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Analytic treatment of periodic orbit systematics for a nonlinear driven oscillator

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A prototype model of nonlinear driven oscillators with a limit cycle is studied. In the fast-relaxation limit dynamics can be reduced to a one-dimensional mapping. The systematics of periodic orbits is investigated.

Recent interest has focused on nonlinear oscillators.[1-5] It is well known[6] that in a fast damping situation the return map exhibits "dimensional reduction" from $d=2$ to $d=1$, and the state of the system may be determined by a single variable, say the phase angle θ. Then a circle map $\theta_{n+1}=f(\theta_n)$ might be defined. In particular, the sine circle map has been investigated in great detail,[6] and the behaviors indicated by this modal, such as mode locking and the transition to chaos, have been found both in numerical simulation and in real experiments for a wide class of physical systems, for example, the Josephson junction, the damped driven pendulum, and so on.

A different kind of circle map, however, occurs in another wide class of models and physical systems, such as the forced Brusselator[3] and some electronic oscillators.[4] This class is characterized as follows: (i) The free nonlinear oscillator has a stable limit cycle, which encloses an unstable stationary point. (ii) The external force is such that above and below a critical strength the circle map $f(\theta)$ exhibits topologically different properties. Two examples are shown in Figs. 1(a) and 1(b). Notice that the nonanalyticity must take place when the external force has the critical strength. These driven oscillators display behaviors different from that of the sine map. The most interesting one is the coexistence of ordering of periodic orbits by the U sequence in strong force situations and mode-locking behavior with rational winding numbers in weak-force situations. However, the connection between these two ways of categorizing periodic orbits has apparently never been discussed. The reason is simple: The connection is too complicated to be revealed by direct solution of the differential equations, and the corresponding simple circle map has up to now never been proposed.

In this Rapid Communication I introduce a prototype oscillator and claim that it contains the essential features of a wide class, and is thus representative. The external force is assumed to be impulsive. Furthermore, the limit of fast relaxation is taken, by which the dimension of the return map is reduced from $d=2$ to $d=1$. Numerical experience shows that a finite relaxation rate apparently gives results close to the limiting case. With these simplifications, the dynamics can be discussed in a fairly transparent manner, and one is, in particular, able to make the connection between a "devil's staircase" region with mode-locking behavior[6] to a region with a unimodal mapping via a more complicated region of parameter space. Besides, in this intermediate region new sequences of periodic orbits occur, and their pattern may be explored systematically.

Since the shape and the location of the limit cycle may be changed by a transformation of variables, a prototype two-dimensional nonlinear oscillator with a stable limit cycle is, in properly scaled polar coordinates,

$$\dot{r}=sr(1-r^2),\ \dot{\theta}=1\ . \tag{1}$$

The parameter s is a measure of the inverse relaxation time for perturbations off the limit cycle $r=1$. This oscillator is subjected to a periodic force in the x direction, with the following evolution equations

$$\dot{x}=sxu-y+2\alpha\sum_n \delta(t-2\pi n\beta)\ ,$$
$$\dot{y}=x+syu\ , \tag{2}$$

with $u=1-x^2-y^2$, and integer n. We make a strobo-

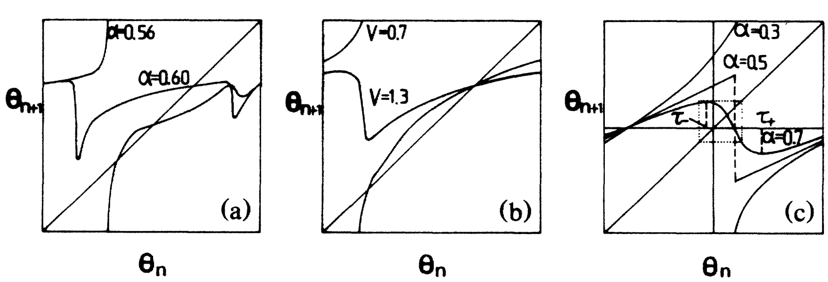

FIG. 1. Circle maps $\theta_{n+1}=f(\theta_n)$ for three models. (a) The forced Brusselator $\dot{x}=2-9x+x^2y+F$, $\dot{y}=8x-x^2y$, $F=\alpha$ if $10n<t<10n+5$, or $F=0$ if $10n+5<t<10(n+1)$, with integer n. The critical strength here is $\alpha_c\approx 0.58$. (b) An electronic oscillator (see Ref. 4) $\ddot{x}+\pi(20x^2-1)\dot{x}+0.5\pi^2x(50x^4-10x^2+1)=V\sum_n\delta(t-4n)$. The critical strength here is $V_c\approx 1.0$. (c) Three topologically different versions of the prototype map (3) with $\beta=0.4$.

