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Algebraic time-ordering techniques and harmonic oscillator with time-dependent frequency

G. Dattoli
Dipartimento Tecnologie Intersettoriali di Base, Divisione Fisica Applicata, Comitato Nazionale per la Ricerca e per lo Sviluppo dell'Energia Nucleare e delle Energie Alternative, Centro Ricerche Energie di Frascati, Cosella Postale 65 I-00044 Frascati, Rome, Italy

S. Solimeno
Dipartimento di Fisica Nucleare, Struttura della Materia e Fisica Applicata, Mostra d'Oltremare pad. 16, I-80125 Napoli, Italy
and Centro Interuniversitario di Elettronica Quantistica e Plasmi, I-80125 Napoli, Italy

A. Torre
Instituto Nazionale di Fisica Nucleare Sezione di Napoli, Mostra d'Oltremare pad. 16, I-80125 Napoli, Italy
and Centro Interuniversitario di Elettronica Quantistica e Plasmi, I-80125 Napoli, Italy
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The Wei-Norman algebraic techniques for time-ordering problems are applied to the study of the evolution of quantum states ruled by a harmonic-oscillator Hamiltonian with a time-dependent frequency. The slowly varying frequency case is studied; the adiabatic theorem is rederived together with higher-order corrections. The analogy with the propagation of a paraxial beam through a self-focusing fiber is also pointed out.

I. INTRODUCTION

Algebraic procedures to deal with time-ordering problems[1] have proved a powerful tool for studying the evolution of dynamical systems whose Hamiltonian can be cast in a time-dependent combination of the generators of a Lie algebra. Renewed interest in these techniques has been recently prompted by the study of the evolution of the SU(2) and SU(1,1) Perelomov coherent states.[2]

From the quantum-mechanical point of view, the problem posed by the evolution of a system consists in finding a suitable representation of the evolution operator. This leads one to exploit suitable disentanglement theorems of the Baker-Hausdorff (BH) type.[3] Several formulas of the BH type have been established for time-independent Hamiltonians, while more general theorems are needed for the time-dependent case. In Refs. 4 and 5 some of the present authors have shown that the most convenient approach to this problem is the Wei-Norman (WN) algebraic method.[1] This approach owes its power to the recourse to group-theoretical concepts, which allows one to recast the generic element of a Lie group as an ordered product of exponentials containing single generators of the group. The WN method implies the solutions of a system of differential equations, whose order depends on the dimension of the Lie algebra.

The time evolution of a harmonic oscillator has been analyzed by Dykhne[6] who has calculated the transition probabilities from an initial state to a final one in the case of an adiabatic variation of $\omega(t)$. In Ref. 7 the evolution for either a prescribed or a random variation of ω has been studied by using the Feynman-Dyson expansion of the evolution operator. More recently, Gerry[8] has considered the evolution of a degenerate parametric oscillator by using Perelomov coherent states.

Here we will reconsider the above problem within a more general framework, by obtaining closed expressions for the matrix elements of the evolution operator. In particular, we will dwell on the solution of the characteristic differential equation (CDE) generated by the WN method, by considering the case of both periodic and slow variations of ω. The use of the realization of the SU(1,1) algebra in terms of the $\hat{q}$ and $\hat{p}$ operators, as suggested by Dirac,[9] will allow us to establish a useful analogy of the oscillator evolution with the propagation of optical beams through a suitable lens combination. Exploiting this analogy, we will obtain the matrix elements for the evolution operator of a two-dimensional oscillator with simple physical considerations. In this context, the possibility of describing a nonuniform optical fiber with a quadratic profile of the refractive index (self-focusing) in terms of an appropriate lens combination will also be pointed out.

The paper is organized into six sections. In Sec. II we discuss the relevance of the SU(1,1) group to the time-dependent harmonic oscillator. In Sec. III we give a simple interpretation of the operators $\exp(\alpha\hat{K}_\pm)$, $\exp(\beta\hat{K}_0)$—$\hat{K}_0,\hat{K}_\pm$ being the SU(1,1) algebra generators relevant to the already quoted realization—in terms of the propagation of a TEM_n two-dimensional beam through an optical system composed of a cylindrical lens and a beam expander. Furthermore, this analogy is used for obtaining, with little mathematical effort, the evolution operator for a degenerate two-dimensional oscillator. In Sec. IV we use a time-dependent basis, coincident with the eigenfunctions of an oscillator with constant frequency $\omega(t)$. A closed expression of the matrix elements of the evolution operator is obtained in terms of the associated Legendre functions, whose argument is a solution of the CDE. Section V is devoted to deriving the asymptotic expansion of the matrix elements of the evolution operator in terms of a smallness parameter ϵ measuring the slowness of the frequency variation. In this context, the evolution of a Glauber coherent state is studied up to the second order in ϵ. Finally, Sec. VI contains some conclusive remarks.

