Simultaneous rational approximations in the study of dynamical systems

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We discuss rational approximations to a single irrational number and pairs of mutually irrational irrational numbers and show that the binary "Farey" tree organization of the rationals extends naturally to a binary organization of pairs of rationals with a common denominator. There are obvious applications to theory and experiments in dynamical systems.

I. INTRODUCTION

In two-frequency dynamical systems, the breakup of Kolmogorov-Arnold-Mosur (KAM) tori or invariant circles can be studied by looking at the stability of the long periodic orbits.[1,2] This involves the approximation of an irrational number by a sequence of rationals.

In three-frequency dynamical systems, we need to simultaneously approximate a pair of mutually irrational irrationals by a sequence of pairs of rationals with a common denominator. Although we are motivated by our desire to understand mode coupling in these multifrequency dynamical systems, the present paper is solely a number-theoretic discussion which does not assume familiarity with dynamical systems and number theory. Results similar to those derived here have been used before[3] and will be discussed in the context of renormalization of mappings in a forthcoming work. However, we felt a need to separate our intended application from the number-theoretic discussion, since we believe that the number theory, by itself, has more general applicability.

We proceed from an overview (Sec. II) by reviewing the standard theory of continued-fraction approximations to a single irrational number in Sec. III and then discuss the somewhat less standard "Farey" approximants in Sec. IV. In Sec. V, we present what we believe are novel ways of investigating simultaneous approximations in a way which we believe will prove to be useful to physicists.

II. OVERVIEW

A central problem in number theory is how to "best" approximate an irrational number σ by a converging sequence of rational approximations

$$\left\{\frac{p_n}{q_n}\right\}_{n=0}^{n=\infty} \equiv \left\{\frac{p_0}{q_0},\frac{p_1}{q_1},\ldots\right\}.$$

The answer to this question was solved long ago by the method of continued fractions.[4] Depending on our definition of "best," the answer is either the "Farey" sequence of rational approximants, discussed briefly in Sec. IV, or the continued-fraction approximations which form a subsequence of intermediate convergents of the Farey sequence.

It is perhaps surprising that the innocent generalization of this problem to the simultaneous rational approximants to several mutually irrational irrational numbers is considered a difficult, essentially unsolved problem in number theory. We hasten to point out that we do not believe we have derived deep new results in the theory of numbers. Our aim is to discuss compactly and elegantly results which should be useful to physicists in a spirit we have not seen before in the literature.

The precise generalization of rational approximants to a single irrational number is to define a sequence of pairs of rational numbers $\{\sigma_n \equiv (p_n/r_n, q_n/r_n)\}_{n=0}^{n=\infty}$, each set with a common denominator r_n which converges to a pair of mutually irrational irrational numbers $\sigma=(\sigma_1,\sigma_2)$. In order to define "best," we have to have a metric which measures how close a rational approximation is to an irrational number. There are two common natural metrics in use. The metric which measures "weak convergence" (called metric of the "first" kind by Khinchin) (Ref. 4) is the ordinary two-dimensional (2D) distance:

$$||\sigma-(p/r,q/r)||_w = |\sigma-(p/r,q/r)| , \tag{2.1}$$

and the metric which measures "strong" convergence (called metric of the "second" kind by Khinchin)

$$||\sigma-(p/r,q/r)||_s = |r\sigma-(p,q)| , \tag{2.2}$$

where $|\ |$ indicates the ordinary Euclidean norm. With these definitions, we say that $\sigma_n = [(p_n/r_n),(q_n/r_n)]$ is a "best" rational approximant if

$$||\sigma_n-\sigma|| < ||(p/r,q/r)-\sigma|| \tag{2.3}$$

for all triplets of integers (p,q,r) for any $r \le r_n$. A rational approximant can only be called best relative to a particular metric.

We now choose σ and a metric and ask how to generate a sequence of best rational approximants σ_n with increasing denominators converging to σ. With these metrics, we are aware of *no* known systematic methods of describing this except a complicated method recently developed by Brentjes.[5] We shall mention his method briefly later when we discuss why it does not provide our application with a suitable solution. Although a suitable solution to the problem is not known, various theorems exist concerning special values of σ and bounds are also known about the size of the norm as a function of the denominator. We will not discuss these theorems since we did not find these theorems useful, nor did we find the direct connection between them and our construction of the simultaneous rational approximation scheme presented in the following sections.

The situation is not hopeless, however, because for our purposes it is not essential to restrict ourselves to the best rational approximants as long as we can obtain strongly convergent sequences. This we can do systematically; aside from prefactors of order 1, these sequences converge as rapidly as the best approximant. The existence of systematically derived, geometrically converging sequences has long been known. This method first attributed to Jacobi is commonly called the Jacobi-Perron algorithm.[6] In many ways, it is a natural generalization of the continued-fraction algorithm. However, since the Jacobi-Perron sequences and various refinements fail to generate either weakly or strongly convergent *best* rational approximants, the algorithms have not gained the recognition that the continued-fraction method has for ordinary rational approximants.

In a sense, what we shall present in Sec. V is yet another Jacobi-Perron algorithm. We believe that from the point of view of naturalness and elegance, this is *the* algorithm of choice for physicists. It turns out this algorithm makes contact with what we believe are the fundamental generalizations of properties of the golden mean, which plays a fundamental role in the theory of rational approximants of a single irrational number. The algorithm presented here also generates a natural extension of the Farey organizations of the rationals which is discussed in Sec. IV.

We will not discuss the applications of this theory to dynamical systems. Rather our wish is to provide a tool which others can use to investigate dynamical systems with several incommensurate frequencies. The application of these ideas should be obvious to those who are studying these questions.

