Dissipation of energy in the damped harmonic oscillator

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A canonical description of a harmonic oscillator with energy dissipation is sought which combines the advantages of the Kanai-Caldirola Hamiltonian and a simple model of strangulation previously considered by the authors. The treatment is in the Heisenberg picture of quantum mechanics or alternatively in classical mechanics. A strangulation β is superimposed on the damping (or growth) γ, represented by an oscillator mass $m(t)=m_0\exp(2\gamma t)$. In general, the expected dissipation of energy occurs only if $\beta>|\gamma|$. If $\beta<|\gamma|$ the attempted strangulation induces a long-term *growth* in energy unless a special initial motion is chosen. A slight deviation from this initial motion results in a temporary decay in energy followed by growth. $\beta<0$ always induces a growth in energy.

I. INTRODUCTION

In the last few decades much attention has been paid to the subject of damping in a quantum-mechanical system, idealized as a harmonic oscillator. In many treatments the damped oscillator is regarded as an open system in which energy is dissipated by interaction with a heat bath or by Brownian motion.[1-11] On the other hand, considerable effort has been made to construct a satisfactory closed canonical model of an oscillator with friction,[12-20] the most generally accepted description being via the Kanai-Caldirola Hamiltonian[13,20]

$$H_{\rm KC}(t)=\tfrac{1}{2}p^2/m(t)+\tfrac{1}{2}m(t)\omega_0^2q^2\ , \tag{1a}$$

in which the mass of the oscillator is given a growth factor

$$m(t)=m_0e^{2\gamma t}\quad(\gamma>0)\ . \tag{1b}$$

This leads to the well-known equations of motion in either classical mechanics or the Heisenberg picture of quantum mechanics,

$$\ddot{q}+2\gamma\dot{q}+\omega_0^2q=0\ , \tag{2a}$$

$$\ddot{p}-2\gamma\dot{p}+\omega_0^2p=0\ . \tag{2b}$$

The disappointing feature of the model is that the energy exhibits only a cyclic variation without dissipation.[21] The expectation value of the energy in a state $|\psi\rangle$ has to be calculated from the ansatz[15,18]

$$E(t)=e^{-2\gamma t}\langle\psi|H_{\rm KC}|\psi\rangle\ . \tag{3}$$

A naturally dissipative system is described by the Hamiltonian

$$H_{\rm diss}(t)=e^{-2\beta t}(\tfrac{1}{2}p^2/m_0+\tfrac{1}{2}m_0\omega_0^2q^2)\quad(\beta>0) \tag{4}$$

for which

$$dH/dt=\partial H/\partial t=-2\beta H\ . \tag{5}$$

Thus, instead of Eq. (3), we have the more satisfactory formula

$$E(t)=\langle\psi|H_{\rm diss}|\psi\rangle=e^{-2\beta t}\langle\psi|H_{\rm diss}(0)|\psi\rangle\ . \tag{6}$$

The disadvantage of the Hamiltonian $H_{\rm diss}$ is that the equation of motion for the coordinate is

$$\ddot{q}+2\beta\dot{q}+\omega^2(t)q=0,\quad \omega(t)=\omega_0e^{-2\beta t}\ , \tag{7}$$

rather than the usual Eq. (2). The solution of Eq. (7) is discussed in Ref. 22. The system may be termed the *strangled* oscillator.

In the present paper we shall examine an exactly solvable model which is more flexible and which can, under favorable conditions, combine some of the advantages of $H_{\rm KC}$ and $H_{\rm diss}$.

II. MODEL FOR DAMPING WITH ENERGY DISSIPATION

Let us consider the modified Kanai-Caldirola Hamiltonian

$$H_{\rm CK}(t)=e^{-2\beta t}H_{\rm KC}(t)\quad(\beta>0) \tag{8}$$

where we attempt to impose some dissipation in an obvious way. We shall examine the asymptotic dependence of $H_{\rm CK}$ on the time for $\beta t\gg1$. The Heisenberg equations (or Hamilton equations in classical mechanics) are

$$\dot{q}=\partial H/\partial p=e^{-2\beta t}p/m(t)=e^{-2(\beta+\gamma)t}p/m_0\ , \tag{9a}$$

$$\dot{p}=-\partial H/\partial q=-e^{-2\beta t}m(t)\omega_0^2q=-e^{-2(\beta-\gamma)t}m_0\omega_0^2q\ . \tag{9b}$$

Elimination of p leads to the equation of motion

$$\ddot{q}+2(\beta+\gamma)\dot{q}+\omega^2(t)q=0,\quad \omega(t)=\omega_0e^{-2\beta t}\ . \tag{10}$$

To investigate the time variation of $H_{\rm CK}$ we introduce the operators

$$L=e^{-2\beta t}[\tfrac{1}{2}p^2/m(t)-\tfrac{1}{2}m(t)\omega_0^2q^2]\ , \tag{11}$$

$$S=\tfrac{1}{2}(qp+pq)\ . \tag{12}$$

Then we obtain the following coupled equations for $H=H_{\rm CK}$:

$$dH/dt=\partial H/\partial t=-2\beta H-2\gamma L\ , \tag{13a}$$

$$dL/dt=\partial L/\partial t+(i\hbar)^{-1}[L,H]$$
$$=-2\beta L-2\gamma H-2\omega_0^2 e^{-4\beta t}S\ , \tag{13b}$$

$$dS/dt=\partial S/\partial t+(i\hbar)^{-1}[S,H]=2L\ . \tag{13c}$$

In the classical case S becomes qp and we replace $(i\hbar)^{-1}[A,H]$ by $(\partial A/\partial q)\partial H/\partial p-(\partial A/\partial p)\partial H/\partial q$, with $A=L$ or S, to obtain formally the same Eqs. (13).

