Fluctuations, viscous fingering, and diffusion-limited aggregation

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A physical interpretation of the diffusion-limited-aggregation (DLA) algorithm for simulating viscous fingering during fluid displacement in a porous medium is given in terms of the material properties of the porous medium. In one dimension, the flow patterns generated by a DLA simulation correspond to a simple exponential distribution of "fluid capacity." Also, for any given distribution of fluid capacity, it is possible to establish a corresponding stochastic algorithm for the moving fluid interface. The discussion of this paper places the DLA simulation of viscous fingering on a physical basis and, more generally, establishes the mathematical equivalence of two models of stochastic interface evolution.

Paterson[1] has shown that the diffusion-limited-aggregation (DLA) process introduced by Witten and Sander[2] can be used to model fingering patterns and sweep efficiencies of two-fluid displacement in a porous medium. The original DLA simulation is only applicable to the case of an inviscid driving fluid displacing a Newtonian fluid (i.e., at the limit of infinite mobility ratio) and when interfacial and wetting effects are negligible. While there have been recent attempts to modify the DLA algorithm to include some of these effects, the precise connection between DLA and the fluid flow problem has yet to be established.

When an inviscid fluid is displacing another fluid of finite viscosity, the fluid interface is unstable to *any* perturbations, however small in magnitude.[3] Even in a macroscopically uniform and statistically homogeneous porous material there are fluctuations in the hydrodynamic conductivities and void structure over the pore scale, and over larger scales, which can initiate finger growth; these fluctuations do not appear in the macroscopic equation governing the flow (Darcy's law). In contrast, in the DLA algorithm, the stochastic nature of the "random walkers" provides the source of fluctuations in the growth of fingers. The aim of this paper is to establish a link between the probabilistic nature of DLA and the fluctuations of microstructure in a porous material. Such a connection will hopefully allow one to devise simple probabilistic models to study more complex problems of fluid flow in porous media.

Consider the displacement of a Newtonian fluid of finite viscosity by an inviscid fluid. We limit the discussion to a displacement process in one dimension, with the viscous fluid withdrawn from the origin and the interfaces between the fluids located at points $x=-l(t)<0$ and $x=r(t)>0$. A constant pressure drop P is maintained between each fluid interface and the origin. Since the pressure satisfies Laplace's equation, the pressure gradients on the left and right of the origin have magnitudes P/l and P/r, respectively. The magnitudes of the volume fluxes v_l and v_r on the left and right of the origin are given by Darcy's law:

$$v_l=(k/\mu)(P/l),\quad v_r=(k/\mu)(P/r)\ , \tag{1}$$

where k is the permeability and μ is the viscosity of the displaced fluid. The instantaneous velocities of the interfaces are given by

$$dl/dt=-v_l/\varphi,\quad dr/dt=-v_r/\varphi\ , \tag{2}$$

where φ is the porosity. In an *ideal* porous medium where there are no fluctuations in material properties, k and φ are constants and the motion of the interfaces can be found by solving Eqs. (1) and (2) for $l(t)$ and $r(t)$. One finds that $l^2(t)-l^2(0)=r^2(t)-r^2(0)=-(2kP/\mu\varphi)t$. In a *statistically homogeneous* rather than *ideal* porous medium the interface will not move in this deterministic manner, as there is a stochastic element in the displacement process. We introduce this stochastic element by *hypothesis A:*

In the next time step Δt, the interface will advance one step Δx with a statistical weight that is proportional to the magnitude of the velocity.

This hypothesis has been proposed by several authors[4] as a growth law or simulation algorithm to produce random structures. We now examine its consequences and its relationship with the statistical properties of the porous medium.

Consider the relative evolution of the left and right interfaces for the one-dimensional displacement described above. According to hypothesis A, the probability of the r interface moving in the next time step is

$$\text{Prob}\{r \text{ interfaces moves}\}=(dr/dt)/[(dr/dt)+(dl/dt)] \tag{3}$$

with a similar equation for the probability of advance of the l interface. Applying Eqs. (1) and (2) to each interface, we find that

$$\begin{aligned}&\text{Prob}\{r \text{ interface moves}\}=l/(r+l)\ ,\\&\text{Prob}\{l \text{ interface moves}\}=r/(r+l)\ .\end{aligned}\tag{4}$$

These results lead to a probabilistic algorithm for the evolution of the two interfaces: select a random number in [0,1] and then move one or the other interface according to Eqs. (4). In a lattice simulation, the interface chosen is advanced by one grid spacing.

