The lowest-order solutions are

$$\Delta t_{j0}(q)=\int_0^q dq'\frac{\partial P(q',h_j,\epsilon t_j)}{\partial h} \tag{35}$$

and

$$\Delta h_{j0}(q)=-\int_0^q dq'\frac{\partial P(q',h_j,\epsilon t_j)}{\partial \lambda}\ , \tag{36}$$

while the first-order corrections are given by

$$\Delta t_{j1}(q)=\int_0^q dq'\left[\Delta t_{j0}(q')\frac{\partial^2 P(q',h_j,\epsilon t_j)}{\partial h\,\partial\lambda}+\Delta h_{j0}(q')\frac{\partial^2 P(q',h_j,\epsilon t_j)}{\partial h^2}\right] \tag{37}$$

and

$$\Delta h_{j1}(q)=-\int_0^q dq'\left[\Delta t_{j0}(q')\frac{\partial^2 P(q',h_j,\epsilon t_j)}{\partial \lambda^2}+\Delta h_{j0}(q')\frac{\partial^2 P(q',h_j,\epsilon t_j)}{\partial h\,\partial\lambda}\right]. \tag{38}$$

A. Period of a step

The period of a step is analyzed by breaking the step into two halves. The first half is the portion from time t_i to some time intermediate between t_i and t_f. The second half is the remainder from the intermediate time to the final time. The exact expression for the period is

$$T=\int_0^{q_{Bi}} dq\frac{\partial P(q,h(q),\epsilon t(q))}{\partial h}+\int_{q_{Bi}}^0 dq\frac{\partial P(q,h(q),\epsilon t(q))}{\partial h}\ , \tag{39}$$

in which q_{Bi} denotes $q_B(h,\lambda_i)$. The dominant term

$$T\approx\int_0^{q_{Bi}} dq\frac{\partial P(q,h_i,\epsilon t_i)}{\partial h}+\int_{q_{Bf}}^0 dq\frac{\partial P(q,h_f,\epsilon t_f)}{\partial h} \tag{40}$$

comes from using the lowest-order expression (35) for the elapsed time. The two terms in this expression were calculated in Sec. III. The results (18), together with the just-mentioned error estimates, give

$$T_b\approx(2\omega)^{-1}\ln|h_b(\lambda_i)/h_i|+(2\omega)^{-1}\ln|h_b(\lambda_f)/h_f| \tag{41}$$

for the time required to encircle lobe b.

The error in this expression arises from the neglect of the correction (37), from the change of limits in the second integral of Eq. (39), and the errors of Eqs. (18). This second error is

$$O(\epsilon T_b)=O(\epsilon\ln[\min\{|h_i|,|h_f|\}])\ ,$$

i.e., it is the amount of movement of the endpoint of the integrand in one period, which is related to the value of h that is of lesser magnitude. The error due to the correction (37) is calculated in Appendix D and given by Eq. (D7). In combination, these results give

$$T_b=(2\omega)^{-1}\ln|h_b(\lambda_i)/h_i|+(2\omega)^{-1}\ln|h_b(\lambda_f)/h_f|+O(\epsilon\ln[\min\{|h_i|,|h_f|\}])+O(\max\{|h_i|^{1/2},|h_f|^{1/2}\})\ , \tag{42a}$$

for the time required to encircle lobe b, and

$$T_a=(2\omega)^{-1}\ln|h_a(\lambda_i)/h_i|+(2\omega)^{-1}\ln|h_a(\lambda_f)/h_f|+O(\epsilon\ln[\min\{|h_i|,|h_f|\}])+O(\max\{|h_i|^{1/2},|h_f|^{1/2}\})\ , \tag{42b}$$

for the time required to encircle lobe a.

B. Single-step energy change

According to Eqs. (5), the energy change in one step is given by

$$\Delta h=-\epsilon\int_{q_l}^{q_u} dq\frac{\partial P(q,h(q),\epsilon t(q))}{\partial\lambda}\ , \tag{43}$$

where the $h(q)$ and $t(q)$ are the actual orbits. The dominant part of this integral is

$$-\epsilon\int_0^0 dq\frac{\partial P(q,0,\epsilon t_i)}{\partial\lambda}\approx-\dot{Y}_a(\lambda_i)\ . \tag{44}$$

There are two sources of error in going from Eq. (43) to Eq. (44). The first is the neglect of the correction (38). As discussed in Appendix D and given by Eq. (D8), this error is found to be of relative size $\epsilon\ln(h)$. The second source of error is the neglect of the nonzero-h corrections to (44). This error is of relative order h. Therefore, the energy change caused by encircling lobe a is given by

$$\Delta h_a=-\dot{Y}_a+O(\epsilon^2\ln[\min\{|h_i|,|h_f|\}])+O(\epsilon\max\{|h_i|,|h_f|\})\ . \tag{45a}$$

The energy change caused by encircling lobe b is similarly found to be given by

$$\Delta h_b = -\dot{Y}_b + O(\epsilon^2 \ln[\min\{|h_i|, |h_f|\}]) + O(\epsilon \max\{|h_i|, |h_f|\}) . \quad (45b)$$

To the order of the calculations, these formulas are valid for both positive and negative energy.

C. Adiabatic-invariant change during one step

The adiabatic-invariant change due to encircling one or both loops is given by

$$J_\alpha^1(h_f,\lambda_f) - J_\alpha^1(h_i,\lambda_i) = Y_\alpha(\lambda_f) - Y_\alpha(\lambda_i) + \delta J_\alpha(h_f,\lambda_f) - \delta J_\alpha(h_i,\lambda_i) + O(\epsilon \max\{|h_i|^{1/2}, |h_f|^{1/2}\}) . \quad (46)$$

The last term in this expression comes from the subtraction of the terms containing the μ_α's. This expression can now be evaluated using the results (42) and (45). The first two terms of Eq. (46) are calculated through first order by Taylor expansion,

$$Y_\alpha(\epsilon t_f) - Y_\alpha(\epsilon t_i) = T_\alpha \dot{Y}_\alpha(\epsilon t_i)[1 + O(\epsilon T_\alpha)] .$$

In the calculation of the second two terms, the time-dependent quantities $\dot{Y}_\alpha$, h_α,ω, and g_α may all be evaluated at $\lambda = \lambda_i$ without affecting the error of these results. Hence, the change of the adiabatic invariant is given by

$$J_\alpha^1(h_f,\lambda_f) - J_\alpha^1(h_i,\lambda_i) = \Delta J_\alpha(h_i,\lambda_i)(1 + O(\epsilon^2 \ln[\min\{|h_i|, |h_f|\}])) + O(\epsilon \max\{|h_i|^{1/2}, |h_f|^{1/2}\}) , \quad (47)$$

where

$$\omega \Delta J_\alpha(h_i,\lambda_i) = [-\dot{Y}_\alpha + (h_i - \dot{Y}_\alpha/2)\ln|h_i/(h_i - \dot{Y}_\alpha)|] \quad (48a)$$

for an encirclement of lobe α ($\alpha = a$ or b), and

$$\omega \Delta J_u(h_i,\lambda_i) = [-2\dot{Y}_a - 2\dot{Y}_b + (2h_i - \dot{Y}_b)\ln|h_i/(h_i - \dot{Y}_b)| + (2h_i - 2\dot{Y}_b - \dot{Y}_a) \times \ln|(h_i - \dot{Y}_b)/(h_i - \dot{Y}_b - \dot{Y}_a)|] \quad (48b)$$

for an encirclement of both lobes starting and ending at vertex u.

V. FINAL VALUE OF THE ADIABATIC INVARIANT

The results of Sec. IV for the change in one step must be combined to obtain the total change of the adiabatic invariant. In this calculation one must account for each interval $t_n < t < t_{n+1}$ for $-\infty < n < \infty$. The choice of t_0 is made as shown in Fig. 2. The time t_0 is either the last, first, or only time that the particle is in region c at vertex u, i.e., it is the time that the particle is at the vertex closest to the separatrix crossing(s). The first part of the calculation will discuss how the value of the adiabatic invariant at time t_0 is related to its value at large time. In the second part the relation of the time t_0 to the crossing time t_x is obtained.

In this first part of the calculation, a particle with energy h_0 at time $t_0 = \lambda_0/\epsilon$ is becoming trapped in region β. At each step the energy changes by $-\dot{Y}_\beta$ according to Sec. IV. Because the particle either just left lobe a, or is about to enter lobe b, the energy satisfies the inequality $0 < h_0 < |\dot{Y}_\beta|$, in which β equals a or b, respectively. It is convenient to define the parameters

$$M_\alpha \equiv -h_0/\dot{Y}_\alpha \quad (\alpha = a, b, \text{ or } c) \quad (49)$$

and the order-unity parameters

$$R_\beta \equiv \dot{Y}_\beta/\dot{Y}_c \quad (\beta = a \text{ or } b) . \quad (50)$$

In terms of these parameters, the aforementioned inequalities are

$$0 < |M_c| < |R_\beta| , \quad (51)$$

in which β equals a or b, respectively.

The approximate expressions for the change of the adiabatic invariant are valid only if the error terms of the change (47) in a single step are small. These errors are small relative to the lowest-order result for the one or two steps during which the separatrix is crossed, if

$$\epsilon \ln|h_{-1}|, \epsilon \ln|h_0|, \epsilon \ln|h_1| \ll 1 .$$

Therefore, this analysis applies to a particle that has either just left lobe a, or is about to enter lobe b, if the crossing parameter satisfies

$$|M_c| \gg O(\exp(-1/\epsilon)/\epsilon) \quad (52a)$$

and

$$|R_\beta| - |M_c| \gg O(\exp(-1/\epsilon)/\epsilon) , \quad (52b)$$

where β equals a or b, respectively. These conditions imply that particles with crossing parameter in a small range that vanishes exponentially with ϵ must be excluded from this analysis.