The elimination of L and S from Eqs. (13) leads to the third-order equation for H,

$$(D+4\beta)[(D+2\beta)^2-4\gamma^2]H=-4\omega_0^2 e^{-4\beta t}(D+2\beta)H\ , \tag{14}$$

where $D=d/dt$. Writing

$$H(t)=e^{-2\beta t}H_0(t)\ , \tag{15}$$

Eq. (14) simplifies to

$$(D+2\beta)(D^2-4\gamma^2)H_0=-4\omega_0^2 e^{-4\beta t}DH_0\ . \tag{16}$$

In the case of the strangled oscillator ($\gamma=0$) Eq. (16) is satisfied simply by $DH_0=0$, i.e., $H(t)=H(0)e^{-2\beta t}$ as given by Eq. (15). In the case $\beta=0$, $H_0=H$ and Eq. (16) reduces to

$$(D^2+4\omega^2)(DH)=0\quad(\omega^2=\omega_0^2-\gamma^2)\ , \tag{17}$$

which is *not* satisfied by $DH=0$, but by the full solution

$$H_{\rm KC}=A\cos(2\omega t)+B\sin(2\omega t)+C\ , \tag{18a}$$

where we may identify (cf. Ref. 18)

$$\begin{aligned}A&=-\gamma(\omega_0/\omega)^2S(0)\ ,\\ B&=-(\gamma/\omega)L(0)\ ,\\ C&=(\omega_0/\omega)^2[H(0)+\gamma S(0)]\ .\end{aligned} \tag{18b}$$

The asymptotic time dependence of H may be examined by making the transformation $x=e^{-2\beta t}$. Then Eq. (16) takes the form

$$x^2\beta^2(xH_0'''+2H_0'')-x(\gamma^2-x^2\omega_0^2)H_0'+\gamma^2H_0=0\ , \tag{19}$$

where a prime indicates differentiation with respect to x. Large values of t correspond to small values of x. However, it should be noted that we cannot sensibly put $\beta=0$ in Eq. (19) since the transformation breaks down in that limit. Seeking a series solution

$$H_0(x)=\sum_{r=0}^{\infty}a_r x^{r+k}\ , \tag{20}$$

we find $k=1$ or $\pm\gamma/\beta$ and hence three possible asymptotic forms for $H=H_{\rm CK}$ given by Eq. (15),

$$\begin{aligned}H^{(1)}(t)&\sim a_0^{(1)}e^{-4\beta t}\ ,\\ H^{(2)}(t)&\sim a_0^{(2)}e^{-2(\gamma+\beta)t}\ ,\\ H^{(3)}(t)&\sim a_0^{(3)}e^{2(\gamma-\beta)t}\ .\end{aligned} \tag{21}$$

These modes of decay (or growth) will be seen again in Sec. V.

III. SOLUTION OF THE EQUATIONS OF MOTION

It is convenient to make the scaling transformation

$$Q=qe^{\gamma t},\quad P=pe^{-\gamma t}\ , \tag{22}$$

then (cf. Ref. 18) the Hamiltonian given by Eq. (8) transforms to

$$K=H+\partial F/\partial t$$
$$=e^{-2\beta t}(\tfrac{1}{2}P^2/m_0+\tfrac{1}{2}m_0\omega_0^2Q^2)+\tfrac{1}{2}\gamma(QP+PQ)\ . \tag{23}$$

A change in time scale[23]

$$t\to\tau=\int_0^t e^{-2\beta t'}dt'=(1-e^{-2\beta t})/(2\beta) \tag{24}$$

transforms the Hamiltonian K to

$$\overline{K}(Q,P,\tau)=\tfrac{1}{2}P^2/m_0+\tfrac{1}{2}m_0\omega_0^2Q^2$$
$$+\tfrac{1}{2}\gamma(1-2\beta\tau)^{-1}(QP+PQ)\ . \tag{25}$$

Let us denote differentiation with respect to τ by a prime, then Eq. (25) gives the equations of motion

$$Q'=(i\hbar)^{-1}[Q,\overline{K}]=P/m_0+\gamma Q/(1-2\beta\tau)\ , \tag{26a}$$

$$P'=(i\hbar)^{-1}[P,\overline{K}]=-m_0\omega_0^2Q-\gamma P/(1-2\beta\tau)\ . \tag{26b}$$

The elimination of P leads to the equation

$$d^2Q/d\tau^2+[\omega_0^2-\gamma(\gamma+2\beta)/(1-2\beta\tau)^2]Q=0\ . \tag{27}$$

We write

$$Q=x^{1/2}v,\quad x=1-2\beta\tau=e^{-2\beta t}\ , \tag{28}$$

then Eq. (27) assumes the form of a Bessel equation

$$\frac{d^2v}{dx^2}+\frac{1}{x}\frac{dv}{dx}+\left[\frac{\omega_0^2}{4\beta^2}-\frac{(\beta+\gamma)^2}{4\beta^2x^2}\right]v=0\ , \tag{29}$$

with the solution

$$v=AJ_\nu(kx)+BJ_{-\nu}(kx)\ , \tag{30a}$$

where (in quantum mechanics) A and B are arbitrary time-independent self-adjoint operators and

$$\nu=\tfrac{1}{2}(\beta+\gamma)/\beta,\quad k=\tfrac{1}{2}(\omega_0/\beta)\ . \tag{30b}$$

