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Exact linear stability analysis of the plane-wave Maxwell-Bloch equations for a ring laser

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We perform an exact linear stability analysis of the Maxwell-Bloch equations for a unidirectional ring laser. The model analyzed in this paper is based on the plane-wave assumption for the cavity field, but it allows arbitrary values of all the other parameters for the cavity and for the active medium. This problem had been solved previously only under exact resonance conditions, or in the uniform-field (mean-field) limit, or after adiabatic elimination of the atomic polarization.

I. INTRODUCTION

The appearance of sustained spontaneous oscillations in the solutions of the Maxwell-Bloch equations for a homogeneously broadened two-level system in a unidirectional ring cavity is a well-established mathematical property of this model. A large number of instabilities have been uncovered in the case of active systems, such as the free-running laser and the laser with an injected signal, and passive systems such as optical bistability.[1-6]

A standard procedure in the search for unstable solutions of a nonlinear set of dynamical equations is based on the linear stability analysis of the stationary state. This leads to a characteristic equation for the eigenvalues of the linearized problem; if at least one eigenvalue has a positive real part, the stationary state is unstable.

In the case of the Maxwell-Bloch equations, the linearized problem is very complicated because of the infinite number of degrees of freedom that are coupled by these partial differential equations. So far, solutions have been found only in a number of limiting cases. The first is the uniform-field limit (also known, in current literature, as the mean-field limit), corresponding to a situation in which $\alpha L \ll 1$, $(1-R) \ll 1$, and the ratio $\alpha L/(1-R)$ is arbitrary; α is the unsaturated gain or absorption coefficient per unit length, L is the length of the atomic sample, and R is the reflectivity coefficient of the mirrors. In this limit, the cavity modes decouple from one another and the characteristic equation can be solved by elementary numerical methods.[7] The second case corresponds to the limit in which the atomic polarization can be eliminated adiabatically and the system develops instabilities of the Ikeda type.[8] The last requires exact resonance between the cavity and the atomic medium or, for driven systems between the cavity, the atoms and the injected external field.[9,10]

Unfortunately, as remarked, for example, in Ref. 4, the method developed in the resonant case does not lend itself to an obvious generalization in the presence of detuning where most instabilities arise. Yet, deriving an analytic solution of the linearized Maxwell-Bloch equations is an obviously desirable aim for the purpose of arriving at a unification of the unstable behaviors of the free-running laser, optical bistability, and the laser with an injected signal.

In this paper we report the results of a study of the linearized Maxwell-Bloch equations and show how one can derive an exact characteristic equation that holds for arbitrary values of αL, R, and the detuning parameter, and without adiabatic elimination of the polarization. On the basis of this result, we can address the important issue of the comparison between theory and experiments in more precise terms than previously possible. Of course, numerous attempts have already been made to match the predictions of the Maxwell-Bloch theory with some of the observed instability phenomena. In the case of optical bistability, this search has met with success, as evidenced, especially, by the experiments of Ref. 11. In the case of the laser, satisfactory results have been obtained with an inhomogeneously broadened medium,[12] and more recently with homogeneously broadened active media, as well.[13]

There are examples of lasers, on the other hand, that develop unstable behavior in the neighborhood of threshold even under near-resonance conditions; this is especially evident in dye lasers,[14] CO_2 lasers, and most solid-state lasers (ruby, semiconductor lasers, etc.), in contrast with the theoretical predictions which place the threshold for self-pulsing at a much higher level than is necessary for ordinary laser action.[15,16] This sharp discrepancy between theory and experiments may be traced to a number of possible causes.

(i) The Maxwell-Bloch model imposes stringent requirements that are usually difficult to meet in laboratory practice; an especially obvious constraint is the assumption that the cavity field can be described with sufficient accuracy by a plane wave, and that the population inversion produced by the pump is uniform both longitudinally and radially.

(ii) The available mathematical procedures that probe the stability of the Maxwell-Bloch equations are usually accompanied by additional constraints, such as, for example, the uniform-field approximation, in an attempt to simplify the analytic and numerical labor.

In this paper we remove the uniform-field approximation when dealing with the linearized plane-wave model, and allow for possible longitudinal variations of the pump parameter, as one finds, for example, in the experiments discussed in Ref. 13. On the other hand, our contribution will not address the important issue of transverse effects. In Sec. II we formulate the linearized analysis of the Maxwell-Bloch equations for the free running laser, and in Sec. III we show that this analysis leads to a standard hypergeometric equation. After imposing the appropriate boundary conditions, in Sec. IV we arrive at the required characteristic equation. In Sec. V we extend our analysis to the cases of optical bistability and the laser with an injected signal. The concluding remarks in Sec. VI generalize our earlier description of optical instabilities in terms of gain and dispersion functions.

II. LINEARIZED ANALYSIS OF THE MAXWELL-BLOCH EQUATIONS

A. Description of the model

The evolution of a traveling-wave field interacting with a homogeneously broadened laser medium in a unidirectional ring cavity is described by the well-known Maxwell-Bloch equations in the plane-wave approximation

$$\frac{\partial \mathscr{F}}{\partial z}+\frac{1}{c}\frac{\partial \mathscr{F}}{\partial t}=-\alpha(z)\mathscr{P} , \tag{2.1a}$$

$$\frac{\partial \mathscr{P}}{\partial t}=\gamma_\perp[\mathscr{F}\mathscr{D}-(1+i\tilde{\delta}_{AC})\mathscr{P}] , \tag{2.1b}$$

$$\frac{\partial \mathscr{D}}{\partial t}=-\gamma_{\parallel}[\tfrac{1}{2}(\mathscr{F}^*\mathscr{P}+\mathscr{F}\mathscr{P}^*)+\mathscr{D}+1] , \tag{2.1c}$$

for the complex field envelope $\mathscr{F}(z,t)$, the atomic polarization envelope $\mathscr{P}(z,t)$, and the population difference $\mathscr{D}(z,t)$; $\tilde{\delta}_{AC}=(\omega_A-\omega_C)/\gamma_\perp$ is the scaled detuning between the atomic transition frequency and one of the cavity modes selected as a reference, and $\gamma_\perp$ and $\gamma_{\parallel}$ are the relaxation rates of the polarization and the population difference, respectively. The quantity $\alpha(z)$ represents the small-signal gain per unit length, which is allowed to vary along the longitudinal direction. Equations (2.1) must be supplemented by the boundary condition

$$\mathscr{F}(0,t)=R\mathscr{F}(L,t-\Delta t) , \tag{2.2}$$

where $\Delta t=(\mathscr{L}-L)/c$, $\mathscr{L}$ is the length of the ring resonator, and L is the length of the active medium.