In Sec. VI it is shown that the crossing parameter is uniformly distributed for an ensemble of particles that has a distribution uniform in the angle variable conjugate to the adiabatic invariant. Hence, only an insignificant fraction of particles are excluded by the condition (52). Moreover, as we will see in this section, the associated singularities are only logarithmic and so are integrable.

To find the total change in the adiabatic invariant, one would like to combine the previous calculation of the change of the adiabatic invariant with the identity

$$J_\beta^1(h_N,\lambda_N) = J_\beta^1(h_0,\lambda_0) + \sum_{n=0}^{N-1} [J_\beta^1(h_{n+1},\lambda_{n+1}) - J_\beta^1(h_n,\lambda_n)] . \quad (53)$$

(In calculating the first term of this sum, the formulas for the adiabatic invariant may have to be extended, because the particle may change regions between vertex 0 and vertex 1. This is no problem because the step-change formulas derived in Sec. IV still apply. A way to avoid this extension is given in Appendix G.) Insertion of the step-

change formulas (48) into (53) is not completely straightforward because the approximate expressions (47) for the terms of this sum have errors that increase with h. Hence, the total error will diverge if the limit $N\to\infty$ is taken immediately. To get around this complication the errors for large but finite N must be estimated.

To begin this procedure, the asymptotic value,

$$J^1_{f\beta}\equiv\lim_{N\to\infty}J^1_\beta(h_N,\lambda_N)\ , \tag{54}$$

is introduced. This limit is not a strict mathematical limit. Rather it refers to times of order $1/\epsilon$, such that the particle is far from the separatrix, and has not crossed the separatrix in the intervening interval. From adiabatic theory we know that for such times the adiabatic invariant is conserved to arbitrarily high order in ϵ. This implies that the error in carrying out the sum to large N rather than to infinite N is given by

$$J^1_{f\beta}-J^1_\beta(h_N,\lambda_N)=O(J_{2\beta}(h_N,\lambda_N))+O(J_{2f\beta})\ . \tag{55}$$

That is, the error is of the order of the next correction in the adiabatic-invariant series. The correction $J_{2\beta}(h_N,\lambda_N)$ is the larger because the terms of the adiabatic series fall off less rapidly as the separatrix is approach. This correction from Eq. (32) with

$$h_N\approx -N\epsilon(dY_\beta/d\lambda)\ , \tag{56}$$

together with Eqs. (53) and (55), indicates that

$$J^1_{f\beta}=J^1_\beta(h_0,\lambda_0)+\sum_{n=0}^{N-1}[J^1_\beta(h_{n+1},\lambda_{n+1})-J^1_\beta(h_n,\lambda_n)]+O(\epsilon/N)\ . \tag{57}$$

The next step is to use the results (47) and (48) for the individual terms in the sum of Eq. (57). The relation (56) implies that the dominant error for large n is the $O(\epsilon h^{1/2})=O(n^{1/2}\epsilon^{3/2})$ term. For small n the dominant error is $O(\epsilon^2\ln(h_{\min}))$, where $h_{\min}$ is the smallest of $|h_{-1}|$, $|h_0|$, and $|h_1|$. Thus

$$J^1_{f\beta}=J^1_\beta(h_0,\lambda_0)+\sum_{n=0}^{N-1}\Delta J^1_\beta(h_n,\lambda_n)+O(N^{3/2}\epsilon^{3/2})+O(\epsilon/N)+O(\epsilon^2\ln|h_{\min}|)\ .$$

This can be further simplified by the replacement $\lambda_n\to\lambda_0$ in the sum. This introduces an error that is $O(N^2\epsilon^2)$, which may be neglected since $N\epsilon$ must be small for the result (48) to be meaningful. Finally, the sum may be extended to $N\to\infty$, without further increase in error. This yields

$$J^1_{f\beta}=J^1_\beta(h_0,\lambda_0)+\sum_{n=0}^{\infty}\Delta J^1_\beta(h_n,\lambda_0)+O(N^{3/2}\epsilon^{3/2})+O(\epsilon/N)+O(\epsilon^2\ln|h_{\min}|)\ . \tag{58}$$

The sums

$$\Delta J^{0,\infty}_\beta(h_0,\lambda_0)\equiv\sum_{n=0}^{\infty}\Delta J^1_\beta(h_0-n\dot Y_\beta,\lambda_0) \tag{59a}$$

and

$$\Delta J^{-\infty,0}_\beta(h_0,\lambda_0)\equiv\sum_{n=-\infty}^{-1}\Delta J^1_\beta(h_0-n\dot Y_\beta,\lambda_0)\ , \tag{59b}$$

in this and later expressions are calculated in Appendix E and given by Eqs. (E3)—(E6). The error in Eq. (58) is minimized by the choice $N=\epsilon^{-1/5}$, which gives error terms that are $O(\epsilon^{6/5})$. As the sum is $O(\epsilon)$, the result

$$J^1_{f\beta}=Y_\beta(\lambda_0)+\delta J_\beta(h_0,\lambda_0)+\Delta J^{0,\infty}_\beta(h_0,\lambda_0)[1+O(\epsilon^{1/5})+O(\epsilon\ln|h_{\min}|)] \tag{60}$$

is obtained.

The time t_0 and the pseudocrossing time t_x are related to h_0 for particles initially with the same value of action. The analog of Eq. (60) for the initial adiabatic invariant for the particle at vertex α is given by

$$J^1_{i\alpha}=Y_\alpha(\lambda_0)+\delta J_\alpha(h_0,\lambda_0)-\Delta J^{-\infty,0}_\alpha(h_0,\lambda_0)[1+O(\epsilon^{1/5})+O(\epsilon\ln|h_{\min}|)]\ ,$$

Taylor expansion, $Y_\alpha(\lambda_0)=Y_\alpha(\lambda_x)+(t_0-t_x)\dot Y_\alpha(\lambda_x)+O(\epsilon^2)$, and the definition of t_x, $J_{i\alpha}=Y_\alpha(\epsilon t_x)$, imply

$$(t_0-t_x)=\{[\Delta J^{-\infty,0}_\alpha-\delta J_\alpha(h_0,\lambda_0)]/\dot Y_\alpha\}\times[1+O(\epsilon^{1/5})+O(\epsilon\ln|h_{\min}|)]\ , \tag{61}$$

Taylor expansion of Eq. (60) and use of the result (61) yield

$$J_{f\beta}=Y_\beta(\lambda_x)+\delta J_{f\beta}[1+O(\epsilon^{1/5})+O(\epsilon\ln|h_{\min}|)]\ , \tag{62}$$

where

$$\delta J_{f\beta}\equiv[\Delta J^{0,\infty}_\beta+\Delta J^{-\infty,0}_\alpha\dot Y_\beta/\dot Y_\alpha+\delta J_\beta(h_0)-\delta J_\alpha(h_0)\dot Y_\beta/\dot Y_\alpha]\ . \tag{63}$$

Specific results for the three different crossings illustrated in Fig. 2 are obtained by inserting the results of Appendix E for the sums ΔJ and the results of Sec. III for the adiabatic invariant derivations δJ into Eq. (63). For an $a\to b$ transition, the result is

$$\delta J_{fb}=G+(\dot Y_b/\omega)[\ln|\Gamma(1+M_b)\Gamma(1-M_a)/(2\pi)|-\tfrac{1}{2}\ln|M_aM_b|+M_a\ln|h_a/\dot Y_a|-M_b\ln|h_b/\dot Y_b|]\ , \tag{64a}$$

for $a\to c$,

$$\delta J_{fc}=G+(\dot Y_c/\omega)[\ln|\Gamma(-M_a)\Gamma(1+M_c)\Gamma(R_b+M_c)/(2\pi)^{3/2}|+\tfrac{1}{2}\ln|h_b/h_a|+(\tfrac{1}{2}+M_a)\ln|h_a/\dot Y_a|-(R_b+2M_c)\ln|h_c/\dot Y_c|]\ , \tag{64b}$$

and for $c \to b$,

$$\delta J_{fb} = G + (\dot{Y}_b/\omega)[\ln|\Gamma(M_b)\Gamma(1-M_c)\Gamma(R_a-M_c)/(2\pi)^{3/2}| + \tfrac{1}{2}\ln|h_a/h_b| + (\tfrac{1}{2}-M_b)\ln|h_b/\dot{Y}_b| - (R_a-2M_c)\ln|h_c/\dot{Y}_c|] . \quad (64c)$$

The parameter $G \equiv (g_b - \dot{Y}_b g_a/\dot{Y}_a)/\omega$ has been introduced in these equations. The various parameters ($\dot{Y}_\alpha$, h_α, and g_α) in Eqs. (63) and (64) are evaluated at $\lambda = \lambda_x$ without increasing the error.

The deviation (64) is a function of the crossing parameter M_c and the five parameters $\dot{Y}_a/\omega$, $\dot{Y}_b/\omega$, h_a/ω, h_b/ω, and G typical of the Hamiltonian. Since these parameters are coordinate independent (see Appendix F), they can be computed from the original Hamiltonian $H(q,p,\epsilon t)$, even if it is not of the type (3). However, application of the procedure of Appendix A to first order is necessary in order to get the value of h_0 with negligible error. (See Sec. VII for an example.) The quantities $\dot{Y}_\alpha/\omega$ give the rate of change of the lobe areas in units of the exponentiation time of the x point. The quantities h_α/ω are related to the amount of time the trajectory is far from the x point. The quantity G is an asymmetry parameter of J_1. In Appendix F it is shown that the quantity G and, hence, the results (63) and (64) are coordinate independent.

Equation (63) shows that there are three parts to the final value of the adiabatic invariant. The first term is the expected $O(1)$ piece discussed in the Introduction. The first two terms inside the braces are $O(\epsilon)$ and $O(\epsilon \ln \epsilon^{-1})$ corrections from the oscillations before and after time t_0. The last two terms inside the braces come from the change in the adiabatic-invariant function at time t_0.