III. CONTINUED FRACTIONS

To place our results in context with well-known theory, we shall first discuss results from the theory of continued fractions. Proofs of our statements can be found in Refs. 4–7 and references therein.

Assume σ is an irrational number and let $\sigma_n = p_n/q_n$ denote a convergent sequence of best rational approximants, with q_n and p_n integers. Let $\{n_I\}$ denote a collection of positive integers and $[n_1, n_2, \ldots]$ be the number

$$[n_1, n_2, \ldots] = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cdots}}} \,. \tag{3.1}$$

The sequence terminates if and only if $[n_1, n_2, \ldots]$ is a rational. With $\sigma = [n_1, n_2, n_3, \ldots]$, $\sigma_0 = 1$, $\sigma_N \equiv p_N/q_N \equiv [n_1, n_2, \ldots, n_N]$, and $\rho_N \equiv [n_{N+1}, n_{N+2}, \ldots, n_\infty]$, the following facts then follow:

(a) σ_N is a sequence of strongly convergent best rational approximants.

(b) ρ_N and n_N are determined by the recursion formula

$$n_{N+1} = [\rho_N^{-1}] \,, \tag{3.2}$$

$$\rho_{N+1} = \rho_N^{-1} - n_{N+1} \,, \tag{3.3}$$

$$\begin{bmatrix} p_N & q_N \\ p_{N+1} & q_{N+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & n_N \end{bmatrix} \begin{bmatrix} p_{N-1} & q_{N-1} \\ p_N & q_N \end{bmatrix} , \tag{3.4}$$

where $[x]$ denotes the largest integer $\leq x$.

It is convenient to introduce the shift operator $s(x)$ which acts on the unit interval by removing the first continued-fraction entry of x. Thus if $x = [n_1, n_2, n_3, \ldots]$

$$s(x) = [n_2, n_3, n_4, \ldots] = x^{-1} - [x^{-1}] \,. \tag{3.5}$$

The inverse to $s(x)$ has an infinite number of branches t_n, where

$$t_n(s(x)) = x \text{ if } [x^{-1}] = n \,. \tag{3.6}$$

Since $s(x)$ is piecewise continuous, we often denote each of the these pieces of $s(x)$ by $s_n(x)$, i.e., $s(x) = s_n(x)$ if $n \leq 1/x < n+1$.

Although continued fractions are extremely useful, there are several properties that are not ideal for us. We list these below:

(i) Continued-fraction representations are not symmetric about $\frac{1}{2}$, violating a fundamental symmetry of the rational numbers on the unit interval.

(ii) The address $[n_1, n_2, n_3, \ldots]$ requires a symbol (i.e., the integer n_N) which has an infinite number of possibilities.

(iii) The shift operator has an infinite number of discontinuities.

(iv) The continued-fraction approximation misses many weakly convergent best rational approximants.

As an example of (iv), $\frac{1}{8}$, $\frac{1}{9}$, . . . , are weakly convergent approximants to 0, while they are completely skipped in the sequence σ_N since $s[\frac{1}{8}] = 0$. To remedy these problems we next discuss the Farey organization of the rational numbers and make a more elegant construction which we will generalize to simultaneous approximants.

IV. FAREY TREE

A. Farey organization of the rationals

We shall now describe the Farey ordering of the rational and irrational numbers.[7–9] This particular way of looking at rational approximants is not familiar to many physicists and appears to generalize quite naturally to simultaneous rational approximants.

A rational number is the ratio of a pair of relatively prime integers. For reasons soon to become clear, we write this explicitly as $p/q = ((p,q))$ where we note the similarity in the notation to an integer vector. The "Farey" sum $\oplus$ of two rationals is the sum of their numerator and denominator. Thus

$$((p_1, q_1)) \oplus ((p_2, q_2)) = ((p_1 + p_2, q_1 + q_2)) \,.$$

Interpreted as an operation on integer vectors, Farey summation is vector addition but is also a "mediant" operation since if $x < y$, then $x < x \oplus y < y$. Based upon the notion of a mediant operation, it is straightforward to encode rational numbers into the binary tree, usually called the Farey tree. This is done in the following steps.

(1) Start from the two trivial rational numbers $((0,1)) = \frac{0}{1}$ and $((1,1)) = \frac{1}{1}$, placing them at each end of a line segment. The two end points will define the left and right boundaries of the tree. The entire line segment is called the level-0 Farey interval. [See Fig. 1(a).]

(2) Perform the mediant operation $\oplus$ on the end points of the level-0 Farey interval, resulting in ((1,2)) at the midpoint. The fraction $\frac{1}{2}$ is defined to be the Farey element of level 0. [See Fig. 1(b).]

(3) The fraction $\frac{1}{2}$ splits the level-0 Farey interval into what we define to be the two level-1 Farey intervals.

(4) Perform the mediant operation on the end points of the level-1 Farey intervals splitting each of the level-1 intervals into two level-2 intervals which are smaller by a factor of 2. Draw an arrow between each element in level 1 ("mother") and the two elements in level 2 that it generates (left and right "daughters"). For example, $\frac{1}{3}$ is the left daughter of $\frac{1}{2}$ being the result of a mediant operation of the left level-1 interval whose end points are ((0,1)) and ((1,2)). Note that only *one* member of each mediant operation is in the previous level. Draw arrows labeled "0" from mother to left daughter and labeled "1" to right daughter. (See Fig. 2.)

(5) Recursively apply (3) and (4) on all new Farey intervals generated. This naturally generates a binary "Farey" tree as shown in Fig. 3.

We now make several important observations whose proofs may not be obvious to the reader:

(i) Every rational between 0 and 1 occurs exactly one in the tree.

