# Measurement of the ratio of the speed of sound to the speed of light

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Measurements of the resonance frequencies of the acoustic modes and of the microwave modes of a single cavity can determine $u/c$, the ratio of the speed of sound of a gas to the speed of light. Such measurements with a monatomic gas would determine the thermodynamic temperature $T$ with unprecedented accuracy. By judicious choices of cavity geometry and resonance modes, $u/c$ can be measured to part-per-million accuracy using cavities whose geometry is known only to parts per thousand. These techniques can also be applied to measurements of the universal gas constant $R$. A measurement of $R$ would also require an accurate determination of the average atomic mass of the monatomic gas.

The application of microwave cavity resonators to metrology was developed by Essen and his colleagues beginning in 1945.[1] Essen's 1950 measurement of the speed of light, carried out with a variable-length cylindrical cavity, had an estimated accuracy of 3 parts in $10^6$, and differs from the recently defined value (299 792 458 m/s) by only 1 part in $10^7$. Stroke refined the microwave cavity technique still further, claiming higher precision;[2] however the microwave techniques were superseded by even more precise techniques for measuring the speed of light.[3]

The application of acoustic cavity techniques to metrology is now under active development. Recently Colclough, Quinn, and Chandler[4] measured the universal gas constant $R$ with an accuracy of 25 parts per million. Colclough *et al.* also used a variable-length cylindrical cavity whose length changes were defined and measured accurately within 2 parts per million.

In 1979 Moldover, Waxman, and Greenspan[5] suggested an alternative approach to acoustic metrology. They noted that the resonance frequencies of the radially symmetric modes of a gas-filled spherical cavity were not affected by volume-preserving deformations of the resonator's shell in the first order of perturbation theory, and that these modes are not subject to viscous damping at the cavity wall. They recommended that these modes be used for accurate speed of sound measurements. A stable resonator of known volume is required; however, the modest requirements for sphericity can be attained in ordinary machine shop practice. The volume of the cavity can be determined at ambient temperatures by filling it with a known mass of mercury. Subsequent work (see Fig. 1) with prototype spherical resonators has demonstrated that the internal consistency of speed of sound measurements can approach a few parts per million and that further improvements can be anticipated.[6]

There are no radially symmetric microwave modes. Thus, although Moldover *et al.*[5] noted that microwave resonances are useful for monitoring thermal expansion of a resonator, they were not aware that microwaves can be used in a straightforward way to determine the volume of an imperfect spherical cavity. The purpose of this article is to point out that the average frequency of a nonradial microwave (or acoustic) multiplet is also independent of volume-preserving deformations of the cavity, in the first order of perturbation theory. Thus the volume of an imperfect cavity can indeed be determined if the nearly degenerate frequencies of a microwave multiplet are resolved sufficiently well that their average frequency can be measured accurately.

If microwaves are used to measure the volume of a spherical acoustic resonator, an acoustic experiment with the same resonator can determine, in effect, $u/c$, the ratio of the speed of sound to the speed of light. The state of the art of acoustic measurements in spherical resonators suggests that this ratio can be measured with an accuracy approaching a few parts per million. Thus, if the average atomic mass of a monatomic gas can be determined with comparable accuracy, it should be possible to improve measurements of $R$ and of the thermodynamic temperature without filling a resonator with mercury and weighting it.

In the remainder of this article, we describe recent experimental and theoretical results on the nonradial acoustic modes of a spherical cavity, and then derive a corresponding theoretical result for the electromagnetic modes. Finally, we speculate on the implications of further advances in the state of the art.