Of special interest is the case of symmetric crossings. In this case both lobes grow equally ($\dot{Y}_a = \dot{Y}_b$) and are similar ($h_a = h_b$ and $g_a = g_b$). No transitions between the two lobes are possible. If $\dot{Y}_a < 0$ particles leave the lobes and end up in region c. If $\dot{Y}_a > 0$ the opposite occurs. In the latter situation the resulting final adiabatic invariant has the remarkably simple form

$$J_{fb} = Y_b(\lambda_x) - (\dot{Y}_b/\omega)\ln|2\sin(\pi h_0/\dot{Y}_b)| , \quad (65)$$

aside from the standard error terms in Eq. (62). The constants h_a and h_b do not enter the formula. For the other situation, $\dot{Y}_a < 0$, the same formula applies if the subscript b is replaced everywhere by c except in the argument of the sine. This formula was first derived by Timofeev[20] for the particular Hamiltonian (2) with $\varphi = 0$.

In the antisymmetric case $\dot{Y}_a = -\dot{Y}_b$ transitions occur only between the lobes. For the case $\dot{Y}_b > 0$, the final value of the adiabatic invariant is given by

$$J_{fb} = Y_b(\lambda_x) + g_c + (\dot{Y}_b/\omega)[\ln|M_b\Gamma^2(M_b)/(2\pi)| - 2M_b\ln|h_c/\dot{Y}_a|] , \quad (66)$$

aside from the standard error terms of Eq. (62). In this expression the definition $M_b \equiv -h_0/\dot{Y}_b$ is used. In this case the constants h_c and g_c remain in the equation.

VI. STATISTICS

The formula (64) for the final-action value depends on the crossing parameter h_0. This parameter corresponds to the initial phase of the particle. In practice the specific phase dependence is not relevant since any distribution in phase space quickly phase mixes to a distribution that is (upon coarse graining) uniform in each infinitesimally thin adiabatic-invariant annulus (see below). The relevant quantities will be the various moments of the final action. In fact, after many jumps only the first two moments are relevant because of the central-limit theorem. To calculate these moments, the distribution function for the crossing parameter is needed.

For an initial particle distribution that is uniform, the distribution can be deduced in the following way. Uniformity of the initial distribution and the fact that Hamiltonian flows conserve area guarantee that the density remains uniform in the small region shown in Fig. 4 near the x point. In this region the number of particles crossing the line $q=0$ between p and $p+dp$ is given by the flux

$$dn = n_0 q\, dp\, dt = n_0 \omega p\, dp\, dt = n_0\, dh_0\, dt .$$

Since the probability $\rho(h_0)dh_0$ is proportional to dn/dt, the probability density $\rho(h_0)$ is uniform.

An immediate consequence of this result is the probability of trapping in different lobes. In the case $\dot{Y}_a > 0$ and $\dot{Y}_b > 0$ particles leave region c and become trapped in lobe a or lobe b. They become trapped in lobe b for $0 < h_0 < \dot{Y}_b$, while they become trapped in lobe a for

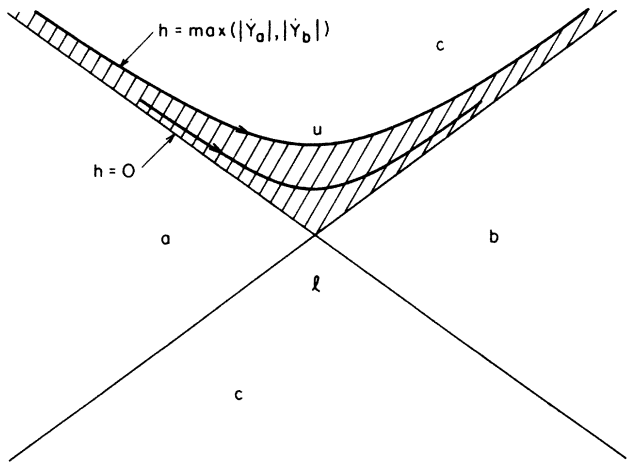

FIG. 4. Region of phase space that has left lobe a or is about to enter lobe b. For the argument of Sec. V, the distribution is nearly constant in this region.

$\dot{Y}_b < h_0 < \dot{Y}_b + \dot{Y}_a$. Since the probability is uniform in h_0, the fraction $R_a = \dot{Y}_a/\dot{Y}_c$ becomes trapped in lobe a, and the fraction $R_b = \dot{Y}_b/\dot{Y}_c$ becomes trapped in lobe b. This result was proven in Refs. 6 and 16 and was stated previously (without proof) as Kruskal's theorem in Ref. 14.

The previous derivation of the probability density can be refined to show that the density $\rho(h_0)$ is also uniform for an initial distribution of particles that is uniform within an adiabatic-invariant annulus, i.e., a region of phase space such that $J_i < J < J_i + \delta J_i$. To be specific, an annulus in region c is chosen, and it is assumed that $\dot{Y}_a > 0$ and $\dot{Y}_b > 0$. This region is illustrated in (q,p) space in Fig. 5(a). The corresponding region in the space of the adiabatic invariant J and its conjugate angle θ is shown in Fig. 5(b). The conjugate angle is chosen to vanish at the vertex ($p=0$) and to have unit period.

To calculate the probability density $\rho(h_0)$, the region in (h_0,t_0) space corresponding to the adiabatic-invariant annulus must be found. To begin, it is assumed that the particles in the adiabatic-invariant annulus are n steps away from the zero vertex at the initial time t_i. Then the region in (h_n,t_n) space is found by evolving the various parts of the annulus around to the vertex. The resulting region in (h_n,t_n) space has the form of a strip, as shown in Fig. 6(a). The left end of the strip comes from the portion of the annulus above the q axis and just to the left of the p axis. The right end of the strip comes from the portion of the annulus above the q axis and just to the right of the p axis. This portion must evolve through nearly a full period before it comes to the vertex. Therefore, the length of the strip is approximately $T_0(h)$.

We now show that the transformation to the variables (h_n,t_n) is canonical. Let γ_1 be $[t=t_i]\times B_1$, where B_1 is the boundary of any domain D_1 in (J,θ), let T be the tube of orbits going through the points of γ_1, let γ_2 be $[q=q_n]\times B_2$, where B_2 is the intersection of T with the plane $q=q_n$, and let D_2 be the domain of (h_n,t_n) bounded by B_2. The Poincaré-Cartan integral-invariant theorem[26] proves that, if γ_1 and γ_2 are two homotopic oriented paths enclosing the same tube of orbits in space (q,p,t), then $I_1 = I_2$, where $I_i = \oint (J\,d\theta - H\,dt)$ with the loop integral over γ_i. Thus, $\oint J\,d\theta$ over γ_1 equals $-\oint H\,dt$ over γ_2. It is easy to check that if γ_1 is positively oriented in (J,θ), then γ_2 is negatively oriented in (h_n,t_n) and vice versa. [This is proven by checking the orientation of the path γ of (h_i,t_i) corresponding to the intersection of T by a plane parallel to (p,t) at a point of B_1 such that $dq/dp=0$.] Therefore, the transformation of the path integrals into oriented-surface integrals shows that the oriented areas of D_1 and D_2 are equal. This proves that the transformation $(\theta,J)\rightarrow(h_n,t_n)$ is canonical.

Subsequent transformations to (h_0,t_0) distort the strip to the form shown in Fig. 6(b). The strip now asymptotes to $h_0=0$ as $t_0\rightarrow\infty$, because particles with $h_0=0$ stay at the x point for an infinite period. Nevertheless, because the transformation to (h_0,t_0) is canonical, the probability density within the strip remains uniform. Hence, the probability density $\rho(h_0)$, found by integrating over t_0, is given by

$$\rho(h_0) \propto t_0(h_0,J_i+\delta J_i) - t_0(h_0,J_i) \approx (\partial t_0/\partial J_i)\delta J_i \ . \tag{67}$$

The function $t_0(h_0,J_i)$ is essentially given by Eq. (62) because t_x and J_i are related by $J_i = Y_c(\epsilon t_x)$. With the chain rule and the fact that $Y_c(\lambda_x)$ is independent of h_0, Eq. (67) can be rewritten

$$\rho(h_0) \propto (\partial t_0/\partial t_x) \ . \tag{68}$$

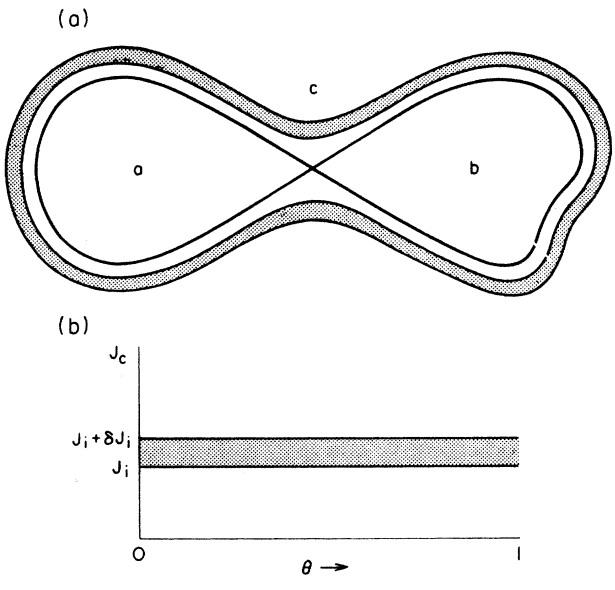

FIG. 5. Adiabatic-invariant annulus in (q,p) and (J,θ) spaces.