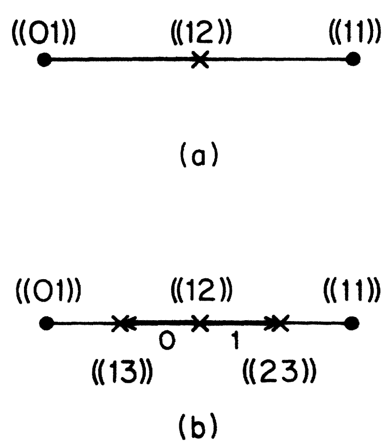

FIG. 1. (a) The line interval bounded by ((0,1)) and ((1,1)) is a level-0 interval of the Farey tree. The first mediant operation generates ((1,2)) and splits the level-0 Farey interval into two level-1 Farey intervals. (b) The mediant operation on the level-1 Farey intervals generates the elements of level 1, ((1,3)) and ((2,3)) which are linked to their mother ((1,2)). Descendants are labeled by arrows.

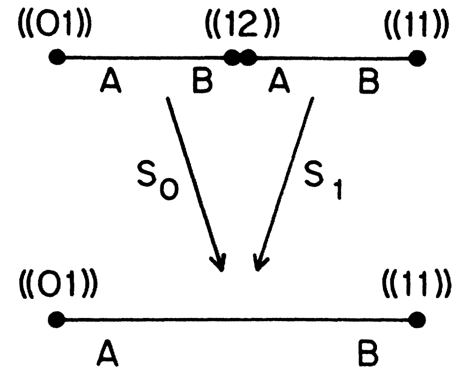

FIG. 2. The shift operations, S_0 and S_1, are illustrated. The labels A and B show how vertices are mapped under the shift operations. Each level-1 Farey interval is mapped into the level-0 Farey interval.

(ii) An element x is on level n of the tree *if and only if* the sum of entries in the continued-fraction representation of x is $n+1$.

(iii) The 2×2 matrix whose rows are the right to left list of integers defining the end points of a Farey interval has determinant 1.

B. Binary Farey address

We now define the binary address associated with a particular rational number. Simply read the path to the rational number from $\frac{1}{2}$, writing down the string of 0's and 1's encountered in going from mother to left or right

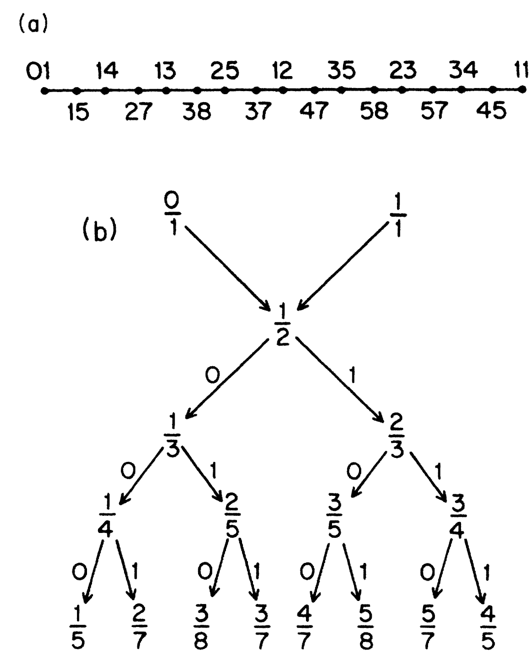

FIG. 3. (a) The elements of the Farey tree through level 3 are shown. (b) A second dimension is added to show the binary organization of rationals and the path of arrows leading to each rational. The entire Farey tree is recovered by the projection of this tree to the unit interval.

daughter. For instance the path to $\frac{3}{8}$ from $\frac{1}{2}$ is left, right, left, so that $\frac{3}{8}=[0,1,0]$ We note the appealing symmetry of the address with respect to reflections about $\frac{1}{2}$:

$$1-[I_0,I_1,I_2,\ldots]=[1-I_0,1-I_1,1-I_2,\ldots] . \quad (4.1)$$

We shall now define an algebraic way of generating this address by generating the *shift map* $s(x)$ with the property that

$$s([I_0,I_1,I_2,\ldots])=[I_1,I_2,I_3,\ldots] . \quad (4.2)$$

The map is obviously 2:1 on the real numbers, maps the unit interval to itself, and maps both the left and right Farey interval of level 1 onto the *entire* interval of level 1. Clearly, there will be two branches to $s(x)$; we define

$$s(x)=\begin{cases} s_0(x) \text{ if } x<\frac{1}{2} \\ s_1(x) \text{ if } x\geq\frac{1}{2} . \end{cases} \quad (4.3)$$

The map $s(x)$ must not only map rationals to rationals but also preserve mediant operations in order to preserve the tree structure. Hence if

$$((p_0,q_0))=((p_1,q_1))\oplus((p_2,q_2))$$

then we also want

$$s((p_0,q_0))=s((p_1,q_1))\oplus s((p_2,q_2)) .$$

But the mediant operation is a simple addition of the 2D integer vectors, hence any linear operation on the integer representation of the fractions will preserve mediant relations. The shift map $s(x)$ can be therefore defined by the action of the two matrices S_0 and S_1 which act on the integers representing the numerator and denominator of the rational number. In order to properly map the top Farey levels, we demand that $S_0(0,1)=(0,1)$, $S_0(1,2)=(1,1)$, $S_1(1,2)=(0,1)$, and $S_1(1,1)=(1,1)$. These matrices are

$$S_0=\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix},\quad S_1=\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} . \quad (4.4)$$

It is simple to show that this induces the transformations of the line interval s_0 and s_1 by the following formula:

$$s_0(x)=\frac{x}{1-x},\quad s_1(x)=2-\frac{1}{x} . \quad (4.5)$$

It is straightforward to show by induction that not only do boundaries of Farey intervals map into boundaries, but that the interior of the Farey intervals map into the interiors. Since it was shown that the shift operations preserve the mediant relations, shift operators map the entire left and right subtrees of rationals onto the whole Farey tree. It is simple to verify that $s(x)$ is indeed the shift operator on the Farey address.