The probabilities given in Eqs. (4) are closely related to the DLA algorithm. If we release an unbiased random walker at the origin, the probability of the walker reaching the r interface (l interface) *before* reaching the l interface (r interface) is given precisely by the first (second) of Eqs. (4).[5] In other words, for a one-dimensional lattice, the DLA algorithm is *mathematically identical* to our hypothesis A. That hypothesis A and DLA are equivalent for a lattice of arbitrary dimensionality is well known.[2] The advantage of hypothesis A is that we have a method for converting a flow equation such as Eq. (1) or its generalization (to include finite mobility, interfacial, or non-Newtonian effects) to a simple stochastic algorithm.[6] The difficulty in two and three dimensions is to derive the analogue of Eq. (1) needed to relate the interfacial velocity to the position of the interface.

For our one-dimensional model we can formally express the probabilistic advance of the two interfaces as follows. Let $P_n(l,r)$ be the probability that the two interfaces be at position l and r at the nth time step. From Eqs. (4) the recurrence equation for P_n is

$$\begin{aligned}P_n(l,r)=&[l/(l+r+1)]P_{n-1}(l,r+1)\\&+[r/(l+r+1)]P_{n-1}(l+1,r)\end{aligned}\tag{5}$$

with the initial and boundary conditions $P_0(l,r)=\delta_{l,L}\delta_{r,R}$;

$P_n(L,R)=0,\ \ n>0,\ \ P_n(l,r)=0$ if $l>l$ or $r>R$.

It is useful to introduce the generating function

$$P(l,r,\xi)\equiv\sum_{n\geq 0}\xi^n P_n(l,r)=\xi^{L-l+R-r}(l+r)h(l,r)\tag{6}$$

and in terms of the ξ-independent function $h(l,r)$ defined by Eq. (6) the difference equation to be solved becomes

$$\begin{aligned}&(l+r)h(l,r)=rh(l+1,r)+lh(l,r+1)\ ,\\&\qquad 0\leq l<L \text{ and } 0\leq r<R\ ,\end{aligned}\tag{7}$$

subject to the boundary conditions: $h(L,R)=(L+R)^{-1}$ and $h(l,r)=0$ for $l>L$ or $r>R$. The numerical solution of Eq. (7) is straightforward.

The quantity of interest in this one-dimensional model is the *displacement inefficiency,* which we define to be the mean position of one interface when the other has reached the origin. We consider only the symmetric case where $L=R=M$ for which the displacement inefficiency is

$$I(M)=M^{-1}\left[\sum_{l\geq 1}lP(l,0,1)\right]\Big/\left[\sum_{l\geq 1}P(l,0,1)\right]\ .\tag{8}$$

The function $I(M)$ is exhibited in Fig. 1. We may interpret $1-I(M)$ as the one-dimensional analogue of the areal sweep efficiency in displacement studies. In one dimension $I(M)\to 0$ as $M\to\infty$ (although the mean length of the region not swept out diverges). In contrast, in two or three dimensions the analogue of $I(M)$ converges to 1 as the size of the region initially filled with the fluid to be displaced grows.

While hypothesis A (and DLA, to which it is equivalent) may be used as a basis for qualitatively correct simulation of unstable displacements, it uses no information about the porous material other than its bulk properties, and so cannot distinguish between microstructurally different porous media. We contend that microstructure is the essence of fingering, and that a correct understanding of the role of microstructure in fingering might lead to the possibility of defining model media for which DLA gives a quantitatively correct description of fingering. We suggest that a key microstructural parameter which controls fingering is the *fluid capacity.* We define the fluid capacity, a dimensionless quantity, to be the void space per specified length λ in one dimension, the void space per specified area λ^2 in two dimensions, and the void space per specified volume λ^3 in a three dimensions. If λ is on the Darcy scale, then in three dimensions fluid capacity becomes identical to porosity φ, and we shall use the symbol φ here to denote fluid capacity. In three dimensions, fluid capacity, like porosity, is constrained to be less than unity; no such constraint exists in one or two dimensions. If λ is on the pore scale, the fluctuations in the fluid capacity correspond to the pore size distribution. In a DLA simulation, we identify λ as the lattice size.