B. Stationary state

The Maxwell-Bloch equations (2.1) are consistent, in general, with a multiplicity of steady states of the form[17]

$$\mathscr{F}(z,t)=\mathscr{F}_{st}(z)e^{-i\delta\omega t} , \tag{2.3a}$$

$$\mathscr{P}(z,t)=\mathscr{P}_{st}(z)e^{-i\delta\omega t} , \tag{2.3b}$$

$$\mathscr{D}(z,t)=\mathscr{D}_{st}(z) , \tag{2.3c}$$

where $\delta\omega$ is the frequency offset between the operating laser frequency and the reference cavity mode, and $\mathscr{F}_{st}(z)$, $\mathscr{P}_{st}(z)$, and $\mathscr{D}_{st}(z)$ are solutions of the equations

$$\frac{d}{dz}\mathscr{F}_{st}=i\frac{\delta\omega}{c}\mathscr{F}_{st}-\alpha(z)\mathscr{P}_{st} , \tag{2.4a}$$

$$0=\mathscr{F}_{st}\mathscr{D}_{st}-(1+i\tilde{\Delta})\mathscr{P}_{st} , \tag{2.4b}$$

$$0=\tfrac{1}{2}(\mathscr{F}^*_{st}\mathscr{P}_{st}+\mathscr{F}_{st}\mathscr{P}^*_{st})+\mathscr{D}_{st}+1 . \tag{2.4c}$$

The detuning parameter $\tilde{\Delta}$ is defined as

$$\tilde{\Delta}=\tilde{\delta}_{AC}-\delta\omega/\gamma_\perp . \tag{2.5}$$

The steady-state atomic variables are given by

$$\mathscr{P}_{st}(z)=-\frac{\mathscr{F}_{st}(z)(1-i\tilde{\Delta})}{1+\tilde{\Delta}^2+|\mathscr{F}_{st}(z)|^2} , \tag{2.6a}$$

$$\mathscr{D}_{st}(z)=-\frac{1+\tilde{\Delta}^2}{1+\tilde{\Delta}^2+|\mathscr{F}_{st}(z)|^2} , \tag{2.6b}$$

while $\mathscr{F}_{st}(z)$ is the solution of Eq. (2.4a) with $\mathscr{P}_{st}(z)$ given by Eq. (2.6a) and subject to the boundary condition

$$\mathscr{F}_{st}(0)=R\mathscr{F}_{st}(L)\exp[i\delta\omega(\mathscr{L}-L)/c] . \tag{2.7}$$

As shown, for example, in Ref. 17 the jth stationary solution for the output intensity is given by ($j=0,\pm1,\pm2,\ldots$)

$$\rho_j^2(L)\equiv|\mathscr{F}_{st}(L)|^2=\frac{2}{1-R^2}[\bar{\alpha}L+(1+\tilde{\Delta}_j^2)\ln R] , \tag{2.8}$$

where

$$\tilde{\Delta}_j=\frac{\tilde{\delta}_{AC}-j\tilde{\alpha}_1}{1+\tilde{\kappa}} , \tag{2.9a}$$

$\tilde{\alpha}_1$ is the intermode spacing in units of $\gamma_\perp$ and $\tilde{\kappa}=c|\ln R|/\mathscr{L}\gamma_\perp$ is the scaled cavity damping rate. The frequency offset of the operating laser line is given by

$$\delta\omega_j=\frac{\kappa\delta_{AC}+j\alpha_1\gamma_\perp}{\gamma_\perp+\kappa} . \tag{2.9b}$$

The parameter $\bar{\alpha}$ is defined as the space average of $\alpha(z)$

$$\bar{\alpha}=\frac{1}{L}\int_0^L dz\,\alpha(z) . \tag{2.9c}$$

C. Linearized equations

In this section we begin our analysis of the linearized laser equations. First, we define the fluctuation variables $\delta\mathscr{F}(z,t),\delta\mathscr{P}(z,t),\delta\mathscr{D}(z,t)$ according to the equations

$$\mathscr{F}(z,t)=[\mathscr{F}_{st}(z)+\delta\mathscr{F}(z,t)]e^{-i\delta\omega t} , \tag{2.10a}$$

$$\mathscr{P}(z,t)=[\mathscr{P}_{st}(z)+\delta\mathscr{P}(z,t)]e^{-i\delta\omega t} , \tag{2.10b}$$

$$\mathscr{D}(z,t)=\mathscr{D}_{st}(z)+\delta\mathscr{D}(z,t) . \tag{2.10c}$$

The resulting fluctuation equations are

$$\frac{\partial}{\partial z}\delta\mathscr{F}+\frac{1}{c}\frac{\partial}{\partial t}\delta\mathscr{F}=i\frac{\delta\omega}{c}\delta\mathscr{F}-\alpha(z)\delta\mathscr{P} , \tag{2.11a}$$

$$\frac{\partial}{\partial t}\delta\mathcal{P}=\gamma_\perp[\mathcal{F}_{st}\delta\mathcal{D}+\mathcal{D}_{st}\delta\mathcal{F}-(1+i\tilde{\Delta})\delta\mathcal{P}] , \quad (2.11b)$$

$$\frac{\partial}{\partial t}\delta\mathcal{D}=-\gamma_\parallel[\tfrac{1}{2}(\mathcal{F}^*_{st}\delta\mathcal{P}+\mathcal{P}_{st}\delta\mathcal{F}^*+\text{c.c.})+\delta\mathcal{D}] . \quad (2.11c)$$