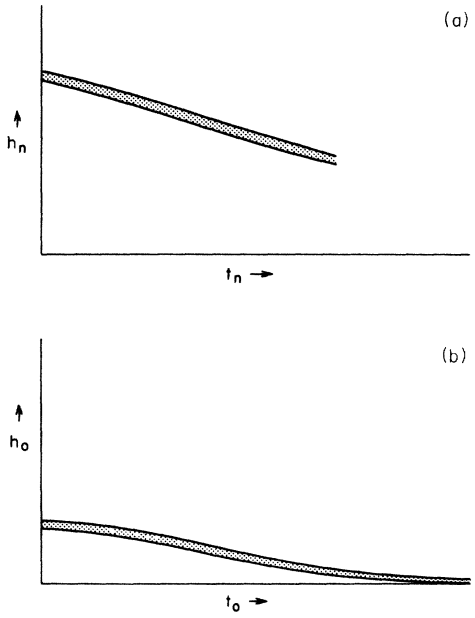

FIG. 6. Adiabatic-invariant annulus in (h,t) space.

The right-hand side of this expression is evaluated by differentiating Eq. (62). This gives

$$\rho(h_0) \propto 1+O(\epsilon \ln |h_{\min}|) . \tag{69}$$

Therefore, the distribution is uniform except for a class of particles with crossing parameter in a small range that vanishes in the limit $\epsilon \to 0$.

With the uniform distribution, the moments of the final adiabatic invariant can be easily calculated for the symmetric case (65). The mean value of the final adiabatic invariant is simply

$$\langle J_{fb} \rangle = Y_b(\lambda_x) , \tag{70}$$

while the mean-square deviation is given by

$$\langle [J_{fb} - Y_b(\lambda_x)]^2 \rangle = (\pi \dot{Y}_b/\omega)^2/12 . \tag{71}$$

These moments are not as easily calculated in other cases. Still the integrals do exist, because the singularities are only logarithmic. Furthermore, inspection shows that the rms spread in nonsymmetric cases if of order $\epsilon \ln(\epsilon)$.

VII. PARTICLE MOTION IN A SLOWLY MODULATED WAVE

A problem occurring in many fields is that of a particle moving in a wave field with slowly varying amplitude and phase velocity. The Hamiltonian for this system is

$$\mathcal{H}(x,v,t) = v^2/2 + A \cos[x-\varphi] ,$$

in which $\dot{A}$ and $\ddot{\varphi}$ are $O(\epsilon)$, and x and v are canonically conjugate variables. At first glance this Hamiltonian appears not to fit within the requirements, since it is not true that $\dot{\varphi} = O(\epsilon)$. However, a transformation to new variables q_0 and p_0 via the generating function

$$F(x,p_0,t) = [x-\varphi(t)]p_0 + \int^t dt' \dot{\varphi}^2/2 - \int^t A(t')dt' ,$$

yields the new Hamiltonian

$$H_0 = \tfrac{1}{2}(p_0 - \dot{\varphi})^2 - 2A \sin^2(q_0/2) , \tag{72}$$

which does have slowly varying parameters.

A second problem with this Hamiltonian is that its contours, shown in Fig. 7, are not of the generic form shown in Fig. 1. This is because the Hamiltonian is periodic in q, and so phase space is a torus in the q direction (see Fig. 8). There is an equivalence, as shown by the labeling in Fig. 7, but some sign confusion is possible if the actions are not chosen correctly.

The transformation of Appendix A to the variables in which the x point is stationary is not needed for the most part, since the parameters $\dot{Y}_a/\omega$, h_a/ω, and G are coordinate independent. However, the transformation does affect the Hamiltonian function, which is needed for calculating the crossing parameter h_0 given the phase-space coordinates. In this case, the choices $A_i = 0$ and $B_i = 1$ may be made for the transformation. According to Appendix A, the first-order coordinates are given by $p_1 = p_0 - \dot{\varphi}$ and $q_1 = q_0$. The first-order Hamiltonian is

$$H_1 = \tfrac{1}{2}p_1^2 - 2A \sin^2(q_1/2) + \epsilon^2 q_1 \frac{d^2\varphi}{d\lambda^2} .$$

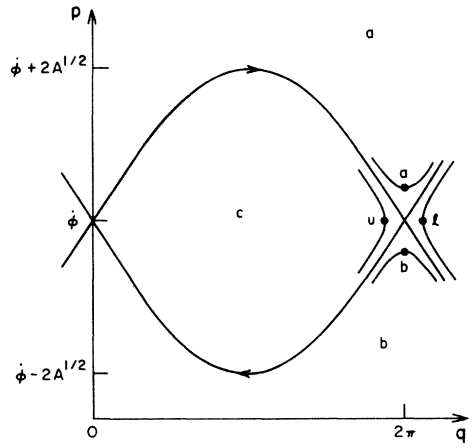

FIG. 7. Contours of the wave Hamiltonian. The regions a, b, and c and the vertices a, b, u, and l are labeled.

With the correct choices, the action in region a or region b is the integral $\int pdq$ over one period along the direction of the flow of the Hamiltonian (72). The standard result is

$$I_a(h,t) = 2\pi\varphi + I'(h,A) , \tag{73}$$

where $I'(h,A) = 8(A/m)^{1/2}E(m)$, $m = 1/(1+h/2A)$, and $E(m)$ is the complete elliptic integral of the second kind. Near the separatrix, where m is near unity, the expression (73) reduces to the standard form (20a), with

$$Y_a = 2\pi\dot{\varphi} + 8A^{1/2} , \tag{74}$$

$$\omega = A^{1/2} , \tag{75}$$

and

$$h_a(t) = 32A . \tag{76}$$

The action in region b has the form (20b), where

$$Y_b = -2\pi\dot{\varphi} + 8A^{1/2} , \tag{77}$$

and

$$h_b = h_a . \tag{78}$$

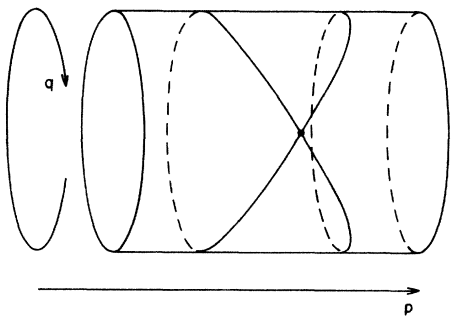

FIG. 8. Illustration of the toroidal structure of phase space for motion in a wave.

The action in region c differs from (73) for large h. Still, it is of the form (20c) with

$$Y_c = 16A^{1/2} \tag{79}$$

for small h.

The correction to the adiabatic invariant of Sec. III C is calculated in the same way. The left-right symmetry of the system causes g_a, g_b, and g_c to vanish. Thus there are no corrections to J_a and J_b. However, the correction to J_1 does not vanish in region c. At the vertex u shown in Fig. 8, the correction is given by

$$J_{1u} = -(2\pi\ddot{\varphi}/\omega\epsilon)\ln|32A/h| \ . \tag{80}$$

To summarize these results, the adiabatic-invariant formulas (28)–(31) hold but with

$$\delta J_a = \delta J_b = (h/\omega)(1+\ln|32A/h|) \tag{81}$$

and

$$\delta J_u = (2h/\omega)(1+\ln|32A/h|) - (2\pi\dot{\varphi}/\omega)\ln|32A/h| \ . \tag{82}$$

The general calculation of Secs. III and IV can now be directly applied to this case. One need only to substitute the appropriate values (73)–(79) in the final result (64). As an aside it may be noted that the errors in the result (64) are smaller for this special case because δH in Eq. (3b) is $O(q^4)$.

The symmetric case is one in which the amplitude varies while $\ddot{\varphi}=0$. For simplicity the frame where $\varphi=0$ is used. Application of the result (64) with the appropriate wave parameters yields

$$J_f = Y_b(t_x) - (4\dot{A}/A)\ln|2\sin(\pi h_0 A^{1/2}/4\dot{A})| \ . \tag{83}$$

This result agrees with Eq. (15) of Ref. 20. To prove this agreement one must note that Timofeev defines the action in region c to be half of that defined here, so that his change ΔI is simply the second term of Eq. (83). As discussed by Timofeev, this result disagrees with that of Ref. 21.

In the antisymmetric case, in which the amplitude is constant while $\dot{\varphi}$ varies, transitions only between regions a and b are possible. This case was previously studied in Ref. 19. In the previous work, the change in the mean momentum $\bar{p}$ was calculated. The mean momentum, which is defined only for particles in regions a and b, is the action integral (19) divided by 2π, except that the integral is along the $+q$ direction rather than the direction of the flow. This makes no difference in region a, but in region b, $\bar{p} = -I_b/2\pi$. Therefore,

$$\begin{aligned}\Delta\bar{p} &= -(J_{fb}+J_{ia})/2\pi \\ &= -8A^{1/2}/\pi - (h_0/\pi A^{1/2})\ln|32A/h_0| \\ &\quad -(2\ddot{\varphi}/\omega)[\tfrac{1}{2}+h_0/(2\pi\ddot{\varphi})]\ln|h_0/(2\pi\ddot{\varphi})| \\ &\quad -\ln|\Gamma(1+h_0/(2\pi\ddot{\varphi}))/(2\pi)^{1/2}|] \ .\end{aligned} \tag{84}$$

In this expression the equality $J_{ia} = Y_a(\lambda_x)$ has been used.

To compare this result with that of Ref. 19, one must note that the quantity ΔW of Eq. (10) of Ref. 19 is (up to a constant factor) $\Delta\bar{p}$, not the energy change. In addition, one must neglect certain higher-order terms in Eq. (13) of Ref. 19. Even so, the result obtained here does not agree with that of Ref. 19. We conclude that the problem lies in the evaluation of the integral in Eq. (10) of Ref. 19.