Since we understand the shift operator, it is simpler to work algebraically with addresses and rational approximants, rather than having to construct a complete tree. Let us assume that the following string of compositions can be made, respecting the convention of intervals in Eq. (4.3). $s_{I_N}s_{I_{N-1}}\cdots s_{I_0}((p,q))=((1,2))$ where I_n is 0 or 1. We then know that the address of $((p,q))=[I_0,I_1,\ldots,I_N]$. Since the shift operator is well defined for irrationals as well as rationals, we can assign an infinite address to an irrational number.

The binary address generates a unique rational approximant to an irrational number. Assume $\sigma=[I_0,I_1,I_2,\ldots]$. We define the Nth Farey approximant σ_N by

$$\sigma_N=[I_0,I_1,\ldots,I_N]=((p_N,q_N)) . \quad (4.6)$$

In order to implement this algorithm, we must understand how to compute $((p_N,q_N))$ given only the address. This is easily done with the operators T_0 and T_1 which are inverses to S_0 and S_1.

We define $T_0=S_0^{-1}$ and $T_1=S_1^{-1}$. The Nth rational approximant is then given by

$$(p_N,q_N)=T_{I_0}T_{I_1}\cdots T_{I_N}(1,2) . \quad (4.7)$$

[The vector (1,2) is of course a column vector.] These formulas can be written with the following recursive algorithms.

Algorithm 4.1: Compute (p_N,q_N) given the binary address $[I_0,I_1,\ldots,I_N]$.

Input: $[I_0I_1,\ldots,I_N]$.

Output: $((p_N,q_N))=[I_0,I_1,\ldots,I_N]$.

Initialize: $M=2\times2$ Identity matrix; $n=0$;

do while $(n<N)\{$

if $(I_n=0)M\leftarrow MT_0$;

if $(I_n=1)M\leftarrow MT_1$;

$n\leftarrow n+1$

$\}$;

$(p_N,q_N)=M(1,2)$.

Algorithm 4.2: Compute the binary address and rational approximations to an arbitrary irrational, σ.

Input: σ .

Output:$[I_0,\ldots,I_N,\ldots]$ and (p_N,q_N) .

Initialize: $M=2\times2$ Identity matrix; $N=0$; $x_0=\sigma$;

do while $(x_N\neq\frac{1}{2})\{$

if $(x_N<\frac{1}{2})I_N=0$;

if $(x_N>\frac{1}{2})I_N=1$;

$x_{N+1}=s_{I_N}x_N$;

$M\leftarrow MT_{I_N}$;

$(p_{N+1},q_{N+1})=M(1,2)$;

$N\leftarrow N+1$

$\}$.

Although we shall not do so, it is straightforward to prove the direct relationship between the Farey address and continued-fraction representation. Except for problems at the two ends of the address, we can convert a continued fraction to a Farey address by expanding each integer n_l into n repeated 0's or 1's, switching between 0 or 1 each time. For instance an address $[\ldots,2,3,4,2,\ldots]_{\rm cf}$ in continued-fraction scheme translates to $[\ldots,11000111100,\ldots]$ in Farey binary scheme. This rule breaks down at the two ends of the address. The precise conversion formula can be found in Refs. 8 and 9.

C. The golden mean

In spite of the length of this section we are not primarily interested in approximations to single irrational numbers. We will therefore not discuss in depth the properties of σ_G, the golden mean, although it plays a major role in the theory of rational approximants. We list, however, several properties that are relevant:

(i) $\sigma_G=(\sqrt{5}-1)/2$.

(ii) $s_0 s_1(\sigma_G)=\sigma_G$ so that $\sigma_G=[1,0,1,0,1,0,1,\ldots]$. Thus σ_G is a fixed point of the reflection composed with the shift operator.

(iii) The Nth rational approximant to the golden mean has the largest denominator at the Nth Farey level. Furthermore, in this special case, the continued-fraction approximants coincide with all the Farey approximants.

(iv) σ_G is the "most irrational" number since it is most difficult to approximate by rationals.

There is another way of assigning a binary address where shift operations are given by $s_0^C \leftarrow s_0$ and $s_1^C \leftarrow 1-s_1$ which was introduced by Cvitanović and Feigenbaum.[9,10] It is equivalent to our scheme and generates the same rational approximants, but has the merit that the golden mean is a fixed point of period 1. However, this address breaks the symmetry around $\frac{1}{2}$ and the natural ordering of rationals. For these aesthetic reasons, we prefer to use our labeling scheme. Next we shall extend these ideas to simultaneous rational approximants.