We assume that ν is not an integer. From Eqs. (22), (28), and (30) we have the full solution of Eq. (10) in the form

$$q=e^{-(\beta+\gamma)t}[AJ_\nu(ke^{-2\beta t})+BJ_{-\nu}(ke^{-2\beta t})]\ , \tag{31a}$$

cf. Refs. 24 and 25, where a different approach is used. The momentum p is calculated from Eq. (9a). In the case when ν is not an integer we find

$$p=m_0\omega_0e^{-(\beta-\gamma)t}[AJ_{\nu+1}(ke^{-2\beta t})+BJ_{-\nu+1}(ke^{-2\beta t})]$$
$$-2m_0(\beta+\gamma)e^{(\beta+\gamma)t}AJ_\nu(ke^{-2\beta t})\ , \tag{31b}$$

where we have used the identity

$$zJ_\nu'(z)=\nu J_\nu(z)-zJ_{\nu+1}(z)\ . \tag{32}$$

If $\nu=n$, an integer, then Eq. (31a) must be replaced by

$$q=e^{-(\beta+\gamma)t}[AJ_n(ke^{-2\beta t})+CY_n(ke^{-2\beta t})] , \tag{33}$$

with an appropriate replacement for Eq. (31b).

IV. LIMITING CASES

A. Limit $\gamma\to 0$

We may easily check that when $\gamma\to 0$, Eq. (31a) yields the solution of Refs. 20 and 22. From Eq. (30b) we see that in this case $\nu\to\frac{1}{2}$ and

$$q=e^{-\beta t}\left[AJ_{1/2}\left[\frac{\omega_0}{2\beta}e^{-2\beta t}\right]+BJ_{-(1/2)}\left[\frac{\omega_0}{2\beta}e^{-2\beta t}\right]\right] . \tag{34}$$

Using the formulas

$$J_{1/2}(z)=(2/\pi z)^{1/2}\sin z , \quad J_{-(1/2)}(z)=(2/\pi z)^{1/2}\cos z , \tag{35}$$

Eq. (34) becomes

$$q=A'\sin\left[\frac{\omega_0}{2\beta}e^{-2\beta t}\right]+B'\cos\left[\frac{\omega_0}{2\beta}e^{-2\beta t}\right] . \tag{36a}$$

Then with τ given by Eq. (24) we may rewrite Eq. (36a) in the form

$$q=A''\cos(\omega_0\tau)+B''\sin(\omega_0\tau) , \tag{36b}$$

which identifies with Eq. (15a) of Ref. 22 when we write

$$A''=q(0) \text{ and } B''=p(0)/(m_0\omega_0) .$$

B. Limit $\beta\to 0$

It is more difficult to see that Eqs. (31) give the well-known damped harmonic oscillator solution in the limit $\beta\to 0$. From Eq. (30b) we see that in this case both ν and k become infinite and we require an asymptotic form of a Bessel function of large argument and large order. The formula required is more specialized than the standard one found by the method of steepest descents.[26] In Appendix A we show how the method of stationary phase[27] may be used to establish that as $\beta\to 0+$

$$J_{\pm\gamma/2\beta}\left[\frac{\omega_0}{2\beta}e^{-2\beta t}\right]\sim 2\left[\frac{\beta}{\pi\omega}\right]^{1/2}\cos\left[\frac{\pm\theta_0\gamma-\omega}{2\beta}+\omega t+\pi/4\right] , \tag{37}$$

where $\cos\theta_0=\pm\gamma/\omega_0$ $(0<\theta_0<\pi)$ and $\omega^2=\omega_0^2-\gamma^2$. Using Eq. (37) it is easy to see that, as $\beta\to 0$, Eq. (31a) leads to

$$q(t)=e^{-\gamma t}[a\cos(\omega t)+b\sin(\omega t)] , \tag{38}$$

with the identification[18] $a=q(0)$, $b=p(0)/(m_0\omega)^2-\gamma q(0)/\omega$.

V. BEHAVIOR OF q, p, V, T, AND $H_{\rm CK}$ FOR LARGE VALUES OF THE TIME

In Sec. IV A, when $\gamma\to 0$, $H_{\rm CK}\to H_{\rm diss}$ giving the strangled motion as discussed in Ref. 22, when in general q and p "freeze" to finite nonzero values (with $\dot{q}\to 0$) as $t\to\infty$. The time dependence of the energy (represented by the Hamiltonian) is very simply

$$H(t)=e^{-2\beta t}[\tfrac{1}{2}p^2(0)/m_0+\tfrac{1}{2}m_0\omega_0^2q^2(0)] , \tag{39a}$$

cf. Eqs. (5) and (6), which may be expressed in the alternative form

$$H(t)=e^{-2\beta t}[\tfrac{1}{2}p^2(\infty)/m_0+\tfrac{1}{2}m_0\omega_0^2q^2(\infty)] \tag{39b}$$

when we use Eqs. (16a) and (16b) of Ref. 22.

On the other hand in Sec. IV B, when $\beta\to 0$, $H_{\rm CK}\to H_{\rm KC}$ giving Eqs. (2a) and (2b), which are the generally accepted equations for damped harmonic motion. A feature of such motion is that $q\to 0$ and $p\to\infty$ as $t\to\infty$ if $\gamma>0$, or $q\to\infty$ and $p\to 0$ if $\gamma<0$. The Hamiltonian is given by Eqs. (18a) and (18b) and shows oscillatory behavior.

In the general case we shall now show that a dissipation in H occurs only if $\beta>|\gamma|$, unless a specially favorable initial motion is chosen. Then dissipation may be induced for $\beta<|\gamma|$.