VIII. DISCUSSION

The present calculation has shown that the final value $J_{f\beta}$ of the adiabatic invariant of a particle that has crossed a separatrix differs from the lowest-order expected value $Y_\beta(\lambda_x)$ by phase-dependent terms of order ϵ and $\epsilon\ln(\epsilon)$. In contrast, the absence of separatrix crossings the adiabatic invariant is preserved to all orders in ϵ. The phase dependence of this new term leads to diffusion of the adiabatic invariant. Particles initially on a single adiabatic-invariant ring are spread over an annulus after separatrix crossing and phase mixing have taken place.

The present calculation is general. It includes the previously studied cases.[19–21] Its application requires that one evaluate a few certain time-dependent functions $(\dot{Y}_a/\omega, \dot{Y}_b/\omega, h_a/\omega, h_b/\omega, G)$ of the Hamiltonian to be analyzed.

IX. ACKNOWLEDGMENTS

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APPENDIX A: TRANSFORMATION TO THE SYSTEM WITH THE FIXED x POINT

The purpose of this appendix is to derive the coordinate transformation that fixes the x point of the given Hamiltonian. Near the x point the given Hamiltonian has the form

$$H_0(q_0,p_0,\lambda_0=\epsilon t_0)$$
$$=\alpha_0(q_0-q_{x0})^2/2+\beta_0(q_0-q_{x0})(p_0-p_{x0})$$
$$+\gamma_0(p_0-p_{x0})^2/2+\delta H_0(q_0,p_0,\lambda_0)\ , \qquad \text{(A1)}$$

in which α_0, β_0, γ_0, q_{x0}, and p_{x0} are functions of λ_0, and δH_0 is of order $(q_0-q_{x0})^3$, $(q_0-q_{x0})^2(p_0-p_{x0})$, $(q_0-q_{x0})(p_0-p_{x0})^2$, $(p_0-p_{x0})^3$, or higher. (In this appendix t_0 denotes the time variable in the original system. It should not be confused with the time the particle is at vertex 0.) The objective is to find new variables $(q,p,\lambda=\epsilon t)$ for which the Hamiltonian has the form

$$H(q,p,\lambda)=\omega(p^2-q^2)/2+\delta H(q,p,\lambda)\ , \qquad \text{(A2)}$$

in which ω is time independent, and δH is of order q^3, q^2p, qp^2, p^3, or higher. The argument is asymptotic. What we will show is that the form (A2) can be achieved to arbitrarily high order in ϵ.

By way of preparation the notation

$$\omega_0\equiv(\beta_0^2-\alpha_0\gamma_0)^{1/2} \qquad \text{(A3)}$$

for the exponentiation rate of orbits near this fixed point is introduced. This rate ω_0 is real because of the assumption that (A1) is an expansion near a (hyperbolic) x point. Moreover, the choice is made that α_0 is negative. This can always be achieved by an interchange of q_0 and p_0.

The first step is a transformation to variables (q_1,p_1) via the generating function

$$F_1(q_0,p_1,\lambda_0)\equiv A_1q_0^2/2+B_1q_0p_1+C_1q_0+D_1p_1\ , \qquad \text{(A4)}$$

in which A_1, B_1, C_1, and D_1 are functions of λ_0. From the usual transformation equations, $q_1=\partial F_1/\partial p_1$ and $p_0=\partial F_1/\partial q_0$, it follows that the relation of the old variables to the new variables is given by

$$q_0=(q_1-D_1)/B_1\ , \qquad \text{(A5a)}$$

and

$$p_0=A_1q_1/B_1+B_1p_1+C_1-A_1D_1/B_1\ . \qquad \text{(A5b)}$$

The new (intermediate) Hamiltonian is given by the usual relation

$$H_{1a}=H_0+\partial F_1/\partial t_0$$
$$=\alpha_0(q_1/B_1-D_1/B_1-q_{x0})^2/2$$
$$+\beta_0(q_1/B_1-D_1/B_1-q_{x0})(A_1q_1/B_1+B_1p_1+C_1-A_1D_1/B_1-p_{x0})$$
$$+\gamma_0(A_1q_1/B_1+B_1p_1+C_1-A_1D_1/B_1-p_{x0})^2/2$$
$$+\delta H_0(q_1/B_1-D_1/B_1,A_1q_1/B_1+B_1p_1+C_1-A_1D_1/B_1,\lambda_0)$$
$$+\dot{A}_1(q_1/B_1-D_1/B_1)^2/2+\dot{B}_1(q_1/B_1-D_1/B_1)p_1+\dot{C}_1(q_1/B_1-D_1/B_1)+\dot{D}_1p_1\ . \qquad \text{(A6)}$$

The last four terms are of order ϵ because they involve time derivatives.

The functions A_1, B_1, C_1, and D_1 are now chosen to put the new Hamiltonian in the desired form to order ϵ. To eliminate the motion of the x point, the choices

$$D_1=-q_{x0}B_1 \qquad \text{(A7a)}$$

and

$$C_1=A_1D_1/B_1+p_{x0} \qquad \text{(A7b)}$$

are made. Next, the cross term is eliminated by the choice

$$A_1=-\beta_0/\gamma_0\ . \qquad \text{(A8)}$$

Finally, the remaining coefficients of q_1^2 and p_1^2 are made equal in magnitude by the choice

$$B_1=(\beta_0^2-\alpha_0\gamma_0)^{1/4}\gamma_0^{-1/2}\ . \qquad \text{(A9)}$$

With these choices, the new Hamiltonian has the form

$$H_{1a}=\omega_0p_1^2/2-(\omega_0+A_1/B_1)q_1^2/2$$
$$+\dot{B}_1q_1p_1/B_1+\omega_0p_1^2/2$$
$$+\delta H_0(q_1/B_1+q_{x0},A_1q_1/B_1+B_1p_1+p_{x0},\lambda_0)$$
$$+(\dot{C}_1/B_1-\dot{A}_1D_1/B_1^2)q_1+\dot{E}_1p_1\ . \qquad \text{(A10)}$$

Terms with no dependence on q_1 or p_1 have been dropped from H_{1a} since they do not affect the motion.

The next step in the calculation is putting H_{1a} in the form (A2). To do so, the displaced x point (q_{x1},p_{x1}), which is a solution of

$$\frac{\partial H_{1a}}{\partial q_1}(q_{x1},p_{x1})=\frac{\partial H_{1a}}{\partial p_1}(q_{x1},p_{x1})=0\ , \qquad \text{(A11)}$$

must be found. From the form (A10) it can be seen that q_{x1} and p_{x1} are order ϵ. Expansion about this point gives

$$H_{1a}(q_1,p_1,\lambda_0)$$
$$=\alpha_{1a}(q_1-q_{x1})^2/2+\beta_{1a}(q_1-q_{x1})(p_1-p_{x1})$$
$$+\gamma_{1a}(p_1-p_{x1})^2/2+\delta H_{1a}(q_1,p_1,\lambda_0)\ , \qquad \text{(A12)}$$

in which $\alpha_{1a}=-\omega_0+O(\epsilon)$, $\beta_{1a}=O(\epsilon)$, $\gamma_{1a}=\omega_0$, and δH_{1a} is of order $(q_1-q_{x1})^3$, $(q_1-q_{x1})^2(p_1-p_{x1})$, $(q_1-q_{x1})(p_1-p_{x1})^2$, $(p_1-p_{x1})^3$, or higher.

The final step in the transformation is the definition of a new time variable to make γ_1 a constant. Straightforward differentiation shows that introduction of a new time variable $t_1(t_0)$ changes the Hamiltonian to

$$H_1=(dt_0/dt_1)H_{1a}\ .$$

Therefore the choice

$$\lambda_1 \equiv \epsilon t_1 = \int^{\lambda_0} d\lambda' \omega_0(\lambda')/\omega \tag{A13}$$

allows the final transformed Hamiltonian to be written

$$H_1(q_1,p_1,\lambda_1) = \alpha_1(q_1-q_{x1})^2/2 + \beta_1(q_1-q_{x1})(p_1-p_{x1}) + \gamma_1(p_1-p_{x1})^2/2 + \delta H_1(q_1,p_1,\lambda_1) , \tag{A14}$$

in which $\alpha_1 = -\omega + O(\epsilon)$, $\beta_1 = O(\epsilon)$, $\gamma_1 = \omega$, and δH_1 is of order $(q_1-q_{x1})^3$, $(q_1-q_{x2})^2(p_1-p_{x1})$, $(q_1-q_{x1})(p_1-p_{x1})^2$, $(p_1-p_{x1})^3$, or higher. Thus, the result of this transformation is to put the Hamiltonian in the form (A2) to order ϵ.

We could take $\omega = 1$, but we keep it explicitly in order to compute easily the five parameters of the theory. In this respect, taking $\omega = \omega_0(\lambda_x)$ may be convenient.

Higher-order calculations are exactly parallel to the lowest-order calculation just presented. With each transformation the deviation of the x point from the origin and the deviation of the quadratic coefficients from the desired values (A2) are diminished by one order in ϵ. Hence we have shown the existence of a series of transformations that put a slowly varying Hamiltonian in the form (A2) to arbitrarily high order.

Another way to get the same result is to look for the most general linear slowly varying canonical transformation that fixes the x point and the quadratic part of the Hamiltonian. This yields a system of coupled ordinary differential equations with respect to time for the coefficients of the transformation. A slowly varying solution can be found perturbatively. Then the change of time that fixes ω can be done.

APPENDIX B: FIRST-ORDER CORRECTION TO THE ADIABATIC INVARIANT

A particle moving in a slowly varying Hamiltonian has an adiabatic invariant J to all orders in the slowness parameter ϵ. In this appendix the first correction J_1 is expressed in terms of definite integrals which can later be evaluated for particular Hamiltonians.