V. GENERALIZED FAREY TREE

A. Definitions and basic results

In this section, we discuss how to generalize the Farey construction of single rational numbers to pairs of relatively prime rationals, i.e., set of 3 integers (p,q,r) where p, q, and r are relatively prime associated with the two "simultaneous rationals," $(p/r,q/r)\equiv((p,q,r))$. We define a "mediant operation" on $((p_1,q_1,r_1))$ and $((p_2,q_2,r_2))$ analogous to the two-dimensional Farey sum:

$$((p_1,q_1,r_1))\oplus((p_2,q_2,r_2))=((p_1+p_2,q_1+q_2,r_1+r_2)) .$$

Read as an operation on triplets of integers, $\oplus$ is of course vector addition. The mediant lies roughly halfway between two rational vectors. We now construct a "tree" of simultaneous rational fractions by the following steps:

(1) We start from the unit square in the plane and the triplet of simultaneous fractions $((0,1,1)),((1,0,1)),((1,1,1))$ placed at each corner of a right triangle as shown in Fig. 4(a). We call this the "level-0" Farey triangle. These vertices will define the tree boundary, just as $((0,1))$ and $((1,1))$ formed the tree boundary in the Farey construction. The rationals in the interior of the triangle are those rationals $((p,q,r))$ for which $p+r\geq r$, $0\leq p\leq r$, and $0\leq q\leq r$, with p, q, and r containing no common divisor. The vectors in the lower left half of the square will be ignored; they can be obtained by a simple reflection operation of the construction outlined below.

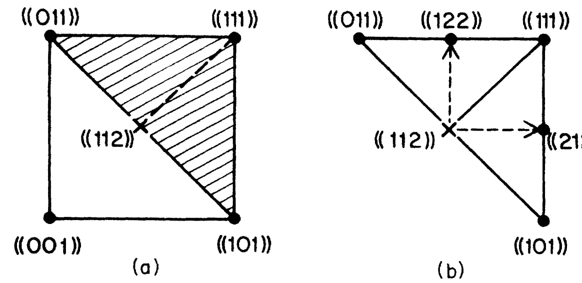

FIG. 4. (a) The shaded area in this schematic diagram denotes the level-0 Farey triangle for the construction of the "generalized Farey tree" and illustrates the first mediant operation, $((0,1,1))\oplus((1,0,1))=((1,1,2))$. (b) The second mediant operation generates two level-1 elements, ((1,2,2)) and ((2,1,2)), which are daughters of ((1,1,2)).

(2) We perform the mediant operation $\oplus$ on the the vertices of the hypotenuse of the Farey triangle, placing $((0,1,1))\oplus((1,0,1))=((1,1,2))$ in the middle of the hypotenuse. The fraction $((1,1,2))$ defines the level-0 element of the tree, analogous to the Farey tree level-0 element.

(3) By connecting ((1,1,2)) and ((1,1,1)) by an arrow, we split the large triangle level-0 triangle into two similar "level-1" Farey triangles, smaller by scale factor $\sqrt{2}$.

(4) We perform the mediant operations on the two hypotenuses of the smaller Farey triangles, generating ((1,2,2)) and ((2,1,2)) as level-1 elements of the tree. The simultaneous fractions ((1,2,2)) and ((2,1,2)) are the two daughters of the level-0 parent ((1,1,2)). [See Fig. 4(b), with the arrow pointing from mother to daughter.] This generates successive levels of daughters and Farey triangles. In contrast to the ordinary Farey construction, the

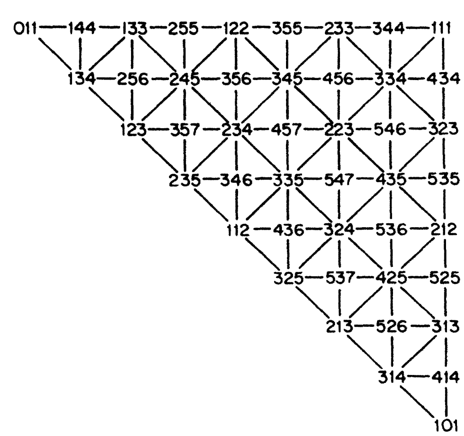

FIG. 5. All elements of the tree up to the fifth level are shown. For clarity, the arrows have been eliminated.

mother *is not* directly involved in the mediant operation; an element is the product of a mediant operation between her grandmother and a more distant parent. The recursive application of (3) and (4) generates the entire tree. All elements of the tree up to fifth level are shown in Fig. 5.

We now make several important observations, which we will shortly prove.

(i) Every pair of simultaneous rationals obeying $p+q\geq r$ appears exactly once in the tree, but those pairs which are not on the boundary of the tree can be reached by *two* distinct paths from ((1,1,2)).

(ii) The 3×3 matrix whose rows consist of the counterclockwise list of vertices of a Farey triangle has determinant 1.

B. Binary representation scheme

The construction of the mediant operation and the analog with the Farey example suggests that we should attempt to find a pair of nonsingular linear fractional transformations, derived from a pair of linear operators on the integer vector defining the simultaneous rationals. Each transformation maps the two level-1 Farey triangles onto the level-0 Farey triangle, maps simultaneous rationals to simultaneous rationals, and preserves mediant operations. [See Fig. 4(a).] If we succeed, we are guaranteed to obtain a binary address in the same manner that an address was generated for the Farey tree since any linear fractional transformation on a pair of rationals preserves the mediant operations. We proceed to construct this binary address.

We shall again take capital letters to refer to matrix operations and lower-case letters refer to mappings of the 2D plane. We further take the lower-case letter to denote the transformation of the plane induced by the matrix of the upper-case letter. For the following discussion, it is helpful to refer to Fig. 6.

The vertices of the level-0 triangle Δ_0 are [(0,1,1),(1,0,1),(1,1,1)]. This triangle is split into two level-1 triangles $\Delta_1^0=[(1,1,1),(0,1,1),(1,1,2)]$ and $\Delta_1^1=[(1,0,1),(1,1,1),(1,1,2)]$. *The order of the listing of vertices is important.* We then define S_0 and S_1 by $S_0\Delta_1^0=\Delta_0$ and $S_1\Delta_1^1=\Delta_0$. Thus, S_0 and S_1 map the vertices of the small triangles into the vertices of the large.