Two additional equations for $\delta\mathcal{F}^*$ and $\delta\mathcal{P}^*$ follow from Eqs. (2.11a) and (2.11b) after complex conjugation. The procedure adopted in this work for the study of the linearized equations follows the usual procedure of linear stability analysis. We seek solutions of the form

$$\begin{bmatrix}\delta\mathcal{F}\\ \delta\mathcal{F}^*\\ \delta\mathcal{P}\\ \delta\mathcal{P}^*\\ \delta\mathcal{D}\end{bmatrix}=e^{\lambda t}\begin{bmatrix}f\\ f^*\\ p\\ p^*\\ d\end{bmatrix} , \quad (2.12)$$

where f, f^*, etc., are solutions of the equations

$$\partial f/\partial z+1/c(\lambda-i\delta\omega)f=-\alpha(z)p , \quad (2.13a)$$

$$\partial f^*/\partial z+1/c(\lambda+i\delta\omega)f=-\alpha(z)p^* , \quad (2.13b)$$

$$(\tilde{\lambda}+1+i\tilde{\Delta})p-\mathcal{F}_{st}d=\mathcal{D}_{st}f , \quad (2.13c)$$

$$(\tilde{\lambda}+1-i\tilde{\Delta})p^*-\mathcal{F}^*_{st}d=\mathcal{D}_{st}f^* , \quad (2.13d)$$

$$(\tilde{\lambda}+\tilde{\gamma})d=-\tfrac{1}{2}\tilde{\gamma}(\mathcal{F}^*_{st}p+\mathcal{P}_{st}f^*+\text{c.c.}) , \quad (2.13e)$$

and where $\tilde{\gamma}=\gamma_\parallel/\gamma_\perp$ and $\tilde{\lambda}=\lambda/\gamma_\perp$. The atomic equations (2.13c)–(2.13e) form a system of algebraic equations that can be solved at once with the result

$$p=T_1(\tilde{\lambda},\tilde{\Delta})f+T_2(\tilde{\lambda},\tilde{\Delta})f^* , \quad (2.14a)$$

$$p^*=T_2^*(\tilde{\lambda},\tilde{\Delta})f+T_1^*(\tilde{\lambda},\tilde{\Delta})f^* , \quad (2.14b)$$

where

$$T_1(\tilde{\lambda},\tilde{\Delta})=-\frac{1}{(1+\tilde{\Delta}^2+|\mathcal{F}_{st}|^2)}\frac{(1+\tilde{\Delta}^2)(\tilde{\lambda}+1-i\tilde{\Delta})-\frac{1}{2}\frac{\tilde{\gamma}}{\tilde{\lambda}+\tilde{\gamma}}|\mathcal{F}_{st}|^2\tilde{\lambda}(1+i\tilde{\Delta})}{(\tilde{\lambda}+1)^2+\tilde{\Delta}^2+\frac{\tilde{\gamma}}{\tilde{\lambda}+\tilde{\gamma}}|\mathcal{F}_{st}|^2(\tilde{\lambda}+1)} \quad (2.15a)$$

and

$$T_2(\tilde{\lambda},\tilde{\Delta})=\frac{1}{2}\frac{\tilde{\gamma}}{\tilde{\lambda}+\tilde{\gamma}}\frac{\mathcal{F}^2_{st}}{(1+\tilde{\Delta}^2+|\mathcal{F}_{st}|^2)}\times\frac{(\tilde{\lambda}+2)(1-i\tilde{\Delta})}{(\tilde{\lambda}+1)^2+\tilde{\Delta}^2+\frac{\tilde{\gamma}}{\tilde{\lambda}+\tilde{\gamma}}|\mathcal{F}_{st}|^2(\tilde{\lambda}+1)} . \quad (2.15b)$$

Next, we substitute p and p^*, given by Eqs. (2.14), on the right-hand side of Eqs. (2.13a) and (2.13b), and obtain the field fluctuation equations

$$df/dz+1/c(\lambda-i\delta\omega)f=-\alpha(z)(T_1f+T_2f^*) , \quad (2.16a)$$

$$df^*/dz+1/c(\lambda+i\delta\omega)f^*=-\alpha(z)(T_2^*f+T_1^*f^*) . \quad (2.16b)$$

Probably the easiest way to proceed at this point is to introduce the polar representation

$$\mathcal{F}(z,t)\equiv\mathcal{F}_{st}(z)+\delta\mathcal{F}(z,t)$$
$$=[\rho_{st}(z)+\delta\rho(z,t)]\exp[i\theta_{st}(z)+\delta\theta(z,t)] . \quad (2.17)$$

To first order in the fluctuation variables, we have

$$\delta\mathcal{F}(z,t)=\exp[i\theta_{st}(z)][\delta\rho(z,t)+i\rho_{st}(z)\delta\theta(z,t)]$$
$$=\exp[i\theta_{st}(z)][\delta\bar{\rho}(z)+i\delta\bar{\theta}(z)]e^{\lambda t} , \quad (2.18)$$

where we have set

$$\delta\rho(z,t)=\delta\bar{\rho}(z)e^{\lambda t} , \quad (2.19a)$$

$$\rho_{st}(z)\delta\theta(z,t)=\rho_{st}(z)\delta\theta(z)e^{\lambda t}=\delta\bar{\theta}(z)e^{\lambda t} . \quad (2.19b)$$

Thus the required polar representation of the field fluctuation variables is

$$f(z)=\exp[i\theta_{st}(z)][\delta\bar{\rho}(z)+i\delta\bar{\theta}(z)] . \quad (2.20)$$