scopic map of the dynamics, focusing on the values x_n, y_n, and θ_n of x, y, and θ immediately after the nth kick at time $t=2\pi n\beta$. In the fast-relaxation limit $s\to\infty$ we will have regained $r=1$ before the force is applied, hence $x_{n+1}=2a+\cos(\theta_n+2\pi\beta)$ and $y_{n+1}=\sin(\theta_n+2\pi\beta)$. By immediate radial relaxation a one-dimensional iteration $\theta_{n+1}=f(\theta_n)$ results, viz.,

$$\tan\theta_{n+1}=\frac{\sin(\theta_n+2\pi\beta)}{2a+\cos(\theta_n+2\pi\beta)} . \tag{3}$$

Without loss of generality we may assume $a\geq 0$, $0<\beta\leq 1$, and $-\pi<\theta_n\leq\pi$. Besides, Eqs. (2) guarantee the uniqueness of the iteration by the implicit requirements that $\sin\theta_{n+1}$ and $\sin(\theta_n+2\pi\beta)$ have the same sign. The iteration, shown in Fig. 1(c), has completely different properties in the three cases $a<\frac{1}{2}$, $a=\frac{1}{2}$, and $a>\frac{1}{2}$. The map is invertible for $a<\frac{1}{2}$ (weak force), piecewise linear for $a=\frac{1}{2}$ and noninvertible for $a>\frac{1}{2}$. The nonanalyticity of map (3) is by no means unphysical. It is universal for a wide class of driven oscillators, since this nonanalyticity corresponds to a force just sufficiently strong to displace the oscillator into the unstable stationary point (the origin in the present model). Note also that since the mapping (3) is symmetric about $\beta=\frac{1}{2}$ it suffices to discuss $\beta\leq\frac{1}{2}$.

The discussion will be centered on superstable orbits, periodic orbits that start at an extremum of the mapping. For the strong-force situation $a>\frac{1}{2}$, the function $f(\theta)$ has a maximum $\theta_m=f(\tau_-)$ and a minimum $-\theta_m=f(\tau_+)$ for $\tau_\pm=\pi-2\pi\beta\pm\cos^{-1}[1/(2a)]$. The superstable orbits are therefore of two types: Class 1 starts from τ_-, while class 2 starts from τ_+. It is also useful to define nonperiodic orbits, called pseudo orbits, of two types: Class 1 starts from τ_- and ends at τ_+, while class 2 starts from τ_+ and ends at τ_-. I associate a point of an orbit with a letter L, M, or R according to whether it falls to the interval $(-\pi,\tau_-]$, (τ_-,τ_+), or $[\tau_+,\pi]$. Therefore, each periodic orbit or pseudo orbit corresponds to a word constructed of L's, M's, and R's. The word of class 1 orbits start with L, while that of class 2 start with R. The words of class 1 pseudo orbits start with L and end by $-R$, while those of class 2 start with R and end by $-L$. The part of a word before the first M is called the prefix. As $a\to\frac{1}{2}$ from above the branch M becomes vertical and both τ_- and τ_+ tend to $\tau=(1-2\beta)\pi$, so that the word of any orbit reduces to its prefix.