Consider a geometrically perfect spherical cavity of radius $a$. If the boundary is a rigid, thermal insulator, the acoustic modes can be described by the eigenfunctions

$$\Psi_{lnm}(\mathbf{r})=j_l(k_{ln}r)Y_{lm}(\theta,\phi)\ , \tag{1}$$

where $j_l(z)$ is the $l$th-order spherical Bessel function and $Y_{lm}$ is a spherical harmonic. The eigenfrequencies $f_{lnm}$ are determined by the condition that the normal derivative of $\Psi$ vanish on the boundary; they can be expressed as $f_{lnm}=uz_{ln}/(2\pi a)$, where $z_{ln}$ is the $n$th root of the equation $j_l'(z)=0$. The eigenfrequencies are independent of the index $m$, thus the modes with indices $\{ln\}$ are $(2l+1)$-fold degenerate. Let the boundary of the cavity be smoothly deformed so that

$$r=a[1-\epsilon f(\theta,\phi)] \; , \tag{2}$$

where $\epsilon$ is a small parameter, and $f(\theta,\phi)$ is a function which describes the perturbed shape. Mehl[7] showed theoretically that although the individual members of a multiplet generally have shape perturbations of order $\epsilon$, the mean frequency shift of the $2l+1$ modes of a multiplet is of order $\epsilon^2$ for volume-preserving deformations. Moldover *et al.*[6] showed experimentally, with a practical resonator, that the mean frequency of the $l=1$ triplets is no more sensitive to geometric perturbations than are the frequencies of the radial ($l=0$) modes. The experimental results are summarized in Fig. 1, which shows fractional deviations of the ratio $u/a$ from a nominal value. Each solid symbol represents the limit of $u/a$ as the pressure of the argon gas used approaches zero for measurements made with a single radial ($l=0$) mode. Each open symbol represents a similar limit based on measurements of the average frequency of a degenerate triplet ($l=1$) set of modes. The mean of the (0,3) through (0,8) values of $\Delta(u/a)/(u/a)$ is $(1.4\pm1.8)\times10^{-6}$. (The quoted error is the rms deviation from the mean.) The corresponding mean of the (1,2) through (1,8) modes is $(5.2\pm1.8)\times10^{-6}$. Thus the two families of modes have mean values of $\Delta(u/a)$ which differ by less than 4 parts in $10^6$. We now argue that comparable measurements with the electromagnetic modes can determine the ratio $c/a$, so that the ratio $u/c$ can be obtained from the two sets of data.

Consider the electromagnetic modes of a cavity resonator with perfectly conducting walls. The electric and magnetic fields can be expressed as linear combinations of eigenfunctions $\mathbf{E}_N$ and $\mathbf{B}_N$, both solutions of the vector

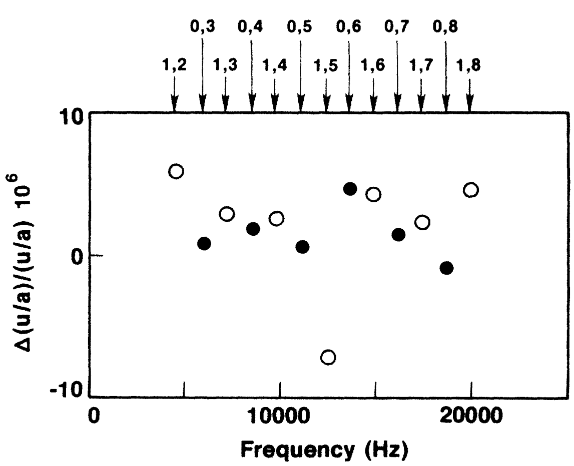


FIG. 1. Fractional deviations (from a nominal value) of the zero-pressure limiting values of $u/a$. The data were determined from acoustic resonances in a one-liter resonator filled with argon near 296 K (Ref. 6). Data from the radially symmetric modes $(0,n)$ are shown as solid circles. The average data from the triply degenerate $(1,n)$ modes are shown as open circles.

Helmholtz equation

$$(\nabla^2+k_N^2)\mathbf{E}_N(\mathbf{r})=\mathbf{0} \; . \tag{3}$$

The fields are related by

$$\begin{aligned} ik_N\mathbf{E}_N &= \nabla\times\mathbf{B}_N \; , \\ -ik_N\mathbf{B}_N &= \nabla\times\mathbf{E}_N \; . \end{aligned} \tag{4}$$

The boundary condition on the cavity wall $S$ is

$$\mathbf{n}\times\mathbf{E}_N=0 \; , \tag{5}$$

which, together with the second of Eqs. (4), implies that the normal component of $\mathbf{B}_N$ vanishes on $S$.