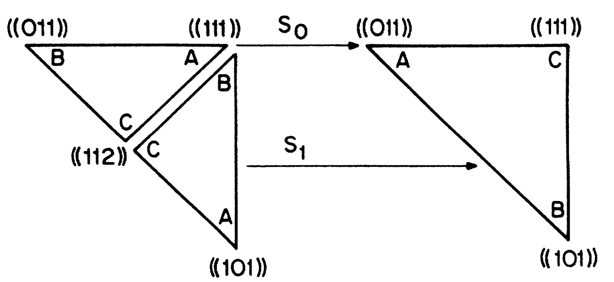

FIG. 6. This diagram illustrates the shift operations s_0 and s_1 which map the two level-1 Farey triangles into the level-0 Farey triangle. The labels A, B, and C show how vertices are mapped under the shift operations.

It is easy to verify that

$$S_0=\begin{bmatrix}-1&0&1\\1&0&0\\0&1&0\end{bmatrix},\quad S_1=\begin{bmatrix}0&1&0\\0&-1&1\\1&0&0\end{bmatrix}. \tag{5.1}$$

Denoting an arbitrary simultaneous rational by $((p,q,r))=(p/r,q/r)$, the operation $s_0((p,q,r))$ corresponds to simple matrix multiplication on the three-dimensional vector (p,q,r), i.e., $s_0((p,q,r))=((S_0(p,q,r)))$. Defining $(x,y)=((p,q,r))=(p/r,q/r)$ it is easily shown that

$$s_0(x,y)=\left[\frac{1-x}{y},\frac{x}{y}\right], \tag{5.2}$$

$$s_1(x,y)=\left[\frac{y}{x},\frac{1-y}{x}\right].$$

We now make the crucial observation that not only the vertices but also the *boundary* and *interior* of the two level-1 Farey triangles map precisely onto the boundary of an interior of the level-0 Farey triangle. That the boundary maps properly is not difficult to see since under s_0, $(y=1)\to(x+y=1)$, $(x+y=1)\to(x=1)$, and $(x=y)\to(y=1)$. The Jacobian of s_0 is $1/y^2$, hence, it is nonsingular in the interior of the domain of s_0. Also since the boundary maps onto the boundary, the interior of the triangle Δ_1^0 is mapped 1:1 onto the interior of the Δ_0 triangle under s_0. Since s_1 can be obtained from s_0 by reflection, analogous statements can be made for s_1.

We can now define a unique map $s(x,y)$ on the triangle Δ_0:

$$s(x,y)=\begin{cases}s_0(x,y) & \text{if } x\leq y\\ s_1(x,y) & \text{if } x>y\end{cases}. \tag{5.3}$$

The domain on the line $x=y$ has been arbitrarily assigned to s_0, slightly breaking the symmetry between x and y.

It is then natural to assign a unique binary address to a pair of simultaneous rationals. Let $((p',q',r'))=s((p,q,r))$. Then it follows that $p'+q'+r'<p+q+r$. It is also easy to verify that if (p,q,r) contains no common divisor, that (p',q',r') also contains no common divisor. Eventually therefore, repeated applications of s to an initial simultaneous rational $((p,q,r))$ obeying $p+q+r\geq 4(p,q\geq 1$ and $r\geq 2)$ must reach ((1,1,2)).

Let us assume that the string of compositions

$$s_{I_N}s_{I_{N-1}}\cdots s_{I_0}((p,q,r))=((1,1,2)). \tag{5.4}$$

For each composition the particular choice s_0 or s_1 was made respecting the choice of domains in Eq. (5.3). We then define $((p,q,r))$ to have the address $((p,q,r))=[I_0,I_1,\ldots,I_N]$. The address shift operator is therefore identified with $s(x)$ since $s[I_0,I_1,\ldots,I_N]=[I_1,I_2,\ldots,I_N]$. One can see that no problems arise extending this definition to irrational numbers since there is no restriction to finite length addresses.

The binary address generates a unique simultaneous rational approximant to a simultaneous irrational analogous to the role of the Farey address in the earlier section. Assume $\sigma=[I_0,\ldots,I_N,\ldots]$ is a simultaneous irrational so that the address does not terminate. Then, the Nth simultaneous rational approximant σ_N is defined to be

$$\sigma_N=[I_0,I_1,\ldots,I_N]=((p_N,q_N,r_N)) . \quad (5.5)$$

In order to actually implement this algorithm, we must understand how to compute $((p,q,r))$ given only the address. This is again done with the operators T_0 and T_1, which are the inverse matrices of S_0 and S_1.

We define $T_0=S_0^{-1}$ and $T_1=S_1^{-1}$ and confirm that

$$T_0=\begin{bmatrix}0&1&0\\0&0&1\\1&1&0\end{bmatrix},\quad T_1=\begin{bmatrix}0&0&1\\1&0&0\\1&1&0\end{bmatrix}, \quad (5.6)$$

and

$$t_0(x,y)=\left[\frac{y}{x+y},\frac{1}{x+y}\right],\quad t_1(x,y)=\left[\frac{1}{x+y},\frac{x}{x+y}\right], \quad (5.7)$$

where $s_0(t_0)$ is identity and $s_1(t_1)$ is identity. The Nth rational approximant is thus given by

$$\begin{bmatrix}p_N\\q_N\\r_N\end{bmatrix}=(T_{I_0}T_{I_1}\cdots T_{I_N})\begin{bmatrix}1\\1\\2\end{bmatrix} \quad (5.8)$$

and $\sigma_N=((p_N,q_N,r_N))$. Note the order in which the matrices T_{I_n} are multiplied.

The whole procedure can be packaged in a neat recursive algorithm. For convenience, we write the algorithms to compute rational approximants from the address and the address from the simultaneous rational or irrational.