To illustrate the role of the fluid capacity, consider a *model* porous medium, which consists of *narrow tubes* of comparable lengths and diameters connecting *chambers* with volumes very much larger than the volumes of the narrow tubes. The narrow tubes give rise to the permeability of the porous medium, while the chambers give rise to the fluid capacity. To apply this model to a real porous medium, one should not necessarily identify the narrow

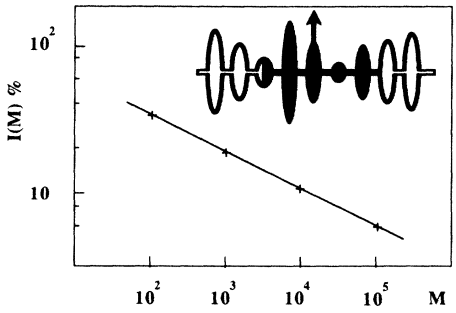

FIG. 1. Displacement inefficiency $I(M)$ [Eq. (8)], obtained by numerical solution of Eq. (7). It appears that $I(M)\propto M^{-1/4}$ as $M\to\infty$, so the mean length not swept is proportional to $M^{3/4}$. This may be compared with the easily derived result that the length not swept is proportional to $M^{1/2}$ when the probabilities of motion of either interface are assumed to be equal. The inset shows the model system simulated in Table I.

tubes as individual pore throats and the chambers as individual pore bodies. The physical nature of the tubes and chambers is determined by the length scale λ. We shall take the volumes of the chambers to be randomly and independently distributed. For a two-fluid displacement in our model porous medium, consider flow within two of the narrow tubes which are adjacent to unfilled chambers, and comparable distances from the pressure source. The interface cannot advance into a new tube until the chamber into which fluid is currently pouring is filled. The flow rates within the tubes, and so the rates of filling of the chambers, may be of comparable magnitude, but all other things being equal, the smallest chamber will fill soonest, so the motion of the interface is heavily governed by the microstructure.

The preceding model is a discrete version of a porous medium in which the fluid capacity φ is a random variable. Can we find a stochastic law for interface growth for such a medium which coincides exactly with the microstructure-insensitive hypothesis A (or DLA)? We propose, for the one-dimensional model, *hypothesis B:*

$$\text{Prob}\{l \text{ interface moves}\} = \text{Prob}\{\,|dl/dt| > |dr/dt|\,\} . \tag{9}$$

With this hypothesis only one interface moves at each time step, even if dl/dt and dr/dt are of comparable magnitude. However, if we consider what happens over a few time steps, we find that both interfaces will advance. We make the plausible assumption that the effective permeabilities experienced by the left and right interfaces are nearly equal, so that all stochastic properties of the flow are attributed to fluctuations in the *fluid capacity* of the porous medium. Equation (2) remains valid with φ now denoting the fluid capacity, and combining this with Eqs. (1) and (9) we have

$$\text{Prob}\{l \text{ interface moves}\} = \text{Prob}\{(l\varphi_l)^{-1} > (r\varphi_r)^{-1}\} . \tag{10}$$

One may establish an exact mathematical equivalence between hypothesis A (which is the same as DLA) and the consequence (10) of hypothesis B, in the particular case in which the fluid capacity φ has an exponential probability density function. Indeed the equivalence is a very general consequence of exponential fluctuations in microstructure, and holds independent of the dimension of the system and the physical interpretation placed upon the random quantity φ. We consider any connected set of sites on a d-dimensional lattice, with a single new site about to be added to the cluster in a random fashion. We label the possible bonds along which the growth may occur (i.e., all bonds adjacent to the cluster) with a coordinate $i=1,2,\ldots,N$. With each of these bonds is associated a deterministic positive number v_i and a positive random variable φ_i. The random variables φ_i are assumed independent. For the one-dimensional fluid displacement problem, $N=2$, with $i=1$ indexing the l interface and $i=2$ the r interface; v_i denotes the volume flux calculated on the basis of Darcy's law (1); and v_i/φ_i is the instantaneous velocity of the appropriate interface. We consider two prescriptions for the probability that the bond labeled i is the bond chosen for growth:

$$\text{Prob}\{\text{bond } i \text{ chosen}\} = v_i \Big/ \sum_{j=1}^{N} v_j , \tag{11}$$

$$\text{Prob}\{\text{bond } i \text{ chosen}\} = \text{Prob}\{v_i/\varphi_i = \max_j\{v_j/\varphi_j\}\} . \tag{12}$$