The simplest part of the parameter space is $a\leq\frac{1}{2}$, in which the system displays quasiperiodic or periodic behavior (Fig. 2). In the latter case mode locking with rational winding numbers P/Q takes place. The widths of the mode-locking intervals (the Arnol'd tongues) increase (from zero) when a increases. At the boundary of this region, i.e., at $a=\frac{1}{2}$, I use $[P/Q]$ to denote the region of β where mode locking with winding number P/Q takes place. If $P/Q=1/[N_1+1/(N_2+\cdots)]$ with positive integers N_j, and the fractions $P_j/Q_j (j=1,2,\dots)$ are the successive approximants, respectively, the corresponding intervals $[P_j/Q_j]$ are called for jth level, and the words for the superstable orbits at both end points of the intervals $[P_j/Q_j]$ can be determined. For example, it is easy to

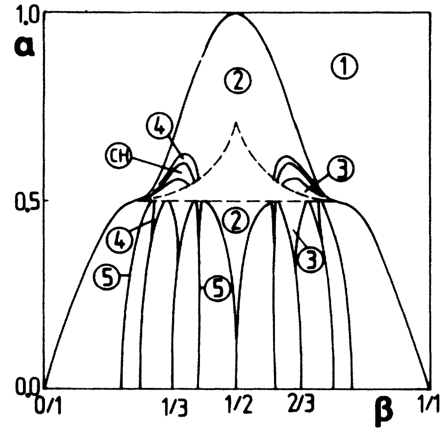

FIG. 2. Location of some periodic (the numbers inside circles indicate the length) and chaotic regions for the map (3). In the intermediate region, marked by dashed lines, the state of the system may depend on the initial condition.

show for the first level that the words W_1 for the left end points and the words $\hat{W}_1$ for the right end points of $[1/N_1]$ are $W_1=RL^{N_1-1}$ and $\hat{W}_1=LRL^{N_1-2}$, respectively.[7] Moreover, one can show[7] that the width of the region P/Q is $r_Q=1/[2(2^Q-1)]$. Then the total measure of the mode-locking regions at $a=\frac{1}{2}$ is $S=\sum_Q r_Q\phi(Q)$, where $\phi(Q)$, Euler's ϕ function,[8] is the number of positive integers less than Q and relative prime to Q. The Liouville formula,[8] $\sum_N\phi(N)x^N/(1-x^N)=x/(1-x)^2$, with $x=\frac{1}{2}$ gives that $S=1$. The devil's staircase is thus complete at $a=\frac{1}{2}$. The dimension of the remainder set is $D=0$ exactly, in contradiction to the critical sine circle map.[6]

For $a>\frac{1}{2}$ the system displays more complicated behavior. Let us denote the line in our parameter space (a,β) for which a superstable orbit (pseudo orbit) of a given type exists, for a trajectory (pseudotrajectory). I use the same word to denote the orbits (pseudo orbits) and the corresponding trajectory (pseudotrajectory). Superstable periodic orbits will only occur for $\frac{1}{4}\leq\beta\leq\frac{3}{4}$.

Consider first the uppermost region $a>a_t$, where a_t is the pseudotrajectory $L-R$, i.e., $a_t=1/[2\sin(\frac{3}{4}-\beta)\pi]$. In this region the iterates will be confined to a region where $f(\theta)$ is unimodal [the dotted square in Fig. 1(c)], and thus all the superstable orbits of the U sequence[9] are found. Note that our notation differs from the notation in Ref. 9, since the relevant branches in our cases are L and M, and the initial point of the superstable orbit is also included in the notation. Thus, these words start with LML^{N_1-2}, with some $N_1\geq 2$. To have the ordering given by the universal sequence one must follow directions in parameter space that intersects each trajectory once [see Figure 3(a)], along $a_t(\beta)$ for instance.

In the intermediate region $\frac{1}{2}<a<a_t$ the transition between the unimodal mapping and the mode-locking behavior takes place. To see the transition let us consider trajectories with words $LML^{N_1-2}M\dots$ in the unimodal region $a>a_t$. When a is lowered, leaving the unimodal region at $a=a_t$, the first iterate crosses over from branch M to branch R. As a result the words will have the prefix