It is now a simple, but lengthy matter to derive the space-dependent equations for $\delta\bar{\rho}(z)$ and $\delta\bar{\theta}(z)$ from Eqs. (2.20) and (2.16). The result is

$$\frac{d}{dz}\delta\bar{\rho}+\frac{\lambda}{c}\delta\bar{\rho}=Q\left[(\tilde{\lambda}+1)\left[1+\tilde{\Delta}^2-\frac{\tilde{\gamma}}{\tilde{\lambda}+\tilde{\gamma}}\rho^2_{st}\right]\delta\bar{\rho}-\tilde{\Delta}\tilde{\lambda}(\tilde{\lambda}+2)\delta\bar{\theta}\right] , \quad (2.21a)$$

$$\frac{d}{dz}\delta\bar{\theta}+\frac{\lambda}{c}\delta\bar{\theta}=Q\left[\tilde{\Delta}(\tilde{\lambda}+2)\left[\tilde{\lambda}+\frac{\tilde{\gamma}}{\tilde{\lambda}+\tilde{\gamma}}\rho^2_{st}\right]\delta\bar{\rho}+\left[(1+\tilde{\Delta}^2)(\tilde{\lambda}+1)+\frac{\tilde{\gamma}}{\tilde{\lambda}+\tilde{\gamma}}\rho^2_{st}\right]\delta\bar{\theta}\right] , \quad (2.21b)$$

where

$$Q=\frac{\alpha(z)}{1+\tilde{\Delta}^2+\rho^2_{st}}\frac{1}{(\tilde{\lambda}+1)^2+\tilde{\Delta}^2+\frac{\tilde{\gamma}}{\tilde{\lambda}+\tilde{\gamma}}\rho^2_{st}(\tilde{\lambda}+1)} . \quad (2.21c)$$

III. SOLUTION OF THE LINEARIZED EQUATIONS

Equations (2.21) are linear equations. They are complicated to solve because of the implicit dependence of the steady-state intensity ρ^2_{st} on the longitudinal coordinate. This problem can be overcome with an extension of the technique used in Refs. 9 and 10. First we introduce the new dependent variables $r(z)$ and $s(z)$ according to the equations

$$\delta\bar{\rho}(z)=r(z)\exp(-\lambda z/c)\ , \tag{3.1a}$$

$$\delta\bar{\theta}(z)=s(z)\exp(-\lambda z/c)\ , \tag{3.1b}$$

with the result

$$\frac{d}{dz}r(z)=Q\left[(\tilde{\lambda}+1)\left[1+\tilde{\Delta}^2-\frac{\tilde{\gamma}}{\tilde{\lambda}+\tilde{\gamma}}\rho_{\rm st}^2\right]r(z)-\tilde{\Delta}\tilde{\lambda}(\tilde{\lambda}+2)s(z)\right]\ , \tag{3.2a}$$

$$\frac{d}{dz}s(z)=Q\left[\tilde{\Delta}(\tilde{\lambda}+2)\left[\tilde{\lambda}+\frac{\tilde{\gamma}}{\tilde{\lambda}+\tilde{\gamma}}\rho_{\rm st}^2\right]r(z)+\left[(1+\tilde{\Delta}^2)(\tilde{\lambda}+1)+\frac{\tilde{\gamma}}{\tilde{\lambda}+\tilde{\gamma}}\rho_{\rm st}^2\right]s(z)\right]\ . \tag{3.2b}$$

Next we change the independent variable from z to

$$x=\rho_{\rm st}^2(z)\ . \tag{3.3}$$

This transformation can be accomplished trivially using the chain rule of differentiation. For this purpose we need to construct the quantity dx/dz which can be obtained using Eqs. (2.4a) and (2.6a) after transforming the steady-state-field amplitude into polar form

$$\mathscr{F}_{\rm st}(z)=\rho_{\rm st}(z)\exp[i\theta_{\rm st}(z)]\ . \tag{3.4}$$

In fact, the steady-state-field equation for the modulus of $\mathscr{F}_{\rm st}(z)$ is

$$\frac{d}{dz}\rho_{\rm st}(z)=\frac{\alpha(z)\rho_{\rm st}(z)}{1+\tilde{\Delta}^2+\rho_{\rm st}^2(z)} \tag{3.5}$$

so that

$$\frac{d}{dz}x=\frac{d}{dz}\rho_{\rm st}^2=2\alpha(z)\frac{x(z)}{1+\tilde{\Delta}^2+x(z)}\ . \tag{3.6}$$

The transformed version of Eqs. (3.2) is

$$\frac{dr}{dx}=\frac{1}{2xP}[(a_1+a_2x)r(x)+b_1s(x)]\ , \tag{3.7a}$$

$$\frac{ds}{dx}=\frac{1}{2xP}[(c_1+c_2x)r(x)+(d_1+d_2x)s(x)]\ , \tag{3.7b}$$

where

$$P(x)=p_1+p_2x \tag{3.8}$$

and

$$p_1=(\tilde{\lambda}+1)^2+\tilde{\Delta}^2,\ \ p_2=\tilde{\gamma}(\tilde{\lambda}+1)/(\tilde{\lambda}+\tilde{\gamma})\ , \tag{3.9a}$$

$$a_1=(1+\tilde{\Delta}^2)(\tilde{\lambda}+1),\ \ a_2=-p_2\ , \tag{3.9b}$$

$$b_1=-\tilde{\Delta}\tilde{\lambda}(\tilde{\lambda}+2)\ , \tag{3.9c}$$

$$c_1=-b_1,\ \ c_2=\tilde{\Delta}\tilde{\gamma}(\tilde{\lambda}+2)/(\tilde{\lambda}+\tilde{\gamma})\ , \tag{3.9d}$$

$$d_1=a_1,\ \ d_2=\tilde{\gamma}/(\tilde{\lambda}+\tilde{\gamma})\ . \tag{3.9e}$$

A remarkable feature of Eqs. (3.7) is that it leads to a decoupled second-order differential equation for $r(x)$ that can be solved analytically in closed form. After deriving the appropriate solution for $r(x)$, the second function $s(x)$ can be obtained as follows:

$$s(x)=1/b_1[2xP\,dr/dx-(a_1+a_2x)r(x)]\ . \tag{3.10}$$

After some elementary calculations, the second-order differential equation for $r(x)$ takes the form

$$\frac{d^2r}{dx^2}+\frac{h_1x+h_2}{x(p_1+p_2x)}\frac{dr}{dx}+\frac{l_1x^2+l_2x+l_3}{[x(p_1+p_2x)]^2}r(x)=0\ , \tag{3.11}$$

where

$$h_1=\tfrac{1}{2}\tilde{\gamma}\frac{5\tilde{\lambda}+4}{\tilde{\lambda}+\tilde{\gamma}}\ , \tag{3.12a}$$