Algorithm 5.1: Compute (p_N,q_N,r_N) given the binary address $[I_0,I_1,\ldots,I_N]$.

Input: $[I_0,I_1,\ldots,I_N]$.

Output: $((p_N,q_N,r_N))=[I_0,I_1,\ldots,I_N]$.

Initialize: $M=3\times 3$ identity matrix; $n=0$;

do while $(n<N)\{$

if $(I_n=0) M\leftarrow MT_0$;

if $(I_n=1) M\leftarrow MT_1$;

$n\leftarrow n+1$

$\}$;

$(p_N,q_N,r_N)=M(1,1,2)$

Algorithm 5.2: Compute the binary address and simultaneousrational approximations to an arbitrary pair of irrationals, σ_x and σ_y.

Input: $\sigma(\sigma_x,\sigma_y)$, where $\sigma_x+\sigma_y\geq 1$.

Output: $[I_0,\ldots,I_N,\ldots]$ and (p_N,q_N,r_N) .

Initialize: $M=3\times 3$ identity matrix; $N=0$;

$(x_0,y_0)=(\sigma_x,\sigma_y)$;

do while $(x_N\neq\frac{1}{2}$ and $y_N\neq\frac{1}{2})\{$

if $(x_N<y_N) I_N=0$;

if $(x_N\geq y_N) I_N=1$;

$(x_{N+1},y_{N+1})=s_{I_N}(x_N,y_N)$;

$M\leftarrow MT_{I_N}$;

$(p_{N+1},q_{N+1},r_{N+1})$

$=M(1,1,2)$;

$N\leftarrow N+1$

$\}$.

These operations can be interpreted geometrically. One visualizes a binary address composed of all 0's by a path of arrows. Begin by drawing an arrow between (1,1,1) to (1,1,2). Then make a right turn of 135°, drawing an arrow from the end point of the previous arrow with a length $1/\sqrt{2}$. The triplet of integers at the tip of the new arrow is a new rational approximant. This process can be repeated recursively and converges to a number we call the "spiral mean," which will be discussed later in Sec. VI. (See Fig. 8.) In an arbitrary address, the process is the same, but an address of 1 indicates a left turn by 135°, rather than a right turn. This completes the picture of the geometric binary tree. There are therefore a few points which need to be made before we go on to discuss fixed points and golden-mean generalizations. In contrast to the ordinary Farey tree, every simultaneous *rational* occurs *twice* in the tree, provided we allow all addresses. The mechanism behind this is seen in Fig. 7, where there are two directed paths to ((3,3,5)). With each copy of ((3,3,5)) is associated a distinct subtree of daughters.

There are two inequivalent paths terminating at each rational. The inequality in our algorithm in Eq. (5.3) will determine which one of the two paths will represent the address; changing the inequality (governing the choice of s_0 or s_1 to map the line $x=y$) chooses the other path.

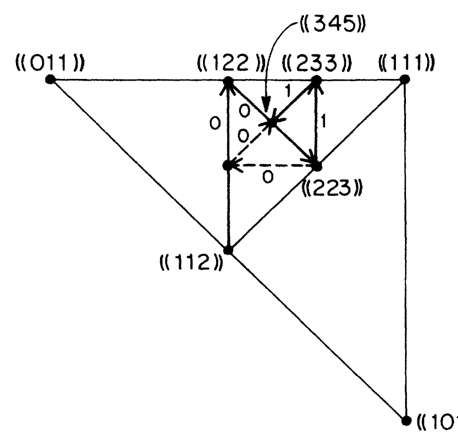

FIG. 7. The directed path to a rational, ((3,4,5))=[0,0,1,1] is shown by successive solid arrows. Dotted arrows show the tail of the second path which leads to the same rational.

For a true simultaneous irrational, there is no ambiguity in the address. Other than being aesthetically objectionable, the double multiplicity in the address of a rational does not seem to pose any problems.

Depending on how the vertices are mapped, we can construct other ways of assigning a binary address. We believe, however, the scheme presented here has the clearest connection to the "generalized golden mean" which we now discuss.

VI. GENERALIZED GOLDEN MEAN

We called the simultaneous irrational whose address is all 0's the spiral mean, since the path to this simultaneous irrational forms a converging spiral. (See Fig. 8.) Since the address is all zeroes, the spiral mean is a fixed point of s_0. Thus $s_0(\sigma_s)=\sigma_s$. It is easy to verify that $\sigma_s=(\tau^{-2},\tau^{-1})$ where τ satisfies $\tau^3-\tau-1=0$. The numerical value of τ is $\tau=1.324718\ldots$. Our choice of a special simultaneous irrational depends on our choice of address. However, this particular simultaneous irrational is more fundamental. We will return to this point after we discuss important properties of the spiral mean and its simultaneous rational approximants.

In order to proceed, we must make some definitions. A *monic polynomial* of degree n is an nth degree polynomial with integer coefficients and the coefficient of x^n is 1. The *algebraic integers* are the roots of monic polynomials. The *minimal polynomial* of an algebraic integer α is the monic polynomial of lowest order with α as a root. The algebraic integers of order n are those whose minimal polynomials are nth order polynomials. It is known that each algebraic integer has a unique minimal polynomial. The roots *conjugate* to an algebraic integer α are the other roots of the minimal polynomial of α. A *Pisot-Vijayaraghavan* (PV) *number* of order n is an algebraic integer of degree n lying outside the unit circle in the complex plane whose conjugate roots are inside the unit circle.[11–13] For each n, there is a unique smallest PV number, x_n. Listed here are the minimal polynomial of the smallest PV number for each degree $n\le 5$:[11] x^2-x-1, x^3-x-1, x^4-x^3-1, $x^5-x^4-x^3+x^2-1$. The numerical value of these are $x_2=(\sqrt{5}-1)/2=1.618359\ldots$, $x_3=1.324718\ldots$, $x_4=1.380277\ldots$, $x_5=1.443269\ldots$. It is known[11] that τ is the smallest PV number for any degree. There is no obvious pattern to the coefficients in these polynomials. We have only been concerned with degree 2 and 3 PV numbers.