The first of these equations is DLA or hypothesis A and the second is hypothesis B. Let f denote the probability density function for the independent random variables φ_i. The event "$v_i/\varphi_i = \max_j\{v_j/\varphi_j\}$" is exactly the same as the event "$\varphi_j \geq (v_j/v_i)\varphi_i$ for all j," so that Eq. (12) becomes

$$\text{Prob}\{\text{bond } i \text{ chosen}\} = \int_0^\infty d\varphi_i f(\varphi_i) \prod_{\substack{j=1 \\ j\neq i}}^{N} \int_{(v_j/v_i)\varphi_i}^\infty d\varphi_j f(\varphi_j) . \tag{13}$$

For the special case in which the random variables φ_i have the exponential density $f(\varphi)=\alpha\exp(-\alpha\varphi)$, where α is an arbitrary constant, the integrals in Eq. (13) are easily evaluated and the probability found to be $v_i/\sum_j v_j$. Hence the two growth laws (11) and (12) are exactly the same, irrespective of the precise values of the deterministic quantities v_i and the value of N, provided that the random variables φ_i are exponentially distributed. This result may be significant for the modeling of a variety of processes in which there is both a deterministic field and a random transport coefficient governing interface motion, but for the particular problem of fluid displacements in porous media leads to the following conclusion: *Hypothesis A (or equivalently the DLA algorithm) will yield displacement results that correspond to a porous medium for which hypothesis B holds, provided that there is an exponential distribution of fluid capacities.*

We conclude by reporting the results of some simulations which we have performed for the model system of identical narrow tubes of negligible volume connecting chambers of randomly distributed volume shown in Fig. 1. These are classical continuum simulations, in which both interfaces are allowed to move simultaneously. The distribution of the chamber volumes is taken to be the fluid capacity distribution $f(\varphi)$. An inviscid fluid displaces the viscous fluid from both sides of the origin at constant driving pressure. In this model system, motion of the inviscid fluid from one tube to a neighboring tube is forbidden until all viscous fluid is displaced from the intervening chamber. The inefficiency I is found as the ratio of the number of empty tubes on one side of the origin when the fluid on the other side reaches the origin to the number of empty tubes initially on one side of the origin. Results of simulations are given in Table I for a selection of fluid capacity probability densities $f(\varphi)$; in each case there are 100 narrow tubes on each side of the origin. For the exponential fluid capacity distribution, we

TABLE I. Displacement inefficiency $I(M)$ for a one-dimensional displacement with $L=R=M=100$. Each result is an average over 200 simulations. The random fluid capacity φ is generated from a random number z, uniformly distributed on (0,1).

Footnote	Fluid capacity	Displacement inefficiency (%)
a	$\varphi=-\ln z$	32.4
b	$\varphi=z$	24.5
c	$\varphi=z^2$	30.6
d	$\varphi=z^3$	34.1
e	$\varphi=(1-\delta+2\delta z)^2$, $\delta=1$	30.6
e	$\varphi=(1-\delta+2\delta z)^2$, $\delta=0.33$	19.8
e	$\varphi=(1-\delta+2\delta z)^2$, $\delta=0.1$	11.2
e	$\varphi=(1-\delta+2\delta z)^2$, $\delta=0$	1.0

[a] On the basis of hypothesis B, this corresponds to DLA.
[b] φ uniformly distributed.
[c] Cylinders with radii uniformly distributed on (0,1).
[d] Spheres with radii uniformly distributed on (0,1).
[e] Cylinders with radii uniformly distributed on $(1-\delta,1+\delta)$, cf. Chen and Wilkinson (Ref. 7).

have proved above that DLA, hypothesis A, and hypothesis B are mathematically identical. Numerical solution of the difference equation (7) yields the exact displacement inefficiency for these three equivalent models, and so can be used to compare the models (and thereby hypothesis B) against the continuum simulation. For the exponential fluid capacity distribution, we find on taking $M=100$ the exact value 33.0% for the displacement inefficiency, which is satisfyingly close to the value of 32.4% found by the simulation.