This calculation is performed using (q,h,λ) as independent variables. To avoid confusion in the meaning of partial derivatives, the usual phase function $J_n(q,p,\lambda)$ and the corresponding function of (q,h,λ) are expressed as distinct functions

$$\mathscr{J}_n(q,h,\lambda) = J_n(q,P(q,h,\lambda),\lambda) . \tag{B1}$$

These functions have the same value but different independent variables. With this convention and the choice that J_0 is the action, it is apparent that $\mathscr{J}_0(h,\lambda) = I(h,\lambda)$ of Eq. (19).

The requirement that J be constant to all orders amounts to saying that each term in the series

$$\epsilon[\partial J_0/\partial\lambda + \{J_1,H\}] + \epsilon^2[\partial J_1/\partial\lambda + \{J_2,H\}] + \cdots \tag{B2}$$

vanishes. This implies

$$\{J_n,H\} = -\partial J_{n-1}/\partial\lambda . \tag{B3}$$

The relation $J_n(q,p,\lambda) = \mathscr{J}_n(q,H(q,p,\lambda),\lambda)$ and Eq. (B3) imply

$$\frac{\partial \mathscr{J}_n(q,h,\lambda)}{\partial q}\frac{\partial H(q,P(q,h,\lambda),\lambda)}{\partial p} = -\frac{\partial \mathscr{J}_{n-1}(q,h,\lambda)}{\partial h}\frac{\partial H(q,P(q,h,\lambda),\lambda)}{\partial \lambda} - \frac{\partial \mathscr{J}_{n-1}(q,h,\lambda)}{\partial \lambda} . \tag{B4}$$

The notation for the partial derivatives, such as

$$\frac{\partial H(q,P(q,h,\lambda),\lambda)}{\partial \lambda}$$

in Eq. (B4), is that the function H is differentiated with respect to λ, holding q and p fixed. Then the independent variable p is replaced by the function $P(q,h,\lambda)$.

The various relations

$$\frac{\partial H(q,P(q,h,\lambda),\lambda)}{\partial p}\frac{\partial P(q,h,\lambda)}{\partial h} = 1 , \tag{B5}$$

$$\frac{\partial H(q,P(q,h,\lambda),\lambda)}{\partial p}\frac{\partial P(q,h,\lambda)}{\partial q} + \frac{\partial H(q,P(q,h,\lambda),\lambda)}{\partial q} = 0 , \tag{B6}$$

and

$$\frac{\partial H(q,P(q,h,\lambda),\lambda)}{\partial p}\frac{\partial P(q,h,\lambda)}{\partial \lambda} + \frac{\partial H(q,P(q,h,\lambda),\lambda)}{\partial \lambda} = 0 , \tag{B7}$$

obtained by differentiating Eq. (4) can be used to simplify Eq. (B4). The result is

$$\frac{\partial \mathscr{J}_n}{\partial q} = \frac{\partial P}{\partial \lambda}\frac{\partial \mathscr{J}_{n-1}}{\partial h} - \frac{\partial P}{\partial h}\frac{\partial \mathscr{J}_{n-1}}{\partial \lambda} . \tag{B8}$$

Equation (B8) can be integrated to obtain the first correction

$$\mathscr{J}_1(q,h,\lambda) = \mathscr{J}_1(q_0,h,\lambda) + \frac{\partial \mathscr{J}_0}{\partial h}\int_{q_0}^{q} dq' \frac{\partial P(q',h,\lambda)}{\partial \lambda} - \frac{\partial \mathscr{J}_0}{\partial \lambda}\int_{q_0}^{q} dq' \frac{\partial P(q',h,\lambda)}{\partial h} \tag{B9}$$

up to the undetermined function $\mathscr{J}_1(q_0,h,\lambda)$. The limit q_0 is an arbitrary point on the particular contour $H(q,p,\lambda) = h$. The integrals over q' must always be along the flow direction from q_0 to q. This specification is necessary since the separate integrals of (B9) are not periodic, even though the sum is periodic. This periodicity follows from the expression (19) for $\mathscr{J}_0(h,\lambda) = I(h,\lambda)$.

Alternatively, one could require single valuedness for $\mathscr{J}_1$. Then Eq. (A9) with $q = q_0$ implies

$$\frac{\partial \mathscr{J}_0}{\partial h}\frac{\partial I}{\partial \lambda} - \frac{\partial \mathscr{J}_0}{\partial \lambda}\frac{\partial I}{\partial h} = 0 ,$$

which allows one to deduce that $\mathscr{J}_0$ must be a function of I. This technique will now be used in the next order to calculate $\mathscr{J}_1(q,h,\lambda)$.

Setting $n=2$ in Eq. (B8) and integrating around a contour of constant H yields the relation

$$0=\oint dq\frac{\partial P(q,h,\lambda)}{\partial\lambda}\frac{\partial \mathcal{J}_1(q,h,\lambda)}{\partial h}-\oint dq\frac{\partial P(q,h,\lambda)}{\partial h}\frac{\partial \mathcal{J}_1(q,h,\lambda)}{\partial\lambda}\ .$$

After considerable algebra this expression reduces to

$$0=\frac{\partial \mathcal{J}_0}{\partial\lambda}\frac{\partial g}{\partial h}-\frac{\partial \mathcal{J}_0}{\partial h}\frac{\partial g}{\partial\lambda}\ , \tag{B10}$$

where

$$g(h,\lambda)\equiv \mathcal{J}_1(q_0,h,\lambda)-\frac{1}{2}\frac{\partial \mathcal{J}_0}{\partial h}\frac{\partial \mathcal{J}_0}{\partial\lambda}+\oint dq\frac{\partial P(q,h,\lambda)}{\partial h}\int_{q_0}^{q}dq'\frac{\partial P(q',h,\lambda)}{\partial\lambda}\ .$$

Equation (B10) indicates that g must be a function of $\mathcal{J}_0$ alone. Thus

$$\mathcal{J}_1(q_0,h,\lambda)=\frac{1}{2}\frac{\partial \mathcal{J}_0}{\partial h}\frac{\partial \mathcal{J}_0}{\partial\lambda}-\oint dq\frac{\partial P(q,h,\lambda)}{\partial h}\int_{q_0}^{q}dq'\frac{\partial P(q',h,\lambda)}{\partial\lambda}+g(\mathcal{J}_0(h,\lambda)) \tag{B11}$$

is determined up to a function of $\mathcal{J}_0$. Since $\mathcal{J}_0$ is an adiabatic invariant, any function of $\mathcal{J}_0$ is an adiabatic invariant.

The function g can be determined by the requirement that the value of J on a contour of J equal the area inside that contour through first order. This requirement is equivalent to the requirement that the transformation between the action-angle variables and the corrected adiabatic-invariant-angle variables be canonical. This requirement fixes the function in a nonarbitrary, coordinate-independent manner. To determine the consequences of this requirement, the equation

$$j=J_0(q,p,\lambda)+\epsilon J_1(q,p,\lambda) \tag{B12}$$

for the contour with $J=j$ is considered. The lowest-order solution is

$$\pi_0(q,j,\lambda)=P(q,h=\mathscr{E}_0(j,\lambda),\lambda)\ ,$$

where $\mathscr{E}_0(j,\lambda)$ is the inversion of J_0, i.e., $\mathscr{E}_0(j,\lambda)$ satisfies $\mathcal{J}_0(\mathscr{E}_0(j,\lambda),\lambda)=j$. The first-order correction is found by expansion,

$$0=\frac{\partial \mathcal{J}_0(q,h,\lambda)}{\partial h}\frac{\partial H}{\partial p}\pi_1(q,j,\lambda)+J_1\ .$$

Therefore,

$$\pi_1(q,j,\lambda)=-\frac{\partial\mathscr{E}_0}{\partial j}\frac{\partial P(q,\mathscr{E}_0(j,\lambda),\lambda)}{\partial h}\mathcal{J}_1(q,\mathscr{E}_0(j,\lambda),\lambda)\ .$$

Thus the change in the area ΔA_1 due to first-order J_1 is given by

$$\Delta A_1(q,j,\lambda)=-\frac{\partial\mathscr{E}_0}{\partial j}\oint dq\frac{\partial P(q,\mathscr{E}_0(j,\lambda),\lambda)}{\partial h}\times\mathcal{J}_1(q,\mathscr{E}_0(j,\lambda),\lambda)\ . \tag{B13}$$

With the previous formulas (B9) and (B11) the evaluation of ΔA_1 is straightforward. The result is

$$\Delta A_1=-g(j)\ .$$

Thus imposition of the requirement that the value of J on a contour of J equal the enclosed area leads to $g(j)=0$. With the further choice $q=q_0$, the result

$$\mathcal{J}_1(q,h,\lambda)=\frac{1}{2}\frac{\partial \mathcal{J}_0}{\partial h}\frac{\partial \mathcal{J}_0}{\partial\lambda}-\oint dq'\frac{\partial P(q',h,\lambda)}{\partial h}\int_{q}^{q'}dq''\frac{\partial P(q'',h,\lambda)}{\partial\lambda}\ . \tag{B14}$$

for the first-order correction to the adiabatic invariant is obtained. From this expression we see that $\epsilon\mathcal{J}_1/\mathcal{J}_0$ is of order δ as stated in the Introduction.

APPENDIX C: ERROR OF THE FIRST CORRECTION TO THE ADIABATIC INVARIANT

As noted in Sec. III C, the first correction to the adiabatic invariant is finite at vertices a and b in the limit $h\to 0$. To assess the error of the total adiabatic-invariant change of Sec. V, it is necessary to know how $J_{1\alpha}$ deviates from g_α ($\alpha=a$ or b) for finite h. The order of $J_{1\alpha}-g_\alpha$ is calculated in this Appendix.

The following calculation of the order of J_{1b+}, defined by Eq. (23a), is actually for an exterior ($h>0$) orbit. It is straightforward, though tedious, to show that the same result is obtained for an interior orbit. However, one must instead use the alternate formalism of Eqs. (10).