For a geometrically perfect sphere of radius $a$, there are two classes of solutions. The eigenfunctions are

$$\mathbf{B}_{lnm}^{e}=j_l(k_{ln}^{e}r)\mathbf{X}_{lm} \; , \tag{6}$$

for the electric modes (denoted by the superscript $e$), and

$$\mathbf{B}_{lmn}^{b}=(i/k_{ln}^{b})\nabla\times[j_l(k_{ln}^{b}r)\mathbf{X}_{lm}] \tag{7}$$

for the magnetic modes (denoted by the superscript $b$). Here the vector spherical harmonics $\mathbf{X}_{lm}$ are defined by[8]

$$\mathbf{X}_{lm}(\theta,\phi)=\frac{\mathbf{r}\times\nabla Y_{lm}}{i\sqrt{l(l+1)}} \; . \tag{8}$$

These functions are tangential to the radial unit vector $\mathbf{n}_r$. The magnetic mode eigenfunction in Eq. (7) has both radial and tangential components given by

$$\nabla\times[j_l(kr)\mathbf{X}_{lm}]=[i\mathbf{n}_r\sqrt{l(l+1)}j_l(kr)Y_{lm} + g_l(kr)\mathbf{n}_r\times\mathbf{X}_{lm}]/r \; , \tag{9}$$

where

$$g_l(z)=\frac{d}{dz}[zj_l(z)]=j_l(z)+zj_l'(z) \; . \tag{10}$$

The frequencies of the unperturbed modes are $f_{ln}^{\sigma}=ck_{ln}^{\sigma}/(2\pi)$, where $k_{ln}^{b}a$ is any solution of $j_l(k_{ln}^{b}a)=0$, and $k_{ln}^{e}a$ is any solution of $g_l(k_{ln}^{e}a)=0$. As in the acoustic case, the unperturbed eigenfrequencies form $(2l+1)$-fold degenerate multiplets.

For the perturbed problem, the fields $\mathbf{E}$ and $\mathbf{B}$ are solutions of the vector Helmholtz equation (3) which satisfy the boundary condition (5) on a perturbed surface $S$. The changes in the eigenfrequencies can be calculated with boundary-shape perturbation theory.[9,10] Let $S$ be given by Eq. (2), with $\epsilon>0$ and $f(\theta,\phi)\geq0$, so that the perturbed surface lies on or within the unperturbed surface.

The boundary condition (5) is equivalent to Feshbach's boundary condition III A

$$\mathbf{n}\times(\nabla\times\mathbf{B}) \; . \tag{11}$$

For this case Feshbach's Eq. (5.11) gives the frequency shift correct to second order:

$$\frac{f^2}{f_N^2}=1+\frac{D_{NN}}{k_N^2P_N}+\sum_{M(\neq N)}\frac{|D_{NM}|^2}{P_NP_Mk_N^2(k_N^2-k_M^2)} \; . \tag{12}$$

The perturbation matrix and normalization parameters in Eq. (12) are given by

$$D_{MN}=\int_S(\mathbf{n}\times\mathbf{B}_M^*)\cdot\nabla\times\mathbf{B}_N dS\ , \tag{13}$$

$$P_N=\int_V |B_N|^2 dV\ . \tag{14}$$

In order to apply Eq. (12) to the degenerate electromagnetic modes, new linear combinations of the eigenfunctions must be chosen so that each submatrix describing a set of degenerate modes is diagonal. For a first-order calculation, the normalization integral can be taken over the unperturbed resonator volume $V_0$. We thus consider a normalized submatrix

$$\Lambda_{mm'}^{\sigma ln}=\frac{D_{\sigma lnm,\sigma lnm'}}{2(k_{nl}^{\sigma})^2P_{ln}^{\sigma}}\ , \tag{15}$$

where

$$2P_{ln}^{e}=a^3[j_l(k_{ln}^{e}a)]^2[1-l(l+1)/(k_{ln}^{e}a)^2]\ ,\qquad 2P_{ln}^{b}=a^3[j_l'(k_{ln}^{b}a)]^2\ . \tag{16}$$