Successive rational approximants to the spiral mean obey

$$p_{N+3}=p_{N+1}+p_N,\quad q_{N+3}=q_{N+1}+q_N\,,\quad r_{N+3}=r_{N+1}+r_N\,, \tag{6.1}$$

with $p_{-3}=1$, $p_{-2}=0$, $p_{-1}=1$, $q_{-3}=0$, $q_{-2}=1$, $q_{-1}=1$, $r_{-3}=1$, $r_{-2}=1$, $r_{-1}=1$. The first 17 of these integer triples are shown in Table I. The asymptotic growth rate of the integers is precisely the largest root of the characteristic polynomial of T_0, namely τ, i.e., the smallest third-order PV number. It is not difficult to compute the proximity of the rational approximants to σ_s. The answer is $\sigma_s-((p_N,q_N,r_N))=(\epsilon_x,\epsilon_y)$ where

$$\epsilon_x=A\left[\frac{\lambda_+}{\tau}\right]^N+A^*\left[\frac{\lambda_-}{\tau}\right]^N,\qquad \epsilon_y=B\left[\frac{\lambda_+}{\tau}\right]^N+B^*\left[\frac{\lambda_-}{\tau}\right]^N, \tag{6.2}$$

where $\lambda_\pm$ are the roots conjugate to τ, the numbers A,B are nonzero complex coefficients of order 1, and the asterisk indicates complex conjugate.

It should now be clear why PV numbers are crucial for strong convergence of the rational approximants: Since r_N diverges with N as τ^N, $r^N(\epsilon_x,\epsilon_y)$ converges only if $|\lambda_\pm|<1$. Matrix recursions generate all the rational approximant schemes which appear to be relevant to our type of rational approximant algorithm. By identical ar-

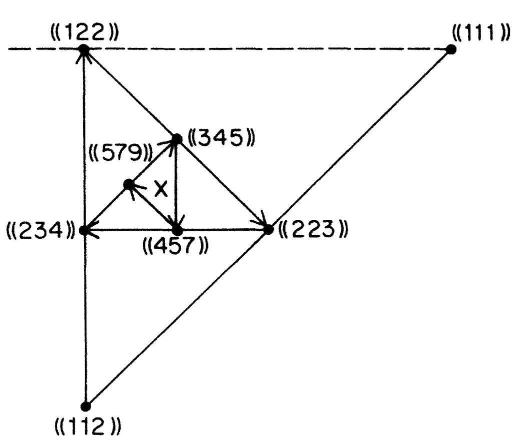

FIG. 8. For the spiral mean , σ_s (denoted by X), the directed path of the simultaneous rational approximants forms a covering spiral.

TABLE I. The first 17 rational approximants of the spiral mean, σ_s, are shown.

N	p_N	q_N	r_N
0	1	1	2
1	1	2	2
2	2	2	3
3	2	3	4
4	3	4	5
5	4	5	7
6	5	7	9
7	7	9	12
8	9	12	16
9	12	16	21
10	16	21	28
11	21	28	37
12	28	37	49
13	37	49	65
14	49	65	86
15	65	86	114
16	86	114	151

guments to that used to derive properties about the spiral mean, the asymptotic growth of the integers forming the rational approximants to any other similar recursive scheme is the leading root of the characteristic polynomial of a matrix M, and is therefore an algebraic integer since a characteristic polynomial is monic. Since the closeness of successive rational approximations to the irrational is given by a formula such as Eq. (6.2), it is seen that there is strong convergence to the simultaneous irrational *if and only if* the leading root of the characteristic polynomial is a PV number. We have thus identified the spiral mean as the cubic irrational with the most robust geometric scaling of the rational approximants.

These properties are certainly shared in one dimension by the golden mean. Using any one of several definitions, we often say that the golden mean is the "most difficult" number to approximate. It seems plausible that this statement can also be made for the spiral mean, but we have not yet proved or seen a precise statement of our assertion.

VII. DISCUSSION

If we are to understand three coupled relatively incommensurate frequencies in a manner analogous to the study of two frequencies whose ratio was the golden mean, the foregoing discussion suggests that the best choice is the spiral mean. The rational approximants to the spiral mean generated by our algorithm are in fact the best rational approximants by the criterion of weak convergence. However, our algorithm does not always give a best rational approximant for arbitrary simultaneous irrationals. No Jacobi-type algorithm is known which satisfies this criterion. There is an algorithm which can extract the best rational approximants.[5] However, the algorithm generates many other approximants as well, and does not appear to be suited for the applications we have in mind. It appears that we must give up either the best rational approximants or give up scaling of approximations. Giving up some of the best rational approximants seems less harmful to analysis if it is scaling and renormalization which will be important to dynamical systems theory.

The Farey triangle construction can be generalized to higher dimension, and for three frequencies and four integers, we work with tetrahedra instead of triangles. We have been able to construct the binary tree in 3D in the same manner as in 2D. We were puzzled to find, however, that a simple, natural extension of our previous analysis of a period 1 fixed point does *not* lead to a characteristic polynomial giving the smallest fourth-order PV number. In fact it does not even lead to a PV number. We do not understand whether this is significant or whether it merely shows that we are not clever enough in generating an acceptable algorithm for a 3D binary tree.

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