Let us consider the asymptotic behavior of $q(t)$ and $p(t)$. Using the asymptotic form

$$J_\nu(ke^{-2\beta t})\sim(\tfrac{1}{2}ke^{-2\beta t})^\nu/\Gamma(\nu+1) \quad (\nu\gtrless 0) , \tag{40}$$

Eqs. (31a) and (31b) give, when $\beta t\gg 1$,

$$q(t)\sim A[\Gamma(\nu+1)]^{-1}(k/2)^\nu e^{-2(\beta+\gamma)t}+B[\Gamma(-\nu+1)]^{-1}(k/2)^{-\nu} , \tag{41a}$$

$$p(t)\sim m_0\omega_0\{A[\Gamma(\nu+2)]^{-1}(k/2)^{\nu+1}e^{-2\beta t}+B[\Gamma(-\nu+2)]^{-1}(k/2)^{-\nu+1}e^{-2(\beta-\gamma)t}\}-2m_0(\beta+\gamma)A[\Gamma(\nu+1)]^{-1}(k/2)^\nu , \tag{41b}$$

where ν and k are given by Eq. (30b). We always suppose $\beta>0$ and we note that as $t\to\infty$

$$q(t)\to q(\infty) \quad (\gamma>0,\ \text{any } \beta/\gamma \text{ or } \gamma<0,\ \beta>|\gamma|) , \tag{42a}$$

$$p(t)\to p(\infty) \quad (\gamma>0,\ \beta>\gamma,\ \text{or } \gamma<0,\ \text{any } \beta/|\gamma|) , \tag{42b}$$

$$q(t)\to\infty \quad (\gamma<0,\ \beta<|\gamma|) , \tag{42c}$$

$$p(t)\to\infty \quad (\gamma>0,\ \beta<\gamma) \tag{42d}$$

where $q(\infty)$ and $p(\infty)$ are constant operators. Equations (42a) and (42b) compare with the limiting case A, whereas Eqs. (42c) and (42d) agree with the case B.

The expectation values of q and p have the following asymptotic forms in terms of the operators A and B:

$$\langle q\rangle \sim \frac{\langle B\rangle}{\Gamma(-\nu+1)}\left[\frac{k}{2}\right]^{-\nu} = \langle q(\infty)\rangle$$

$$(\gamma>0,\ \text{any } \beta/\gamma \text{ or } \gamma<0,\ \beta>|\gamma|)\,, \tag{43a}$$

$$\langle q\rangle \sim \frac{\langle A\rangle}{\Gamma(\nu+1)}\left[\frac{k}{2}\right]^{\nu} e^{2(|\gamma|-\beta)t}\ \ (\gamma<0,\ \beta<|\gamma|)\,, \tag{43b}$$

$$\langle p\rangle \sim -\frac{2m_0(\beta+\gamma)\langle A\rangle}{\Gamma(\nu+1)}\left[\frac{k}{2}\right]^{\nu} = \langle p(\infty)\rangle$$

$$(\gamma>0,\ \beta>\gamma,\ \text{or } \gamma<0,\ \text{any } \beta/|\gamma|)\,, \tag{43c}$$

$$\langle p\rangle \sim \frac{m_0\omega_0\langle B\rangle}{\Gamma(-\nu+2)}\left[\frac{k}{2}\right]^{-\nu+1} e^{2(\gamma-\beta)t}$$

$$(\gamma>0, \beta<\gamma)\,. \tag{43d}$$

We have taken expectation values since we are concerned primarily with quantum mechanics. On squaring Eqs. (41a) and (41b) for the operators q and p before taking expectation values we obtain corresponding results for $\langle q\rangle$ and $\langle p\rangle$ in terms of $\langle A\rangle$ and $\langle B\rangle$. Similar results apply for the uncertainties ΔA and ΔB, thus

$$\Delta q \sim \frac{\Delta B}{|\Gamma(-\nu+1)|}\left[\frac{k}{2}\right]^{-\nu} = \Delta q(\infty)$$

$$(\gamma>0,\ \text{any } \beta/\gamma \text{ or } \gamma<0,\ \beta>|\gamma|)\,, \tag{44a}$$

$$\Delta q \sim \frac{\Delta A}{|\Gamma(\nu+1)|}\left[\frac{k}{2}\right]^{\nu} e^{2(|\gamma|-\beta)t}$$

$$(\gamma<0,\ \beta<|\gamma|)\,, \tag{44b}$$

$$\Delta p \sim \frac{2m_0|\beta+\gamma|\Delta A}{|\Gamma(\nu+1)|}\left[\frac{k}{2}\right]^{\nu} = \Delta p(\infty)$$

$$(\gamma>0,\ \beta>\gamma \text{ or } \gamma<0,\ \text{any } \beta/|\gamma|)\,, \tag{44c}$$

$$\Delta p \sim \frac{m_0\omega_0 B}{|\Gamma(-\nu+2)|}\left[\frac{k}{2}\right]^{-\nu+1} e^{2(\gamma-\beta)t}\ \ (\gamma>0,\ \beta<\gamma)\,. \tag{44d}$$

From Eqs. (44a) and (44c)

$$\Delta q\Delta p \sim \frac{2m_0(\beta+\gamma)}{|\Gamma(\nu+1)\Gamma(-\nu+1)|}\Delta A\Delta B$$

$$= \Delta q(\infty)\Delta p(\infty)\ \ (\gamma>0,\ \beta>\gamma)\,. \tag{45a}$$

From Eqs. (44a) and (44d), however, we obtain

$$\Delta q\Delta p \sim \frac{m_0\omega_0}{|\Gamma(-\nu+1)\Gamma(-\nu+2)|}\left[\frac{k}{2}\right]^{-2\nu+1}(\Delta B)^2$$

$$\times e^{2(\gamma-\beta)t}\ \ (\gamma>0,\ \beta<\gamma)\,. \tag{45b}$$

Results similar to Eqs. (45a) and (45b) hold in the case $\gamma<0$.