To calculate J_{1b+}, a Taylor expansion of Eq. (23a) is used,

$$J_{1b+}(h)\approx J_{1b+}(0)-h\left[\int_0^{q_B}dq\frac{\partial^2P(q,0,\lambda)}{\partial h^2}\int_0^q dq'\frac{\partial P(q',0,\lambda)}{\partial\lambda}+\int_0^{q_B}dq\frac{\partial P(q,0,\lambda)}{\partial h}\times\int_0^q dq'\frac{\partial^2P(q',0,\lambda)}{\partial\lambda\partial h}\right]\ . \tag{C1}$$

Estimates of the integrands follow from Eqs. (8) and (9). We find $\partial P/\partial\lambda=O(P_0^2)$, $\partial P/\partial h=O(1/P_0)$, $\partial^2P/\partial h^2=O(1/P_0^3)$, and $\partial^2P/\partial h\partial\lambda=O(1)$. Therefore,

$$J_{1b+}(h)=J_{1b+}(0)+O(h)\ . \tag{C2}$$

An identical analysis applies to J_{1b-}. Hence

$$J_{1b}=g_b+O(h)\ . \tag{C3}$$

APPENDIX D: ERROR OF THE SINGLE-STEP PERIOD AND ENERGY CHANGE

Errors in the lowest-order, single-step period and energy change arise from the corrections (37) and (38) to the orbit. In this appendix, the order of these errors is calculated.

The correction to the period for half of a step is given by $\epsilon\Delta t_{j1}(q_{Bj})$, where Δt_{j1} is given by Eq. (37). There are two terms in Eq. (37) to be analyzed. Accordingly, the division

$$\Delta t_{j1}(q_{Bj})=t'_{j1}(q_{Bj})+t''_{j1}(q_{Bj}) \tag{D1}$$

is defined, where

$$t'_{j1}(q_{Bj})\equiv\int_0^{q_{Bj}}dq\,\Delta t_{j0}(q)\frac{\partial^2 P(q,h_j,\epsilon t_j)}{\partial h\,\partial\lambda} \tag{D2}$$

and

$$t''_{j1}(q_{Bj})\equiv\int_0^{q_{Bj}}dq\,\Delta h_{j0}(q)\frac{\partial^2 P(q,h_j,\epsilon t_j)}{\partial h^2}\ . \tag{D3}$$

The expression (D2) can be bounded by replacing Δt_0 with T_{0bu} as defined in Eq. (11) and moving the partial derivative with respect to λ outside the integral. This gives

$$t'_1(q_{Bj})=O(T_{0bu}\partial T_{0bu}/\partial\lambda)=O(\ln|h_j|)\ . \tag{D4}$$

The remaining term has the form

$$t''_{j1}(q_{Bj})\equiv-\int_0^{q_{Bj}}dq\frac{\partial^2 P(q,h_j,\lambda_j)}{\partial h^2}\times\int_0^q dq'\frac{\partial P(q',h_j,\lambda_j)}{\partial\lambda}\ . \tag{D5}$$

The estimation of this integral is identical to the estimation of the first integral in Eq. (C1). Hence

$$t''_{j1}=O(1)\ . \tag{D6}$$

In combination, Eqs. (D1), (D4), and (D10) imply an error

$$\epsilon\Delta t_{j1}(q_{Bj})=O(\epsilon\ln|h_j|) \tag{D7}$$

in the period.

The analysis of the error due to the correction (38) proceeds in a similar fashion. The result for the error is

$$\epsilon^2\Delta h_{j1}(q_{Bj})=O(\epsilon^2\ln|h_j|)\ . \tag{D8}$$

Hence the relative error is given by

$$\epsilon\Delta h_{j1}(q_{Bj})/\Delta h_{j0}(q_{Bj})=O(\epsilon\ln|h_j|)\ . \tag{D9}$$

From this equation it follows that the relative error is of order unity if the period of the particle, $O(\ln|h_j|)$, is of order $1/\epsilon$, i.e., the system changes significantly in that period.

APPENDIX E: TOTAL ADIABATIC INVARIANT CHANGE FOR A HALF-INFINITE SEQUENCE OF STEPS

In Sec. V the total change in the adiabatic invariant is written in terms of the sums (59). In this appendix these sums are calculated. First the sum for an orbit encircling both loops is calculated. Specialization then yields the formulas for orbits encircling a single loop.

The sums for region c at vertex u can be written in terms of the order-unity quantity,

$$\Delta_n\equiv-\omega\Delta J_u(h_0-n\dot Y_c,\lambda_0)/\dot Y_c\ , \tag{E1}$$

which takes the form

$$\Delta_n=2+2(M_c+n+R_b/2)\times\ln|(M_c+n)/(M_c+n+R_b)|+2(M_c+n+\tfrac{1}{2}+R_b/2)\times\ln|(M_c+n+R_b)/(M_c+n+1)|\ , \tag{E2}$$

in terms of the quantities M_c and R_b defined by Eqs. (49) and (50).

The total change is obtained by summing the individual terms,

$$\Delta J_u^{0,\infty}=-\frac{\dot Y_c}{\omega}\lim_{N\to\infty}\sum_{n=0}^{N}\Delta_n\ .$$

This sum can be calculated with standard techniques including (i) identifying telescoping terms, (ii) writing the remaining sums in terms of the Γ function, and (iii) applying Stirling's formula for large N. The result is

$$\Delta J_u^{0,\infty}=-(\dot Y_c/\omega)\times[(R_b+2M_c)\ln|M_c|-2M_c-\ln|\Gamma(1+M_c)\Gamma(R_b+M_c)/(2\pi)|]\ . \tag{E3}$$

Time-reversal arguments yield the result

$$\Delta J_u^{-\infty,0}=-(\dot Y_c/\omega)\times[(R_a-2M_c)\ln|M_c|+2M_c-\ln|\Gamma(1-M_c)\Gamma(R_a-M_c)/(2\pi)|]\ . \tag{E4}$$

for the particle about to be trapped in loop b as shown in Fig. 2(c).

The single-loop sums are readily obtained from these expressions by specialization. The result

$$\Delta J_b^{0,\infty}=-(\dot Y_b/\omega)[-M_b+(\tfrac{1}{2}+M_b)\ln|M_b|-\ln|\Gamma(1+M_b)/(2\pi)^{1/2}|] \tag{E5}$$

for a particle entering loop b as shown in Figs. 2(a) and 2(c), is obtained from Eq. (E3) by setting $\dot Y_a=0$ and using one-half of the total. These are the symmetries that produce the single-loop change (48a) from the double-loop change (48b). Similarly, one can obtain from Eq. (E4) the result

$$\Delta J_a^{-\infty,0}=-(\dot Y_a/\omega)[M_a+(\tfrac{1}{2}-M_a)\ln|M_a|-\ln|\Gamma(1-M_a)/(2\pi)^{1/2}|] \tag{E6}$$

for a particle leaving loop a as shown in Figs. 2(a) and 2(b).

APPENDIX F: COORDINATE DEPENDENCE OF THE ADIABATIC INVARIANT CHANGE

In the analysis of the adiabatic-invariant change, a coordinate transformation $q(q_0,p_0,\lambda)$ and $p(q_0,p_0,\lambda)$ was

introduced to put the Hamiltonian in the form (3). This transformation is not unique. The generating function for the general linear transformation, cf. (A4), would also have a term with p_1^2. Moreover, the exponentiation rate ω, chosen to be constant in Appendix A, is arbitrary. The purpose of this appendix is to show that, in fact, the result (64) does not depend on the particular coordinate system.

The adiabatic-invariant change (64) can be written in terms of the five quantities $\dot{Y}_a/\omega$, $\dot{Y}_b/\omega$, h_a/ω, h_b/ω, and $G \equiv g_b - \dot{Y}_b g_a/\dot{Y}_a = g_c - \dot{Y}_c g_a/\dot{Y}_a = g_b - \dot{Y}_b g_c/\dot{Y}_a$, and the crossing parameter $h_0/\dot{Y}_c$. The transformation (A13) of the time variable does not affect any of these quantities because g_α, H/ω, and $\omega^{-1}\partial/\partial t$ are invariant with respect to the choice of ω. It remains, therefore, to check whether these quantities depend on the canonical transformation.

The first four quantities are independent of the particular canonical transformation because they can be defined in terms of the action functions $I_\alpha(h,\lambda)$, which are invariant under canonical transformation. Specifically,

$$\dot{Y}_a(\epsilon t_x)/\omega \equiv \omega^{-1}\frac{\partial I_\alpha(0,\epsilon t_x)}{\partial t} ,$$

and

$$h_\alpha/\omega \equiv \lim_{h\to 0}[h \exp(\omega \partial I_\alpha/\partial h)/\omega] .$$

The parameter g_α is defined to be the value of the correction $J_{1\alpha}$ on the line $p=0$ in the limit $h\to 0$. Hence the value of g_α does depend on the particular choice of coordinate system. However, g_α enters the result (64) only via G. Therefore, the appearance of the g_α's does not imply coordinate dependence if G can be shown to be coordinate independent.

To investigate the dependence of g_b on the coordinate system, a canonical transformation to new coordinates will be introduced. This transformation will be required to preserve the form (3). It will then be shown that the value of G is independent of the choice of the coordinate system.