$$h_2=\tilde{\lambda}(\tilde{\lambda}+1-\tilde{\Delta}^2)\ , \tag{3.12b}$$

$$l_1=\frac{1}{4}\left[\frac{\tilde{\gamma}}{\tilde{\lambda}+\tilde{\gamma}}\right]^2(\tilde{\lambda}+1)(2\tilde{\lambda}+1)\ , \tag{3.12c}$$

$$l_2=\frac{1}{4}\frac{\tilde{\gamma}}{\tilde{\lambda}+\tilde{\gamma}}[(\tilde{\lambda}+1)(2\tilde{\lambda}^2+3\tilde{\lambda}+2)+\tilde{\Delta}^2(\tilde{\lambda}^3+3\tilde{\lambda}^2+5\tilde{\lambda}+2)]\ , \tag{3.12d}$$

$$l_3=\tfrac{1}{4}[(1+\tilde{\Delta}^2)^2(\tilde{\lambda}+1)^2+\tilde{\Delta}^2\tilde{\lambda}^2(\tilde{\lambda}+2)^2]\ . \tag{3.12e}$$

Equation (3.11) can be recognized as the Riemann equation. With the further change of independent variable

$$\xi=p_2x/(p_2x+p_1) \tag{3.13}$$

the regular singular points of Eq. (3.11)

$$x=0,\ \ x=-p_1/p_2,\ \ x=\infty \tag{3.14a}$$

can be mapped into the canonical singular points

$$\xi=0,\ \ \xi=\infty,\ \ \xi=1\ , \tag{3.14b}$$

respectively. Thus, Eqs. (3.11) takes the form

$$\frac{d^2r}{d\xi^2}+\frac{t_1\xi+t_2}{\xi(1-\xi)}\frac{dr}{d\xi}+\frac{t_3\xi^2+t_4\xi+t_5}{[\xi(1-\xi)]^2}r=0\ , \tag{3.15}$$

where

$$t_1=(h_1p_1-h_2p_2-2p_1p_2)/p_1p_2\ , \tag{3.16a}$$

$$t_2=h_2/p_1\ , \tag{3.16b}$$

$$t_3=(l_1p_1^2-l_2p_1p_2+l_3p_2^2)/(p_1p_2)^2=0\ , \tag{3.16c}$$

$$t_4=(l_2p_1p_2-2l_3p_2^2)/(p_1p_2)^2\ , \tag{3.16d}$$

$$t_5=l_3/p_1^2\ . \tag{3.16e}$$

Equation (3.15) can be reduced to a standard hypergeometric equation whose linearly independent solutions are

$$r_1(\xi)=\xi^{\alpha}(1-\xi)^{\beta}F(a,b;c;\xi) , \quad (3.17a)$$

$$r_2(\xi)=\xi^{\alpha}(1-\xi)^{\beta}\xi^{1-c}F(a-c+1,b-c+1;2-c;\xi) , \quad (3.17b)$$

where α is either one of the two roots

$$\alpha_{\pm}=[(\tilde{\lambda}+1)(1+\tilde{\Delta}^2)\pm\tilde{\Delta}\tilde{\lambda}(\tilde{\lambda}+2)]/[(\tilde{\lambda}+1)^2+\tilde{\Delta}^2] , \quad (3.18a)$$

and β is either

$$\beta_+=\tfrac{1}{2}(2\tilde{\lambda}+1)/(\tilde{\lambda}+1)$$

or (3.18b)

$$\beta_-=\tfrac{1}{2} .$$

With the selections $\alpha=\alpha_+$ and $\beta=\beta_-$ the parameters of the hypergeometric function take the form

$$a=\tfrac{1}{2}(\tilde{\lambda}+2)(\tilde{\lambda}+1+\tilde{\Delta}^2+i\tilde{\lambda}\tilde{\Delta})/[(\tilde{\lambda}+1)^2+\tilde{\Delta}^2] , \quad (3.19a)$$

$$b=\tfrac{1}{2}(\tilde{\lambda}+2)[-\tilde{\lambda}\tilde{\Delta}^2/(\tilde{\lambda}+1)+i\tilde{\lambda}\tilde{\Delta}]/[(\tilde{\lambda}+1)^2+\tilde{\Delta}^2] , \quad (3.19b)$$

$$c=1+i\tilde{\lambda}\tilde{\Delta}(\tilde{\lambda}+2)/[(\tilde{\lambda}+1)^2+\tilde{\Delta}^2] . \quad (3.19c)$$

Thus the required solutions of Eqs. (3.2) are

$$r(x)=K_1\Phi_1(x,\tilde{\lambda})+K_2\Phi_2(x,\tilde{\lambda}) , \quad (3.20a)$$

$$s(x)=K_1\Psi_1(x,\tilde{\lambda})+K_2\Psi_2(x,\tilde{\lambda}) , \quad (3.20b)$$

where $\Phi_i(x,\tilde{\lambda})$ and $\Psi_1(x,\tilde{\lambda})$ $(i=1,2)$ are combinations of hypergeometric functions whose explicit form is given in the Appendix.

IV. BOUNDARY CONDITIONS AND THE CHARACTERISTIC EQUATION

The boundary condition (2.2) for the cavity field implies the following constraint for the field fluctuation variable $\delta\mathcal{F}$:

$$\delta\mathcal{F}(0,t)=R\,\delta\mathcal{F}(0,t-(\mathcal{L}-L)/c)\exp[i\delta\omega(\mathcal{L}-L)/c] . \quad (4.1)$$

According to Eq. (2.12), we have set $\delta\mathcal{F}(z,t)=\exp(\lambda t)f(z)$ so that the space-dependent part of the fluctuation variable must satisfy the relation

$$f(0)=Rf(L)\exp[-\lambda(\mathcal{L}-L)/c]\exp[i\delta\omega(\mathcal{L}-L)/c] . \quad (4.2)$$

On the other hand, according to Eq. (2.20), the function $f(z)$ and the fluctuation variables $\delta\bar{\rho}(z)$ and $\delta\bar{\theta}(z)$ are related by

$$f(z)=\exp[i\theta_{st}(z)][\delta\bar{\rho}(z)+i\delta\bar{\theta}(z)] . \quad (4.3)$$