From Eqs. (31a) and (32b) as shown in Appendix B,

$$[q,p]=i\hbar \to [A,B]=i\hbar\alpha,\ \ \alpha=\frac{\Gamma(\nu+1)\Gamma(-\nu+1)}{2m_0(\beta+\gamma)}\,, \tag{46}$$

which gives rise to the Heisenberg uncertainty relation

$$\Delta A\Delta B \geq \tfrac{1}{2}\hbar|\alpha|\,. \tag{47}$$

Equation (45a) may be written in the form

$$\Delta q\Delta p \sim |\alpha|^{-1}\Delta A\Delta B \geq \tfrac{1}{2}\hbar\,. \tag{48}$$

If both the equalities $\Delta q(0)\Delta p(0)=\frac{1}{2}\hbar|\alpha|$, $\Delta A\Delta B=\frac{1}{2}\hbar|\alpha|$ hold, then $\Delta q\Delta p=\frac{1}{2}\hbar$ still holds as $t\to\infty$. An initially coherent state can remain coherent at all times in the case $\beta>|\gamma|$, but not otherwise. As may be seen from Eqs. (45a) and (45b), the cases $\beta>|\gamma|$ and $\beta<|\gamma|$ are fundamentally different and we shall now discuss them in turn.

A. Case $\beta>|\gamma|$

Let us take the Hamiltonian $H=H_{\rm CK}$ given by Eq. (8),

$$H=T+V=\frac{1}{2m_0}p^2e^{-2(\beta+\gamma)t}+\tfrac{1}{2}m_0\omega_0^2q^2e^{-2(\beta-\gamma)t}\,, \tag{49}$$

substitute the asymptotic forms given by Eqs. (41a) and (41b) for q and p and take expectation values. Then for times such that $\beta t\gg 1$ the following dissipative asymptotic forms are found:

$$\langle H(t)\rangle \sim \tfrac{1}{2}m_0\omega_0^2\langle q^2(\infty)\rangle e^{-2(\beta-\gamma)t}$$

$$=\langle V(t)\rangle\ \ (\gamma>0)\,, \tag{50a}$$

$$\langle H(t)\rangle \sim (2m_0)^{-1}\langle p^2(\infty)\rangle e^{-2(\beta-|\gamma|)t}$$

$$=\langle T(t)\rangle\ \ (\gamma<0)\,, \tag{50b}$$

where, cf. Eqs. (43a) and (43c),

$$\langle q^2(\infty)\rangle=[\Gamma(-\nu+1)]^{-2}(k/2)^{-2\nu}\langle B^2\rangle\,, \tag{50c}$$

$$\langle p^2(\infty)\rangle=4m_0^2(\beta-|\gamma|)^2[\Gamma(\nu+1)]^{-2}(k/2)^{2\nu}\langle A^2\rangle\,. \tag{50d}$$

When once Eqs. (42a) and (42b) have been established, then Eqs. (50a) and (50b) are obvious from Eq. (49). These decays may be identified with the second and third modes of Eq. (21).

B. Case $\beta<|\gamma|$

Rather surprisingly, we find that if $\beta<|\gamma|$, an imposed strangulation produces in general just the opposite effect. Let us consider first the classical aspect. Substituting Eqs. (41a) and (41b) into Eq. (49), with $\gamma>0$, we see that the last part of q leads to a growth in H proportional to $e^{2(\gamma-\beta)t}$, the third mode of Eq. (21). To avoid this growth we have to take $B=0$; similarly, if $\gamma<0$, we have to take $A=0$. Then we find

$$H(t)\sim(2m_0)^{-1}p^2(\infty)e^{-2(\beta+\gamma)t}\quad(\gamma>0,\ B=0,\ \beta t>>1)\ , \tag{51a}$$

$$H(t)\sim\tfrac{1}{2}m_0\omega_0^2q^2(\infty)e^{-2(\beta+|\gamma|)t}\quad(\gamma<0,\ A=0,\ \beta t>>1)\ , \tag{51b}$$

i.e., a switch to the second mode of Eq. (21). Provided the initial motion is chosen carefully, an infinitesimal strangulation induces the Kanai-Caldirola Hamiltonian ultimately to decay at the proper logarithmic rate $\dot{H}/H=-2\gamma$, as in Eq. (5). In this case Eq. (10) gives an equation of motion almost identical with the standard Eq. (2a) for damped motion. The Hamiltonian may be considered to represent the energy and Eqs. (51a) and (51b) give

$$E\approx E_0e^{-2(|\gamma|+\beta)t}\quad(\beta t>>1,\ \beta<<|\gamma|)\ , \tag{52a}$$

$$E_0=\tfrac{1}{2}p^2(\infty)/m_0\quad(\gamma>0)$$

or (52b)

$$E_0=\tfrac{1}{2}m_0\omega_0^2q^2(\infty)\quad(\gamma<0)\ .$$

This replaces the unsatisfactory distinction between Hamiltonian and energy in Eq. (3).

There are, however, two serious difficulties. Firstly, the smaller we make β the longer we must wait for Eq. (52) to apply and by this time the solution of Eq. (10) may differ appreciably from the solution of Eq. (2a). Secondly, we must consider the practicability of making $B=0$ (or $A=0$) by a choice of the initial motion. To make $B=0$ we need to choose the initial velocity to satisfy the relation

$$\begin{aligned}\dot{q}(0)&=p(0)/m_0\\&=-\omega_0[J_\nu'(k)/J_\nu(k)+\nu/k]q(0)\ ,\end{aligned} \tag{53}$$

where we have used Eq. (32). Such precision is not feasible. If Eq. (53) is not exactly satisfied there may be asymptotic decay for a time, followed by growth as we shall discuss next.