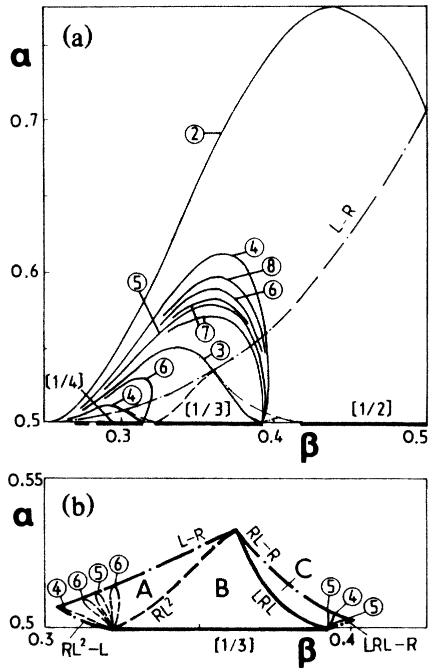

FIG. 3. (a) Some trajectories of class 1 for superstable orbits of U sequence (solid lines). Numbers inside circles indicate the period. Dot-dashed lines indicate pseudotrajectories. (b) The pentagon connected to the interval [1/3]. The dashed lines are trajectories of class 2 which do not belong to the U sequence.

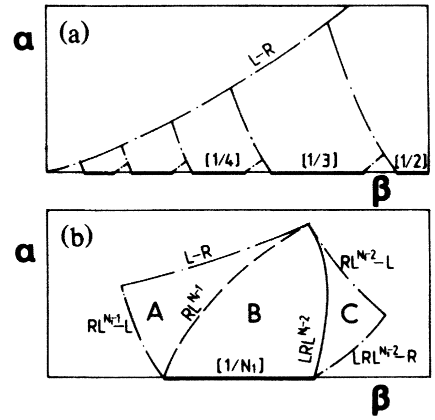

FIG. 4. (a) Pentagons connected to intervals of the first level (sketched only). (b) Blowup of the pentagon immediately above the interval $[1/N_1]$.

LRL^{N_1-2} below $\alpha=\alpha_t$. When $\alpha=\frac{1}{2}$ only the prefix survives; thus the period becomes N_1, and all these trajectories terminate at the right-hand end points of $[1/N_1]$, an interval of the first level. Figure 3(a) bears this out.

In the intermediate region many new sequences of period orbits occur. Some of them are shown in Fig. 3(b). From these periodic orbits one will also find period-doubling bifurcations with the usual Feigenbaum convergence rate[10] $\delta=4.6692\ldots$. In order to explore the pattern of the new sequences of trajectories in this intermediate region systematically, we consider the trajectories which terminate at the intervals of the first level, $[1/N_1]$. The pseudotrajectories $LRL^{N_1-2}-R$, $RL^{N_1-2}-L$, $L-R$, $RL^{N_1-1}-L$, and the interval $[1/N_1]$ enclose, as sketched in Fig. 4, a "pentagon" in parameter space. As an example, the pentagon for $N_1=3$ is shown in Fig. 3(b). At the point of intersection between $L-R$ and $RL^{N_1-2}-L$, a superstable orbit must necessarily be present, and thus the trajectory LRL^{N_1-2} of class 1, and that of class 2, RL^{N_1-1}, must also pass through this point. The period of the corresponding orbits is clearly N_1. As $\alpha\to\frac{1}{2}$ these two trajectories terminate at the right and left end point of $[1/N_1]$, respectively. They divide the pentagon into three triangular areas A, B, and C (Fig. 4). All the trajectories of class 2 in A have words with prefix RL^{N_1-1}, and they must terminate at the left end point of $[1/N_1]$. Correspondingly, all the trajectories of class 1 in C have words with prefix LRL^{N_1-2}, and they must terminate at the right end point of $[1/N_1]$. There is no trajectory in the middle area B at all.[11]

So far I have not discussed trajectories of class 1 in A and of class 2 in C. When they leave these regions they will enter small pentagons of second order. These second-order pentagons are constructed on second-level intervals $[P_2/Q_2]$ in a similar way as the above first-order pentagons were constructed on first-level intervals. This process will continue, and the set of pentagons corresponding to all intervals $[P/Q]$ will thereby form a hierarchy in parameter space. The words of the trajectories in successive levels have longer and longer prefixes. A more complete discussion, in which also the nature of the new sequential ordering of periodic orbits is given, will be published elsewhere.[11]

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