It follows that the polar representation of the boundary condition takes the form

$$\delta\bar{\rho}(0)+i\delta\bar{\theta}(0)=R\exp[-\lambda(\mathcal{L}-L)/c][\delta\bar{\rho}(L)+i\delta\bar{\theta}(L)] , \quad (4.4)$$

or, in terms of $r(z)$ and $s(z)$

$$r(0)+is(0)=R\exp(-\lambda\mathcal{L}/c)[r(L)+is(L)] . \quad (4.5)$$

The boundary condition for $f^*(z)$ leads to

$$r(0)-is(0)=R\exp(-\lambda\mathcal{L}/c)[r(L)-is(L)] , \quad (4.6)$$

so that, after proper addition and subtraction of Eqs. (4.5) and (4.6), we arrive at the required constraints for the functions $r(z)$ and $s(z)$

$$r(0)=R\exp(-\lambda\mathcal{L}/c)r(L) , \quad (4.7a)$$

$$s(0)=R\exp(-\lambda\mathcal{L}/c)s(L) , \quad (4.7b)$$

With the help of Eqs. (3.20a) and (3.20b) these boundary conditions can also be put into the form

$$K_1\Phi_1(\rho_{st}^2(0),\tilde{\lambda})+K_2\Phi_2(\rho_{st}^2(0),\tilde{\lambda})=R\exp(-\lambda\mathcal{L}/c)[K_1\Phi_1(\rho_{st}^2(L),\tilde{\lambda})+K_2\Phi_2(\rho_{st}^2(L),\tilde{\lambda})] , \quad (4.8a)$$

$$K_1\Psi_1(\rho_{st}^2(0),\tilde{\lambda})+K_2\Psi_2(\rho_{st}^2(0),\tilde{\lambda})=R\exp(-\lambda\mathcal{L}/c)[K_1\Psi_1(\rho_{st}^2(L),\tilde{\lambda})+K_2\Psi_2(\rho_{st}^2(L),\tilde{\lambda})] . \quad (4.8b)$$

Equations (4.8) form a system of two homogeneous linear equations for the weighting factors K_1 and K_2 which allows nontrivial solutions if and only if the determinant of the coefficients vanishes identically. This solvability condition which takes the form

$$[\Phi_{10}-R\exp(-\lambda\mathcal{L}/c)\Phi_{1L}][\Psi_{20}-R\exp(-\lambda\mathcal{L}/c)\Psi_{2L}]-[\Phi_{20}-R\exp(-\lambda\mathcal{L}/c)\Phi_{2L}][\Psi_{10}-R\exp(-\lambda\mathcal{L}/c)\Psi_{1L}]=0 , \quad (4.9)$$

provides the required characteristic equation for the eigenvalues of the linearized problem. The symbols Φ_{10}, Φ_{1L}, etc., are shorthand notations for $\Phi_1(\rho_{st}^2(0),\tilde{\lambda})$, $\Phi_1(\rho_{st}^2(L),\tilde{\lambda})$, etc., where the input and output steady-state intensity are given by [see Eqs. (2.8) and (2.9)]

$$\rho_j^2(0)=R^2\rho_j^2(L) , \quad (4.10a)$$

$$\rho_j^2(L)=2/(1-R^2)[\bar{\alpha}L+(1+\tilde{\Delta}^2)\ln R] , \quad (4.10b)$$

and

$$\tilde{\Delta}_j=(\tilde{\delta}_{AC}-j\tilde{\alpha}_1)/(1+\tilde{\kappa}),\ \ j=0,\pm1,\pm2,\ldots . \quad (4.10c)$$

V. GENERALIZATION TO EXTERNALLY DRIVEN SYSTEMS

In the preceding sections we have discussed the case of the free-running laser. This analysis can be generalized without difficulty to include the laser with an injected signal and optical bistability. In both cases, a coherent external field with a carrier frequency ω_0 is injected into the cavity. While for a free-running laser we have chosen the frequency of a selected cavity mode as the carrier frequency of the cavity field, in the case of driven systems it is more convenient to select ω_0 as the reference carrier frequency. Thus, the Maxwell-Bloch equations take the form

$$\frac{\partial \mathscr{F}}{\partial z}+\frac{1}{c}\frac{\partial \mathscr{F}}{\partial t}=-\alpha(z)\mathscr{P} , \tag{5.1a}$$

$$\frac{\partial \mathscr{P}}{\partial t}=\gamma_\perp[\mathscr{F}\mathscr{D}-(1+i\tilde{\Delta})\mathscr{P}] , \tag{5.1b}$$

$$\frac{\partial \mathscr{D}}{\partial t}=-\gamma_{\parallel}[\tfrac{1}{2}(\mathscr{F}^*\mathscr{P}+\mathscr{F}\mathscr{P}^*)+\mathscr{D}+1] , \tag{5.1c}$$

where $\tilde{\Delta}$ is the atomic detuning parameter

$$\tilde{\Delta}=(\omega_A-\omega_0)/\gamma_\perp \tag{5.2}$$

and, in the case of optical bistability, $\alpha(z)$ is (usually) constant and negative, i.e., $-\alpha$ represents the unsaturated field absorption coefficient per unit length. The boundary condition is

$$\mathscr{F}(0,t)=(1-R)Y+Re^{-i\delta_0}\mathscr{F}(L,t-\Delta t) , \tag{5.3}$$

where Y is the normalized amplitude of the input field, δ_0 is the cavity detuning parameter

$$\delta_0=\frac{\omega_C-\omega_0}{c/\mathscr{L}} \tag{5.4}$$

and ω_C is the cavity frequency that lies nearest to ω_0. The steady-state solutions are given by Eqs. (2.3) with $\delta\omega=0$. A detailed description of the stationary solutions of Eqs. (5.1) for optical bistability and the laser with injected signal can be found in Refs. 3 and 18, respectively.