Turning our attention to quantum mechanics, let us suppose that $\gamma>0$ and the system is initially in a slightly fuzzy eigenstate of the operator B with eigenvalue zero. Since B is independent of the time the state remains stationary and

$$\langle B\rangle=0,\quad \Delta B=\epsilon\quad(\epsilon\ \text{small})\ \text{for all}\ t\ . \tag{54a}$$

The operator A has a certain expectation value with large dispersion

$$\langle A\rangle=\bar{A},\quad \Delta A\geq\tfrac{1}{2}\hbar|\alpha|/\epsilon\quad\text{for all}\ t\ , \tag{54b}$$

where we have used Eq. (47). It follows that

$$\langle B^2\rangle=\epsilon^2,\quad\langle A^2\rangle>\tfrac{1}{4}\hbar^2\alpha^2/\epsilon^2\ . \tag{55}$$

We square the asymptotic forms of q and p given by Eqs. (41a) and (41b) and substitute into Eq. (49). Taking expectation values and using Eqs. (55), together with the fact that $\langle AB+BA\rangle$ has some constant value which does not depend appreciably on ϵ, we obtain

$$\begin{aligned}\langle H\rangle\sim&(x/\epsilon^2)e^{-2(\beta+\gamma)t}+ye^{-4\beta t}+z\epsilon^2e^{2(\gamma-3\beta)t}\\&+(u/\epsilon^2)e^{-2(3\beta+\gamma)t}+ve^{-4\beta t}+w\epsilon^2e^{2(\gamma-\beta)t}\ ,\end{aligned} \tag{56}$$

where x,y,z,u,v,w are constants independent of ϵ. The third and sixth terms grow, but by making ϵ sufficiently small they can be made not to exceed any agreed amount up to a fixed time t_2, greater than a time t_1 that satisfies $\beta t_1>>1$. Then during the time interval (t_1,t_2) the expectation value of H_{CK} decays according to the first term in Eq. (56). By making $\epsilon\to0$ we can make $t_2\to\infty$. $\beta<<\gamma$ gives the correct decay rate of 2γ as discussed in the classical case. Similarly if $\gamma<0$ the roles of A and B may be reversed to obtain a decay at rate $2|\gamma|$.

The difficulties that we noted in the classical case are still present: how to construct an eigenstate of the operator B (or A), how long to wait for decay to occur and how well does $q(t)$ represent damped harmonic motion? A characteristic of damped harmonic motion is that $q\to0$ as $t\to\infty$ and with our model we see from Eq. (41a) that this occurs if $B=0$. This is an indication that the q motion is essentially right.

Looking back at the classical case, it is clear that if we make B small rather than zero by only approximately satisfying Eq. (53), then H_{CK} will decay in a time interval (t_1,t_2) and then grow.

VI. EFFECT OF $\beta<0$

We have seen that, unless we prepare the system very precisely in a state favorable to decay, a slight strangulation $0<\beta<|\gamma|$ leads to growth. This has led us to investigate the effect of $\beta<0$. It might happen that the attempted imposition of a growth leads to decay.

We take, instead of Eq. (8)

$$H=e^{2|\beta|t}H_{\text{KC}}\quad(\beta<0)\ . \tag{57}$$

Equations (31a) and (31b) still give $q(t)$ and $p(t)$ and Eqs. (30b) give k (<0) and ν (assumed not an integer). Thus

$$q=e^{(|\beta|-\gamma)t}[AJ_\nu(ke^{2|\beta|t})+BJ_{-\nu}(ke^{2|\beta|t})]\ , \tag{58a}$$

$$\begin{aligned}p=&m_0\omega_0e^{(|\beta|+\gamma)t}[AJ_{\nu+1}(ke^{2|\beta|t})+BJ_{-\nu+1}(ke^{2|\beta|t})]\\&+2m_0(|\beta|-\gamma)e^{-(|\beta|-\gamma)t}AJ_\nu(ke^{2|\beta|t})\ ,\end{aligned} \tag{58b}$$

where

$$k=-\tfrac{1}{2}(\omega_0/|\beta|),\quad \nu=\tfrac{1}{2}(|\beta|-\gamma)/|\beta|\ . \tag{58c}$$

As $t\to\infty$ we need the asymptotic form of $J_\nu(z)$ for large $|z|$:

$$J_\nu(z)\sim2(\pi z)^{-1/2}\cos(z-\tfrac{1}{2}\nu\pi)\quad(\nu\gtrless0)\ . \tag{59}$$

Then Eqs. (58a), (58b), and (59) give the following asymptotic forms for $|\beta|t>>1$:

TABLE I. Asymptotic behavior of $q(t)$ and $p(t)$ from Eqs. (41a) and (41b) for $\gamma>0$ and $\beta<\gamma$. For $\gamma<0$ and $\beta<|\gamma|$ B is replaced by A and the roles of q and p are interchanged.

	Behavior for large values of the time ($\beta t \gg 1$, $\gamma>0$)
General motion ($B=0$)	$q \to q(\infty)$ (finite, nonzero) $\dot{q}\to 0$ as for H_{diss} $p\to\infty$ as for H_{KC}
Special motion ($B=0$ or $\langle B\rangle=0$, $\Delta B=\epsilon$)	$q\to 0$, $\dot{q}\to 0$ as for H_{KC} $p\to p(\infty)$ (finite, nonzero) as for H_{diss}

$$q(t) \sim 2(\pi|k|)^{-1/2} e^{-\gamma}[A\cos(ke^{2|\beta|t}-\tfrac{1}{2}\nu\pi)+B\cos(ke^{2|\beta|t}+\tfrac{1}{2}\nu\pi)] , \tag{60a}$$

$$p(t) \sim 2m_0\omega_0(\pi|k|)^{-1/2} e^{\gamma t}\{A\cos[ke^{2|\beta|t}-\tfrac{1}{2}(\nu+1)\pi]+B\cos[ke^{2|\beta|t}+\tfrac{1}{2}(\nu-1)\pi] + (2A/\omega_0)(|\beta|-\gamma)\cos(ke^{2|\beta|t}-\tfrac{1}{2}\nu\pi)\} . \tag{60b}$$