II. HARMONIC OSCILLATOR WITH TIME-DEPENDENT FREQUENCY AND SU(1,1) GROUP

Let us consider a quantum-mechanical harmonic oscillator with a prescribed time-dependent frequency $\omega(t)$, whose Hamiltonian reads

$$\hat{H}(t)=\tfrac{1}{2}\hat{p}^2+\tfrac{1}{2}\omega^2(t)\hat{q}^2 . \tag{1}$$

We will use dimensionless quantities and, in particular, we will assume $\hbar=1$. For much of the following discussion, it is sufficient to require that the frequency evolution be such that $\omega(t)$ is an analytic function.

The operator $\hat{U}(t,t_0)$ transforming the state of the system at the generic initial time t_0 into the state at the time t can be represented as a time-ordered exponential function of $\hat{H}$,

$$\hat{U}(t,t_0)=\hat{T}\exp\left[-i\int_{t_0}^{t}\hat{H}(t')dt'\right], \tag{2}$$

$\hat{T}$ being the Feynman-Dyson time-ordering operator. Since $\hat{H}(t)$ does not commute in general with $\hat{H}(t')$, except for the trivial case of constant frequency, the right-hand side of Eq. (2) is represented by a nonterminating series of commutators. To get around this difficulty, Wei and Norman have developed a method involving the algebraic structure of the Hamiltonian operator. We will illustrate the WN method in the following for the algebra realized by embedding the $\hat{q}$ and $\hat{p}$ operators appearing in (1) as follows:

$$\hat{K}_+=\frac{i}{2}\hat{q}^2,\ \ \hat{K}_-=\frac{i}{2}\hat{p}^2,\ \ \hat{K}_0=\frac{i}{4}(\hat{p}\hat{q}+\hat{q}\hat{p}) . \tag{3}$$

The relevant algebraic structure, displayed by the commutators

$$[\hat{K}_0,\hat{K}_\pm]=\pm\hat{K}_\pm,\ \ [\hat{K}_+,\hat{K}_-]=-2\hat{K}_0 , \tag{4}$$

is immediately recognized as that of the SU(1,1) group.

The realization (3) of the SU(1,1) algebra is not unique; the more widespread realization involves the harmonic-oscillator annihilation and creation operators, namely,[8]

$$\hat{K}'_+=\tfrac{1}{2}(\hat{a}^\dagger)^2,\ \ \hat{K}'_-=\tfrac{1}{2}\hat{a}^2,\ \ \hat{K}'_0=\tfrac{1}{4}(\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}) . \tag{5}$$

Expressing $\hat{a}$ and $\hat{a}^\dagger$ in terms of $\hat{q}$ and $\hat{p}$,

$$\hat{a}=\frac{1}{\sqrt{2}}(\hat{q}+i\hat{p}),\ \ \hat{a}^\dagger=\frac{1}{\sqrt{2}}(\hat{q}-i\hat{p}), \tag{6}$$

we can easily show that the two realizations (3) and (5) are connected by the linear transformation

$$\begin{bmatrix}\hat{K}_0\\ \hat{K}_+\\ \hat{K}_-\end{bmatrix}=\frac{1}{2}\begin{bmatrix}0 & -1 & 1\\ 2i & i & i\\ 2i & -i & -i\end{bmatrix}\begin{bmatrix}\hat{K}'_0\\ \hat{K}'_+\\ \hat{K}'_-\end{bmatrix} . \tag{7}$$