The linear stability analysis can be carried out in an entirely similar way as done for the free-running laser, and leads again to the same equations (3.7). Here, of course, we must use the definition of $\tilde{\Delta}$ given by Eq. (5.2). The boundary conditions for $r(z)$ and $s(z)$ instead are more complicated. They take the form

$$r(0)=Re^{-\lambda\mathscr{L}/c}[\cos(\zeta)r(L)-\sin(\zeta)s(L)] , \tag{5.5a}$$

$$s(0)=Re^{-\lambda\mathscr{L}/c}[\sin(\zeta)r(L)-\cos(\zeta)s(L)] , \tag{5.5b}$$

where

$$\zeta=\theta_{\rm st}(L)-\theta_{\rm st}(0)-\delta_0 . \tag{5.6}$$

Hence, the characteristic equation takes the form

$$\{\Phi_{10}-Re^{-\lambda\mathscr{L}/c}[\cos(\zeta)\Phi_{1L}-\sin(\zeta)\Psi_{1L}]\}\{\Psi_{20}-Re^{-\lambda\mathscr{L}/c}[\sin(\zeta)\Phi_{2L}-\cos(\zeta)\Psi_{2L}]\}$$
$$-\{\Phi_{20}-Re^{-\lambda\mathscr{L}/c}[\cos(\zeta)\Phi_{2L}-\sin(\zeta)\Psi_{2L}]\}\{\Psi_{10}-Re^{-\lambda\mathscr{L}/c}[\sin(\zeta)\Phi_{1L}-\cos(\zeta)\Psi_{1L}]\}=0 . \tag{5.7}$$

Note that Eq. (4.9) is a special case of Eq. (5.7) obtained by setting $\zeta=0$ and interpreting the detuning parameter $\tilde{\Delta}$ according to Eq. (2.9a).

VI. DISCUSSION AND CONCLUSIONS

With the parameters of the hypergeometric functions derived in Sec. III, Eq. (4.9) or (5.7) can be solved numerically for the unknown eigenvalues $\tilde{\lambda}$. A possible strategy is as follows. Let

$$\eta=R\exp(-\tilde{\lambda}\mathscr{L}/c) , \tag{6.1a}$$

$$W_1=\Phi_{1L}\Psi_{2L}-\Phi_{2L}\Psi_{1L} , \tag{6.1b}$$

$$W_2=\Phi_{20}\Psi_{1L}+\Phi_{2L}\Psi_{10}-\Phi_{10}\Psi_{2L}-\Phi_{1L}\Psi_{20} , \tag{6.1c}$$

$$W_3=\Phi_{10}\Psi_{20}-\Phi_{20}\Psi_{10} , \tag{6.1d}$$

and write Eq. (4.9) or (5.7) in the form of the quadratic equation

$$W_1\eta^2+W_2\eta+W_3=0 \tag{6.2}$$

whose solutions are

$$\eta_\pm=\frac{-W_2\pm(W_2^2-4W_1W_3)^{1/2}}{2W_1}=Re^{-\lambda_\pm\mathscr{L}/c} . \tag{6.3}$$

An alternative implicit form of the eigenvalue equation follows directly from Eq. (6.3):

$$\tilde{\lambda}_{\pm,n}=-i\tilde{\alpha}_n-\tilde{\kappa}-(c/\mathscr{L}\gamma_\perp)\ln\eta_\pm(\tilde{\lambda}_{\pm,n})$$
$$n=0,\pm1,\pm2,\ldots , \tag{6.4}$$

where $\tilde{\kappa}$ is the cavity linewidth in units of $\gamma_\perp$

$$\tilde{\kappa}=\frac{c\,|\ln R\,|}{\mathscr{L}\gamma_\perp} . \tag{6.5}$$

The index n labels the different cavity resonances, with $n=0$ corresponding to the resonant mode (usually the cavity mode that lies nearest the center of the atomic line), and

$$\tilde{\alpha}_n=2\pi cn/\mathscr{L}\gamma_\perp . \tag{6.6}$$

In our numerical investigations we have adopted a graphical technique for calculating the roots of Eq. (6.4) which is, of course, equivalent to two real equations of the form $\mathrm{Re}[F(\tilde{\lambda})]=0$ and $\mathrm{Im}[F(\tilde{\lambda})]=0$. The technique is based on the selection of an initial guess for the unknown variable $\mathrm{Re}(\tilde{\lambda}_{\pm,n})$, followed by the graphical display of the two curves $\mathrm{Re}[F(\tilde{\lambda})]$ and $\mathrm{Im}[F(\tilde{\lambda})]$ as functions of $\mathrm{Im}(\tilde{\lambda}_{\pm,n})$ over an appropriate range. The objective is to force $\mathrm{Re}[F(\tilde{\lambda})]$ and $\mathrm{Im}[F(\tilde{\lambda})]$ to cross the horizontal $\mathrm{Im}(\tilde{\lambda}_{\pm,n})$ axis in the same place in order to insure that both $\mathrm{Re}[F(\tilde{\lambda})]$ and $\mathrm{Im}[F(\tilde{\lambda})]$ will vanish simultaneously. This is easily accomplished in a few manual iterations by varying the initial guess $\mathrm{Re}(\tilde{\lambda}_{\pm,n})$, practically to any required level of accuracy. As a test of the correctness of the numerical construction, we have reconstructed the exact eigenvalues of the resonant Maxwell-Bloch equations which had been calculated in an entirely different way.[10,19] A survey of the stability properties of the exact Maxwell-Bloch problem shows that even away from the uniform-field limit, laser instabilities develop only at much higher pumping levels than are needed to produce laser action, so that no major differences, except for quantitative details, have become apparent relative to the results discussed previously in Refs. 10 and 17. Thus, as anticipated in the Introduction, the results of the recent investigations of the plane-wave Maxwell-Bloch equations can be interpreted as a strong indication that realistically low thresholds for unstable behaviors may well require additional physical inputs, or the removal of some of the traditional assumptions (e.g., the plane-wave approximation).

An interesting aspect of our analysis is that as a consequence of Eq. (3.3), the gain parameter $\alpha(z)$ disappears from the linearized equations [see Eqs. (3.7)]. Thus, on the basis of Eq. (2.8), we see that the gain parameter $\alpha(z)$ influences the stationary solutions and their stability only by way of its space average $\bar{\alpha}$ defined by Eq. (2.9c). This implies that a longitudinal space variation in the gain does not change the qualitative structure of the results obtained in the case of a uniform gain, with respect to the stationary behavior and the emergence of instabilities.