We note that as $t\to\infty$, $q\to 0$, $p\to\infty$ if $\gamma>0$, or $q\to\infty$, $p\to 0$ if $\gamma<0$, which is the same behavior as for the Kanai-Caldirola oscillator or the damped oscillator. Unlike the case when $\beta>0$, the A and B terms in both q and p have similar time dependence and we cannot prevent a growth in H by putting either A or B equal to zero. Substituting Eqs. (60a) and (60b) into Eq. (49) we obtain

$$H(t) \sim F_{\text{osc}}(t)e^{2|\beta|t} \quad (|\beta|t \gg 1) , \tag{61}$$

where $F_{\text{osc}}(t)$ is rapidly oscillating, but bounded. Thus $\beta<0$ cannot lead to energy dissipation.

VII. CONCLUSION

As discussed in Sec. I, the Kanai-Caldirola oscillator described by the Hamiltonian H_{KC} of Eqs. (1a) and (1b) has the same equations of motion for q and p as the damped harmonic oscillator, yet the cycle-averaged expectation value of H_{KC} remains constant.[21] We expect this situation to be unstable with regard to time-dependent modifications in H_{KC}. We have studied the effect of multiplying H_{KC} by the factor $e^{-2\beta t}$, where $\beta(>0)$ is referred to as a *strangulation*. This could be expected to induce a decay in H. Unless a very special initial motion is arranged decay occurs, in fact, only if $\beta>|\gamma|$.

An interesting situation arises when $\beta<|\gamma|$. In Table I we compare the asymptotic behavior of q and p given by Eqs. (41a) and (41b) in the general motion ($B\neq 0$) and the special motion ($B=0$) when $\gamma>0$. In the general motion q shows strangled oscillator behavior and p behaves as in damped harmonic motion. If we arranged that $B=0$ (or the equivalent in quantum mechanics) q goes to zero, as in

TABLE II. Asymptotic behavior of $\langle H_{\text{CK}}(t)\rangle$ for $|\beta|t\gg 1$. $H_{\text{CK}}=e^{-2\beta t}H_{\text{KC}}$, where H_{KC} is the Kanai-Caldirola Hamiltonian given by Eqs. (1a) and (1b). $\langle q_\infty^2\rangle$ and $\langle p_\infty^2\rangle$ are given by Eqs. (50c) and (50d).

		Case A: $\beta>|\gamma|$	Case B: $\beta<|\gamma|$
$\beta>0$	$\gamma>0$	$\frac{1}{2}m_0\omega_0^2\langle q_\infty^2\rangle e^{-2(\beta-\gamma)t}$ Decay	General motion ($B\neq 0$) $\frac{1}{2}m_0\omega_0^2\langle q_\infty^2\rangle e^{2(\gamma-\beta)t}$ Growth Special motion ($\langle B\rangle=0$, $\Delta B=\epsilon$) $\frac{1}{2m_0}\langle p_\infty^2\rangle e^{-2(\beta+\gamma)t}$ ($t_1<t<t_2$, $t_2\to\infty$ as $\epsilon\to 0$) Decay
	$\gamma<0$	$\frac{1}{2m}\langle p_\infty^2\rangle e^{-2(\beta-|\gamma|)t}$ Decay	General motion ($A\neq 0$) $\frac{1}{2m_0}\langle p_\infty^2\rangle e^{2(|\gamma|-\beta)t}$ Growth Special motion ($\langle A\rangle=0$, $\Delta A=\epsilon$) $\frac{1}{2}m_0\omega_0^2\langle q_\infty^2\rangle e^{-2(\beta+|\gamma|)t}$ ($t_1<t<t_2$, $t_2\to\infty$ as $\epsilon\to 0$) Decay
$\beta=0$		Kanai-Caldirola oscillator $\langle H_{\text{KC}}(t)\rangle_{\text{cycle-av}}=\text{const}$ for all values of t	
$\beta<0$		$\langle H_{\text{CK}}(t)\rangle_{\text{cycle-av}} \sim \langle H_{\text{CK}}(0)\rangle e^{2|\beta|t}$ Growth	

damped motion, while p goes to a finite nonzero limit as in the case of the strangled oscillator. If $\gamma < 0$, then the roles of q and p interchange. The situation is summarized in Table I.

However, our primary interest is in a decaying Hamiltonian H_{CK}, so that it may represent the energy. In Table II we summarize the behavior of $\langle H_{CK}(t)\rangle$ for $\beta t \gg 1$. For completeness we include the cases $\beta=0$ and $\beta<0$.

The surprising outcome of our investigation is that $\beta > |\gamma|$ is necessary to ensure decay when the initial state of the oscillator is arbitrary. A smaller value of β actually causes $\langle H(t)\rangle$ to grow, as would a negative value of β. If, however, the oscillator is prepared in a favorable state, then the Hamiltonian oscillates [cf. Eq. (18) for H_{KC}] and shows a decay after a sufficient lapse of time. The smaller β is, the longer we must wait for the eventual exponential decay. Obviously, there is continuity from H_{KC} (i.e., $H_{CK}, \beta=0$) to H_{CK} ($\beta \ll \omega_0$) for finite times.