It is sufficient to consider only linear transformations, since g_b is the value of J_1 at the x point, and therefore, is unaffected by higher-order terms. A general, linear, canonical transformation is affected by the generating function

$$F = Aq^2/2 + BqP + CP^2/2 . \tag{F1}$$

The transformation due to this generating function is given by

$$q = Q/B - CP/B \tag{F2a}$$

and

$$p = AQ/B + (B - AC/B)P . \tag{F2b}$$

The new Hamiltonian, $K = H + \partial F/\partial t$, has the form

$$K = \{\omega[(B-AC/B)^2 - C^2/B^2] + \dot{A}C^2/B^2 + \dot{C} - 2\dot{B}C/B\}P^2/2 + \{\omega[(B-AC/B)A/B + C/B^2] - \dot{A}C/B^2 + \dot{B}/B\}PQ - [\omega(1-A^2)/B^2 - \dot{A}/B^2]Q^2/2 . \tag{F3}$$

Preservation of the form (3) leads to the requirements

$$\omega[(B-AC/B)^2 - C^2/B^2 + (A^2-1)/B^2] = 2\dot{B}C/B - \dot{C} - \dot{A}C^2/B^2 - \dot{A}/B^2 \tag{F4a}$$

and

$$\omega[(B-AC/B)A/B + C/B^2] = \dot{A}C/B^2 - \dot{B}/B . \tag{F4b}$$

To lowest order in ϵ, these equations imply that

$$B^2 = 1 - A^2 \tag{F5a}$$

and

$$C = -A . \tag{F5b}$$

Hence there is a one-parameter family of coordinate transformations that preserve the form (3) for the Hamiltonian. We must show that G is the same regardless of which member of this set of transformations is chosen.

The vertices in the new coordinate system are the intersection of the orbits with the lines $Q=0$ and $P=0$ (see Fig. 9). We define q_b' to be the value of q at which the orbit intersects the $P=0$ line. Since J_1 depends on h and q, $J_{1b}(q_b) \neq J_{1b}(q_b')$. The difference follows readily from the application of Eq. (B.14),

$$J_{1b}(q_b',h,\lambda) - J_{1b}(q_b,h,\lambda) = \oint dq \frac{\partial P(q,h,\lambda)}{\partial h}\left[\int_{q_b}^{q} dq' \frac{\partial P(q',h,\lambda)}{\partial \lambda} - \int_{q_b'}^{q} dq' \frac{\partial P(q',h,\lambda)}{\partial \lambda}\right] .$$

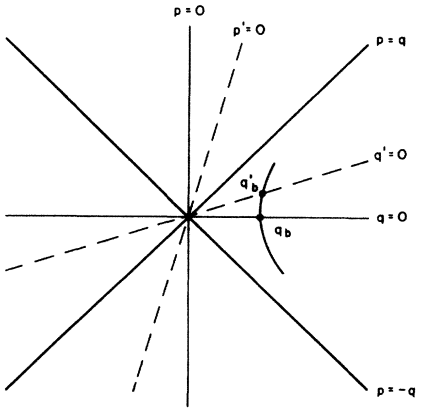

FIG. 9. The lines $q'=0$ and $p'=0$ of the alternate coordinate system.

This expression is simplified by subtracting the quantity in the parentheses and using the identity

$$\int_{q_b'}^{q_b} dq' \frac{\partial P}{\partial \lambda} = \frac{dY_b}{d\lambda} - \int_{q_b}^{q_b'} dq' \frac{\partial P}{\partial \lambda} .$$

The simplified expression is

$$[\mathscr{J}_{1b}(q_b',h,\lambda) - \mathscr{J}_{1b}(q_b,h,\lambda)]/(dY_b/d\lambda)$$
$$= -\int_{q_b}^{q_b'} dq \frac{\partial P}{\partial h} + T_b (dY_b/d\lambda)^{-1} \int_{q_b}^{q_b'} dq \frac{\partial P}{\partial \lambda} . \quad \text{(F6)}$$

The relation between the new value of

$$g_b' \equiv \lim_{h\to 0} \mathscr{J}_{1b}(q_b',h,\lambda)$$

and g_b is found by evaluating (F6) in the limit $h \to 0$. In this limit

$$q_b = |2h/\omega|^{1/2} ,$$

and

$$q_b' = |2h/\omega|^{1/2}/B .$$

The latter expression follows from the definition of q_b' and Eqs. (F2). For the integrands of Eq. (F6), the lowest-order terms of Eqs. (8) and (9) may be used. The result of taking the limit $h \to 0$ of Eq. (F6) is

$$(g_b' - g_b)/(dY_b/d\lambda) = -2\ln|B|/\omega . \quad \text{(F7)}$$

The analysis of lobe-a is identical. Hence,

$$(g_a' - g_a)/(dY_a/d\lambda) = -2\ln|B|/\omega . \quad \text{(F8)}$$

In combination, Eqs. (F7) and (F8) imply that

$$g_b - \dot{Y}_b g_a/\dot{Y}_a = g_b' - \dot{Y}_b g_a'/\dot{Y}_a .$$

Therefore, the quantity G is coordinate independent.

The remaining quantity to consider is the crossing parameter $h_0/\dot{Y}_c$. It is necessarily coordinate dependent to some extent, because h is continually changing. However, from the lowest-order expressions of the perturbation theory, (33)—(36), it can be shown that the change $\epsilon \Delta h_0(q_b')$ in h_0 from using the (q',p') coordinate system is $O(\epsilon h_0^{3/2}) = O(\epsilon^{5/2})$ provided q_b' is of order unity. This gives a relative change in the crossing parameter of order $\epsilon^{3/2}$, which is insignificant.

APPENDIX G: ALTERNATIVE COMPUTATION OF $J_{f\beta}$

The purpose of this appendix is to show how one can avoid the extension of the adiabatic invariants referenced after Eq. (53) and to derive slightly differently Eqs. (63) and (64).

Let $n(j)$ $(j=\alpha,\beta)$ be the vertex in region j closest to the separatrix crossing, and let $\mathscr{J}_\alpha \equiv J_{i\alpha}$ and $\mathscr{J}_\beta \equiv J_{f\beta}$. Then

$$\mathscr{J}_j = J_j(h_{n(j)},\lambda_{n(j)}) + \eta_j \Delta J^j \quad \text{(G1)}$$

for $j=\alpha,\beta$, where

$$\Delta J^\alpha \equiv \Delta J_\alpha^{-\infty,n(\alpha)} ,$$

$$\Delta J^\beta \equiv \Delta J_\beta^{n(\beta),\infty} ,$$

$$\eta_\alpha \equiv -1 ,$$

and

$$\eta_\beta \equiv 1 .$$

Taylor expansion gives

$$\mathscr{J}_j = Y_j(\lambda_x) + (t_{n(j)} - t_x)\dot{Y}_j + \delta J_j(h_{n(j)},t_x) + \eta_j \Delta J^j . \quad \text{(G2)}$$

This equation for $j=\alpha$ yields

$$t_{n(\alpha)} - t_x = [\Delta J^\alpha - \delta J_\alpha(h_{n(\alpha)},t_x)]/\dot{Y}_\alpha . \quad \text{(G3)}$$

In order to compute $J_{f\beta}$ we need to know $t_{n(\beta)} - t_x = (t_{n(\alpha)} - t_x) + (t_{n(\beta)} - t_{n(\alpha)})$. The first term is given by Eq. (G3), while the second term is computed from Eq. (42). One finally gets Eqs. (63) and (64) with the same $\Delta J_\beta^{0,\infty}$ and $\Delta J_\alpha^{-\infty,0}$. One at least of these two quantities appears as made up of two pieces: ΔJ^j and the contribution from the step linking vertex $n(j)$ to vertex 0.

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[25]The divergence of $\partial P/\partial h$ on a trajectory appears to lead to difficulties in the evaluation of integrals such as occur in Eqs. (11) and (37), for example. These difficulties can be removed either by defining for each lobe a new momentum variable whose contours are nowhere parallel to those of H, or by patching the two formalisms (5) and (10) to cover all of phase space without singularities. For clarity and brevity, this was not done. In any case, it is easy to show that the apparently divergent integrals converge and give the right answer if the principal value is used.

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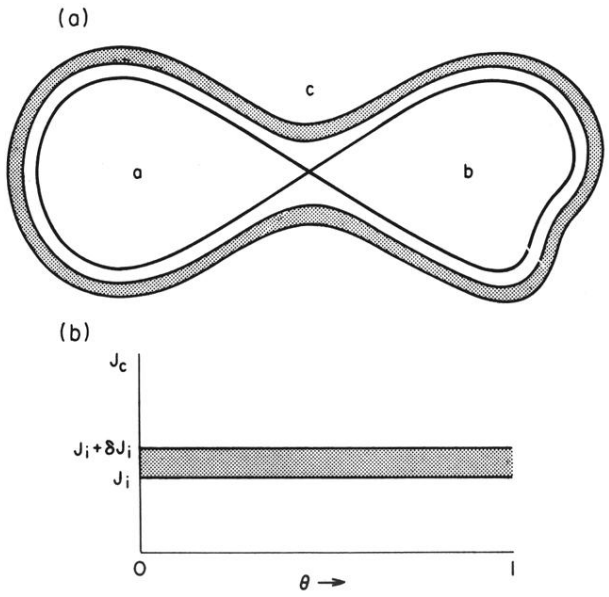

FIG. 5. Adiabatic-invariant annulus in (q,p) and (J,θ) spaces.

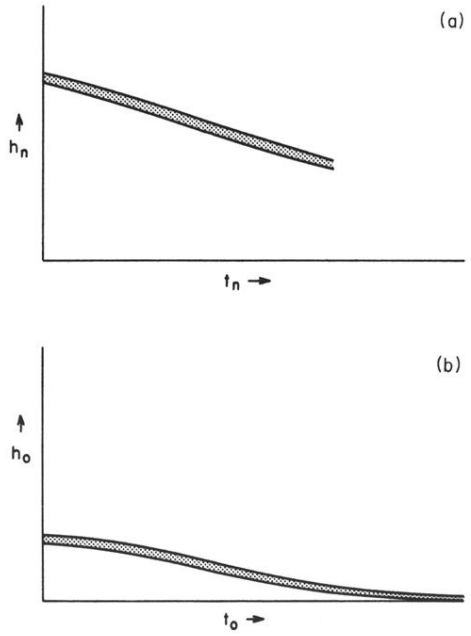

FIG. 6. Adiabatic-invariant annulus in (h,t) space.