While DLA and hypothesis A are equivalent, they coincide with hypothesis B only for the particular case of an exponential distribution of fluid capacity. For any porous medium in which fingering is governed by fluctuations in the fluid capacity, a DLA simulation will correctly describe the fingering if and only if there is a length scale λ for which the distribution of fluid capacity is well approximated by an exponential. It should be remembered that the probability density function for fluid capacity need not be scale invariant, so that a density appropriate at one length scale λ may not be appropriate at a different length scale. We have obtained by simulation of the model system shown in Fig. 1 the displacement efficiency for various forms of the fluid capacity probability density function $f(\varphi)$. The results in Table I are consistent with the simulations of Chen and Wilkinson.[7] The discussion given in this paper gives the DLA simulation of viscous fingering a physical basis in terms of microscopic fluctuations in the porous medium. Provided that one can identify the physical origin of relevant fluctuations, the arguments advanced here in the context of viscous fingering can be used to quantify connections between DLA and other physical phenomena.

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[3]In two dimensions, the flow equations for a porous medium coincide with those governing flow in a Hele-Shaw cell [Nature (London) **58**, 34 (1898)], i.e., between two closely spaced parallel plates. The hydrodynamic instability when a less viscous fluid displaces a more viscous one is common to both systems; for the Hele-Shaw cell it is called the Saffman-Taylor instability. There are many papers in the literature on this subject; a recent one is D. Bensimon, Phys. Rev. A **33**, 1302 (1986). Despite the mathematical equivalence of their governing equations, the Hele-Shaw cell is a poor model for porous media because of its complete lack of the microstructure wherein most of the salient features of porous medium hydrodynamics reside. Papers on DLA and the Saffman-Taylor problem include L. P. Kadanoff, J. Stat. Phys. **39**, 267 (1985); C. Tang, Phys. Rev. A **31**, 1977 (1985); S. K. Sarker, *ibid.* **32**, 3114 (1985); and J. Nittmann, G. Daccord, and H. E. Stanley, Nature (London) **314**, 141 (1985).

[4]J. D. Sherwood and J. Nittmann, J. Phys. (Paris) **47**, 15 (1986); A. J. DeGregoria, Phys. Fluids **28**, 2933 (1985); M. J. King and H. Scher, SPE Report No. 14366 (unpublished).

[5]This is just the classical solution of the "gambler's ruin" problem (see, e.g., W. Feller, *An Introduction to Probability Theory and Its Applications,* 3rd ed. (Wiley, New York, 1968), Vol. 1. The solution of this one-dimensional problem may be thought of as a simple example of an elegant procedure described by L. A. Turkevich and H. Scher, Phys. Rev. Lett. **55**, 1026 (1985), for calculating growth pattern probabilities in DLA.

[6]For the displacement of a power-law fluid by an inviscid fluid, Eq. (1) is replaced by $v_r=B/r^\beta$, where β is the power-law index, and Eq. (4) by Prob{r interface moves}$=(dr/dt)/[(dr/dt)+(dl/dt)]=l^\beta/(r^\beta+l^\beta)$. For the case of finite mobility ratios $\gamma\equiv m_1/m_2$ ($m=k/\mu$, subscript 1 denotes the driving fluid), Eq. (1) becomes $v_r=m_1P/[R+(\gamma-1)r]$. Here the driving fluid occupies the intervals $-L<x<l(t)$ and $r(t)<x<R$, while the displaced fluid is in $-l(t)<x<r(t)$; the points $x=L$ and $x=R$ are held at a pressure P and the origin is taken at zero pressure. Equation (4) becomes Prob{r interface moves}$=[L+(\gamma-1)l]/[L+R+(\gamma-1)(r+l)]$. From these examples, it is easy to see how to generate the appropriate stochastic algorithm for a given physical displacement process. The interpretation of each stochastic algorithm in terms of fluctuations in the fluid capacity is found in principle from Eq. (13).

[7]J.-D. Chen and D. Wilkinson, Phys. Rev. Lett. **55**, 1892 (1985).