As a final point, we generalize the description of optical instabilities in terms of gain and dispersion functions, formulated in Ref. 7 under uniform-field conditions. For this purpose, we focus on the boundary of the instability domain in the space of the parameters. This is defined by the condition $\mathrm{Re}\tilde{\lambda}=0$, i.e.,

$$\tilde{\lambda}_{\pm,n}=i\tilde{\nu}_{\pm,n} \;, \tag{6.7}$$

with $\tilde{\nu}_{\pm,n}$ real. If we substitute the ansatz (6.7) into Eq. (6.4) and equate the real and imaginary parts separately, we obtain two equations of the type

$$1=\mathscr{G}_{\pm}(\tilde{\nu}_{\pm,n}) \;, \tag{6.8a}$$

$$(\tilde{\nu}_{\pm,n}+\tilde{\alpha}_n)/\tilde{\kappa}=\mathscr{D}_{\pm}(\tilde{\nu}_{\pm,n}) \;, \tag{6.8b}$$

where the gain functions $\mathscr{G}_{\pm}$ are defined by

$$\mathscr{G}_{\pm}(\tilde{\nu})=\mathrm{Re}[\ln\eta_{\pm}(i\tilde{\nu})]/\ln R \;, \tag{6.9a}$$

and the dispersion functions $\mathscr{D}_{\pm}$ are given by

$$\mathscr{D}_{\pm}(\tilde{\nu})=\mathrm{Im}[\ln\eta_{\pm}(i\tilde{\nu})]/\ln R \;. \tag{6.9b}$$

The plus or minus signs must be selected concurrently in Eqs. (6.8a) and (6.8b). The stability boundary in the space of the system parameters can be constructed from Eqs. (6.7) with the following procedure. For the sake of concreteness, we fix the values of all the parameters except for one, which we denote by the symbol δ, and search for the value of δ that lies on the stability boundary. Also, for definiteness, we consider the positive sign option in Eqs. (6.8a) and (6.8b). For a chosen value of δ, we can solve Eq. (6.8b) graphically with respect to $\tilde{\nu}$ by seeking the intersects of the curve $\mathscr{D}_{+}$ with the set of parallel equispaced straight lines $(\tilde{\nu}+\tilde{\alpha}_n)/\tilde{\kappa}$. Let $\bar{\nu}(\delta)$ denote a solution; if the value of $\mathscr{G}_{+}(\bar{\nu}(\delta))$ is exactly equal to unity, the selected value of δ lies on the stability boundary. If this is not the case, one gradually varies δ until the condition $\mathscr{G}_{\pm}(\bar{\nu})=1$ is satisfied. This can be done in a few iterations within any prescribed accuracy.

The functions $\mathscr{G}_{\pm}$ are even with respect to $\bar{\nu}$, while the functions $\mathscr{D}_{\pm}$ are odd. Thus, if $\bar{\nu}$ is a solution of Eq. (6.8b) for $n=\bar{n}$, also $-\bar{\nu}$ is a solution of the same equation for $n=-\bar{n}$. Note that if we consider the resonant mode $\tilde{\alpha}_n=0$, the corresponding solutions of Eq. (6.8b) for $\bar{\nu}>0$ requires that $\mathscr{D}_{\pm}(\bar{\nu})$ be positive. If we consider also the nonresonant modes $\tilde{\alpha}_n\neq 0$, solutions of Eq. (6.8b) for $\bar{\nu}>0$ exist also when $\mathscr{D}_{\pm}(\bar{\nu})$ is negative.

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APPENDIX

In this Appendix we list the explicit expressions of the functions $\Phi_i(x,\tilde{\lambda})$ and $\Psi_i(x,\tilde{\lambda})$ with $i=1,2$ to complete the discussion of Sec. IV. These functions are given by

$$\Phi_1(x,\tilde{\lambda})=\left[\frac{p_2x}{p_2x+p_1}\right]^{\alpha}\left[\frac{p_1}{p_2x+p_1}\right]^{\beta}F\left[a,b;c;\frac{p_2x}{p_2x+p_1}\right], \tag{A1}$$

$$\Phi_2(x,\tilde{\lambda})=\left[\frac{p_2x}{p_2x+p_1}\right]^{\alpha+1-c}\left[\frac{p_1}{p_2x+p_1}\right]^{\beta}F\left[a-c+1,b-c+1;2-c;\frac{p_2x}{p_2x+p_1}\right], \tag{A2}$$

$$\Psi_1(x,\tilde{\lambda})=\left[\frac{p_2x}{p_2x+p_1}\right]^{\alpha}\left[\frac{p_1}{p_2x+p_1}\right]^{\beta}\left[F\left[a,b;c;\frac{p_2x}{p_2x+p_1}\right]\left[\frac{2p_1}{b_1}p_2x\left[\frac{\alpha}{p_2x}-\frac{\beta}{p_1}\right]-\frac{a_1+a_2x}{b_1}\right]\right.$$
$$\left.+\frac{2p_1p_2x}{b_1(p_2x+p_1)}\frac{ab}{c}F\left[a+1,b+1;c+1;\frac{p_2x}{p_2x+p_1}\right]\right], \tag{A3}$$

$$\Psi_2(x,\tilde{\lambda})=\left[\frac{p_2x}{p_2x+p_1}\right]^{\alpha-c+1}\left[\frac{p_1}{p_2x+p_1}\right]^{\beta}$$

$$\times\left\{F\left[a-c+1,b-c+1;2-c;\frac{p_2x}{p_2x+p_1}\right]\left[\frac{2p_1}{b_1}p_2x\left[\frac{\alpha-c+1}{p_2x}-\frac{\beta}{p_1}\right]-\frac{a_1+a_2x}{b_1}\right]\right.$$

$$\left.+\frac{2p_1p_2x}{b_1(p_2x+p_1)}\frac{(a-c+1)(b-c+1)}{2-c}F\left[a-c+2,b-c+2;3-c;\frac{p_2x}{p_2x+p_1}\right]\right\}. \quad \text{(A4)}$$

The relevant symbols are defined in Sec. III and $F(a,b;c;z)$ denotes the usual hypergeometric function.

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