Let the eigenvalues of $\Lambda^{\sigma ln}$ be $\lambda_s^{\sigma ln}$, where $-l\le s\le l$. From Eq. (12) the first-order frequency shifts are given by

$$\Delta f_{ln}^{\sigma}/f_{ln}^{\sigma}=\lambda_s^{\sigma ln}\ . \tag{17}$$

Calculations of the frequency shifts for specific modes will be considered elsewhere.[11] Here we focus on the average shift for the set of $2l+1$ modes with indices $\sigma ln$. The mean fractional shift is given by

$$(2l+1)\langle\Delta f_{ln}^{\sigma}\rangle/f_{nl}^{\sigma}=\sum_{s=-l}^{l}\lambda_s^{\sigma ln}=\sum_{m=-l}^{l}\Lambda_{mm}^{\sigma ln}\ . \tag{18}$$

The second equality holds because the sum of the eigenvalues is equal to the trace of the matrix, which is invariant under the diagonalizing transformation.

The matrix elements (13) can be simplified by using Eq. (4) for the curl of $\mathbf{B}_N$, leading to

$$\begin{aligned}D_{MN}&=ik_N\int_S\mathbf{n}\times\mathbf{B}_M^*\cdot\mathbf{E}_N dS\\&=ik_N\int_S\mathbf{n}\cdot(\mathbf{B}_M^*\times\mathbf{E}_N)dS\\&=-ik_N\int_{\Delta V}\nabla\cdot(\mathbf{B}_M^*\times\mathbf{E}_N)dV\ .\end{aligned} \tag{19}$$

The second equality follows from a vector identity, and the third follows from the use of the divergence theorem. The volume integral is over $\Delta V$, the region between the unperturbed and perturbed boundaries. Further simplification by expansion and the use of Eqs. (4) leads to

$$D_{MN}=\int_{\Delta V}(k_N^2\mathbf{B}_M^*\cdot\mathbf{B}_N-k_Mk_N\mathbf{E}_M^*\cdot\mathbf{E}_N)dV\ . \tag{20}$$

The diagonal form of Eq. (20) appears in the frequently quoted "perturbation formula."[12]

With the eigenfunctions given by Eqs. (4), (6), and (7), the diagonal matrix elements are

$$D_{MN}=\pm k^2\int_{\Delta V}dV\{[j_l(kr)]^2[\,|\mathbf{X}_{lm}|^2-l(l+1)|Y_{lm}/(kr)|^2]-[g_l(kr)/(kr)]^2|\mathbf{X}_{lm}|^2\}\ , \tag{21}$$

where the plus sign is for the electric modes and the minus sign is for the magnetic modes, and $k$ stands for the appropriate unperturbed eigenvalue $k_{ln}^{\sigma}$. The diagonal matrix elements must be summed over $m$ in taking the trace in Eq. (18). According to the addition theorem for spherical harmonics, the sums of both $|Y_{lm}|^2$ and $|\mathbf{X}_{lm}|^2$ equal $(2l+1)/(4\pi)$. The mean frequency shift thus simplifies to

$$\frac{\langle\Delta f_{ln}^{\sigma}\rangle}{f_{nl}^{\sigma}}=\pm\frac{1}{4\pi P_{ln}^{\sigma}}\int_{\Delta V}dV\{[j_l(kr)]^2[1-l(l+1)/(kr)^2]-[g_l(kr)/(kr)]^2\}\ . \tag{22}$$

Near the boundary, the functions in Eq. (22) differ from their values at $r=a$ only by terms of order $\epsilon$. The integration volume in Eq. (22) is itself of order $\epsilon$. Thus, for a first-order calculation, the fields can be approximated by $j_l(kr)\approx j_l(ka)$ and $g_l(kr)\approx 0$ for the electric modes and $j_l(kr)\approx 0$ and $g_l(kr)\approx kaj_l(ka)$ for the magnetic modes. These cancel similar terms in the normalization constants, leading to

$$\frac{\langle\Delta f_{ln}^{\sigma}\rangle}{f_{nl}^{\sigma}}=\frac{\epsilon}{4\pi}\int d\Omega f(\theta,\phi)+O(\epsilon^2)\ . \tag{23}$$