As discussed in Ref. 22, the Hamiltonian H_{diss} given by Eq. (4) provides a satisfactory energy decay but Eq. (7) shows that damped harmonic motion as in Eq. (2a) is not followed. The Hamiltonian H_{CK} given by Eq. (8) with $\beta \ll \gamma$ provides a more satisfactory model because according to Eq. (10) the correct damped motion ensues for small values of the time and, if the motion is started off in the right way, the Hamiltonian will eventually decay to zero.

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APPENDIX A: TO ESTABLISH EQ. (37)

The following proof is for $J_{\gamma/2\beta}(\omega_0 e^{-2\beta t}/2\beta)$; that for $J_{-\gamma/2\beta}$ is similar.

A well-known integral representation is[26]

$$J_\nu(z)=\frac{1}{\pi}\int_0^\pi \cos(\nu\theta - z\sin\theta)d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\sinh\phi-\nu\phi}d\phi\ , \tag{A1}$$

which gives

$$\pi J_\nu(\omega_0 e^{-2\beta t}/2\beta)=I_1-\sin(\gamma\pi/2\beta)I_2\ , \tag{A2}$$

where

$$I_1=\int_0^\pi \cos\left[\frac{\gamma\theta}{2\beta}-\frac{\omega_0}{2\beta}e^{-2\beta t}\sin\theta\right]d\theta\ , \tag{A3a}$$

$$I_2=\int_0^\infty \exp\left[-\frac{\gamma\phi}{2\beta}-\frac{\omega_0}{2\beta}e^{-2\beta t}\sinh\phi\right]d\phi\ . \tag{A3b}$$

As $\beta\to 0$, $e^{-2\beta t}\sim 1-2\beta t+O(\beta^2)$, so that

$$I_1=J_1-J_2+O(\beta)\ , \tag{A4a}$$

$$J_1=\int_0^\pi \cos\left[\frac{1}{2\beta}(\gamma\theta-\omega_0\sin\theta)\right]\times\cos(\omega_0 t\sin\theta)d\theta\ , \tag{A4b}$$

$$J_2=\int_0^\pi \sin\left[\frac{1}{2\beta}(\gamma\theta-\omega_0\sin\theta)\right]\times\sin(\omega_0 t\sin\theta)d\theta\ . \tag{A4c}$$

We assume $\gamma<\omega_0$ (undercritical damping), then the function $h(\theta)=\gamma\theta-\omega_0\sin\theta$ has a unique maximum in $(0,\pi)$ at θ_0 given by

$$\omega_0\cos\theta_0=\gamma\ . \tag{A5}$$

Also

$$\omega_0\sin\theta_0=(\omega_0^2-\gamma^2)^{1/2}=\omega\ . \tag{A6}$$

Hence the method of stationary phase[27] gives that for $\beta\to 0+$

$$J_1\sim\left[\frac{4\pi\beta}{h''(\theta_0)}\right]^{1/2}\cos(\omega_0 t\sin\theta_0)\times\cos\left[\frac{1}{2\beta}h(\theta_0)+\frac{\pi}{4}\right]. \tag{A7a}$$

Similarly

$$J_2\sim\left[\frac{4\pi\beta}{h''(\theta_0)}\right]^{1/2}\sin(\omega_0 t\sin\theta_0)\times\sin\left[\frac{1}{2\beta}h(\theta_0)+\frac{\pi}{4}\right], \tag{A7b}$$

so that

$$J_1-J_2\sim\left[\frac{4\pi\beta}{h''(\theta_0)}\right]^{1/2}\cos\left[\frac{1}{2\beta}h(\theta_0)+\omega_0 t\sin\theta_0+\frac{\pi}{4}\right]. \tag{A8}$$

Since $h''(\theta_0)=\omega_0\sin\theta_0$, Eqs. (A4), (A6), and (A8) give for $\beta\to 0+$

$$I_1\sim\left[\frac{4\pi\beta}{\omega}\right]^{1/2}\cos\left[\frac{1}{2\beta}(\gamma\theta_0-\omega)+\omega t+\frac{\pi}{4}\right]. \tag{A9}$$

Also, clearly,

$$I_2<\int_0^\infty \exp\left[-\frac{\gamma\phi}{2\beta}\right]d\phi=2\beta/\gamma\ . \tag{A10}$$

Hence from Eq. (A2), as $\beta\to 0+$,

$$J_{\gamma/2\beta}\left[\frac{\omega_0}{2\beta}e^{-2\beta t}\right]\sim 2\left[\frac{\beta}{\pi\omega}\right]^{1/2}\times\cos\left[\frac{\gamma\theta_0-\omega}{2\beta}+\omega t+\frac{\pi}{4}\right], \tag{A11}$$

where $\theta_0=\cos^{-1}(\gamma/\omega_0)$, $0<\theta_0<\pi$.

APPENDIX B: TO ESTABLISH EQ. (46)

At all times

$$[q(t),p(t)]=i\hbar \ . \tag{B1}$$

Substituting Eqs. (31a) and (31b) into Eq. (B1) gives

$$m_0\omega_0 e^{-2\beta t}(J_\nu J_{-\nu+1}-J_{-\nu}J_{\nu+1})[A,B] \\ +2m_0(\beta+\gamma)J_\nu J_{-\nu}[A,B]=i\hbar \ , \tag{B2}$$

where all the Bessel functions have argument $ke^{-2\beta t}$. Since the operators A and B are time independent we can let $t \rightarrow \infty$ and use Eq. (40). Then we find for $\beta t \gg 1$

$$m_0\omega_0 k\nu(\nu^2-1)^{-1}e^{-4\beta t}[A,B]+2m_0(\beta+\gamma)[A,B] \\ =i\hbar\Gamma(\nu+1)\Gamma(-\nu+1) \ . \tag{B3}$$

Equation (46) follows when $t \rightarrow \infty$.

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