The right-hand side of this expression is equal to one-third of the fractional decrease of the volume of the resonator. Thus the average frequency shift of each multiplet will be the same up to order $\epsilon^2$ for all perturbations which have the same change in volume, including the perturbation for which $f(\theta,\phi)\equiv 1$, i.e., a uniform contraction. It follows that the average frequency shift of each multiplet is zero in order $\epsilon$ for all perturbations which do not change the volume. This is the same result that was demonstrated for acoustic modes earlier.[7]

For spherical resonators, the main contribution to the resonance linewidths is the thermal boundary layer effect for the acoustic modes, and the analogous skin-depth effect for electromagnetic modes. The electromagnetic modes in an aluminum cavity are typically 10 times sharper than the radially symmetric acoustic modes when the resonator is filled with argon at normal temperature and pressure. We thus expect that experimental techniques similar to those that we have used for acoustic measurements will suffice to measure electromagnetic resonance frequencies with an accuracy of one part in $10^7$. Combined with the acoustic measurements, the ratio $u/c$ can be determined to an accuracy limited by the acoustic measurement to a few parts in $10^6$.

To measure $u/c$ to 1 part in $10^7$ using resonator techniques, one would have to know the ratio of the cavity volume sensed by the acoustic field to the cavity volume sensed by the electromagnetic field to 3 parts in $10^7$. This puts stringent requirements on the mechanical and chemical nature of the resonator's surfaces (e.g., oxide layers would have to be very thin or very well characterized). Furthermore, one would have to characterize the deviations from Ohm's law in the microwave penetration layer at the level of 1% as well as the deviations from the Navier-Stokes equations and their boundary conditions at the level of $0.1\mu/\lambda$. (Here $\mu$ is the mean free path and $\lambda$ is the wavelength of sound.) It is not likely that these advances will occur in the near future.

The possibility of accurately measuring $u/c$ leads us to speculate on the practicality of replacing the defined temperature of the triple point of water by a defined value of the gas constant $R$. At the present time, the Kelvin is defined as 1/273.16 of the temperature of the triple point of water ($T_t = 273.16$ K). Comparisons of triple-point cells[13] indicate that this definition can be realized with a precision on the order of 50 $\mu$K ($\approx 2\times 10^{-7} T_t$). At present, the universal gas constant $R$ is measured at the temperature $T_t$ by, for example, measuring the speed of sound of a monatomic gas and using the relation $R = 3u^2 M/(5T_t)$, where $M$ is the atomic mass of the gas. In order to compete with the present scheme, the temperature of a gas-filled resonator would have to be definable to a precision exceeding that of reproducibility of the realized temperature of the triple point of water. One might imagine doing this by measuring the acoustic and microwave resonance frequencies of a resonator filled with a gas of known atomic mass, for example, $^4$He. An accuracy within 1 part in $10^7$ would be required to compete with the existing practice. (As mentioned in the preceding paragraph, it is not likely that such accuracy will be achieved in the near future.) If such a great advance in the state of the art were achieved, one could then consider defining the gas constant to be a particular value $R_0$. The temperature of a resonator would then be determined from the relation $T = 3u^2 M/(5R_0)$ and $T_t$ would lose its special status. (Of course triple-point cells would continue to be extremely useful devices for the calibration of practical thermometers.) Such a change in metrology would in some ways be analogous to the 1983 agreement to define the speed of light and to abandon the special status of the wavelength of the $2p_{10}$-$5d_5$ transition of Kr.[86] Of course, a competitor to this conceptually attractive scheme would be further refinement of the definition of the triple point of water and its realization, perhaps by specifying more precisely the isotopic composition of the water used, the size and shape of the ice crystals, dissolved gases, etc.

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