The wave functions of the harmonic oscillator realize two irreducible unitary representations of the SU(1,1) algebra, generated by the operators (3) or (5), in correspondence with the two possible values of the Bargman index k, $k=\frac{1}{4},\frac{3}{4}$, as deduced from the expression of the Casimir invariant relevant to the realizations (3) and (5), i.e.,

$$\hat{C}=-\tfrac{3}{16}\hat{I}=k(k-1)\hat{I} . \tag{8}$$

The states with even n span the space of the irreducible unitary representation with $k=\frac{1}{4}$ and the corresponding states with odd n realize the representation with $k=\frac{3}{4}$. In particular, with the notation $|n,\frac{1}{4}\rangle=|2n\rangle$ and $|n,\frac{3}{4}\rangle=|2n+1\rangle$, we have

$$\begin{aligned}\hat{K}'_0|n,k\rangle&=(n+k)|n,k\rangle ,\\ \hat{K}'_+|n,k\rangle&=[(n+1)(n+2k)]^{1/2}|n+1,k\rangle ,\\ \hat{K}'_-|n,k\rangle&=[n(n+2k-1)]^{1/2}|n-1,k\rangle .\end{aligned} \tag{9}$$

The corresponding relations for $\hat{K}_0$, $\hat{K}_+$, and $\hat{K}_-$ can be obtained by using Eqs. (6).

Once having identified the group structure of the Hamiltonian (1), we can deal with the time-ordered exponential (2) by using the already quoted algebraic-ordering technique, according to which we can express the evolution operator as the ordered product of three uncoupled exponentials,

$$\hat{U}(t,t_0)=\exp(2u\hat{K}_0)\exp(v\hat{K}_+)\exp(-w\hat{K}_-) , \tag{10}$$

where u,v,w satisfy the system of nonlinear differential equations

$$\begin{aligned}\dot{u}&=-v\exp(2u) ,\\ \dot{v}&=-\dot{u}v-\omega^2\exp(-2u) ,\\ \dot{w}&=\exp(2u) ,\end{aligned} \tag{11}$$

with the initial conditions $u(t_0)=v(t_0)=w(t_0)=0$, a dot denoting a time derivative.

The system (11) can be transformed into a Riccati equation for the function $s=\dot{u}$

$$\dot{s}(t)-s^2(t)-\omega^2(t)=0,\ \ s(t_0)=0 . \tag{12}$$

This is the standard CDE of the WN procedure, relevant to the split three-dimensional Lie algebra. It turns out, however, to be more convenient to introduce the auxiliary functions $\mathscr{U}=\exp(-u)$ and $\mathscr{W}=w\exp(-u)$, which are two independent integrals of the equation

$$\ddot{y}(t)+\omega^2(t)y(t)=0 , \tag{13}$$

satisfying the initial conditions $\mathscr{U}(t_0)=\dot{\mathscr{W}}(t_0)=1$, $\dot{\mathscr{U}}(t_0)=\mathscr{W}(t_0)=0$. In particular, we have

$$u=-\ln\mathscr{U},\quad w=\frac{\mathscr{W}}{\mathscr{U}},\quad v=\mathscr{U}\dot{\mathscr{U}}\ . \tag{14}$$

$\mathscr{U}$ and $\mathscr{W}$ are related by the Wronskian relation

$$\mathscr{U}\dot{\mathscr{W}}-\dot{\mathscr{U}}\mathscr{W}=1\ . \tag{15}$$

Although both Eqs. (12) and (13) cannot be solved analytically for a generic function $\omega(t)$, we can, however, investigate the asymptotic behavior of the functions $\mathscr{U}(t)$ and $\mathscr{W}(t)$ in the hypothesis that ω tends to constant values $\omega_{\pm}$ for $t\rightarrow\pm\infty$ and $t_0\rightarrow-\infty$. It follows indeed from (13) and (15) that

$$\begin{aligned}
&\mathscr{U}(t)\underset{t\rightarrow\infty}{\rightarrow}\mathscr{U}_{\infty}\cos(\omega_{+}t+\phi)\,,\\
&\mathscr{W}(t)\underset{t\rightarrow\infty}{\rightarrow}\frac{1}{\mathscr{U}_{\infty}\omega_{+}}\sin(\omega_{+}t+\phi)\,,\\
&\mathscr{U}+i\mathscr{W}\underset{t\rightarrow\infty}{\rightarrow}\frac{\mathscr{U}_{\infty}}{2}\left[1+\frac{1}{\mathscr{U}_{\infty}^{2}\omega_{+}}\right]\\
&\qquad\times\{\exp[i(\omega_{+}t+\phi)]\\
&\qquad\qquad+\rho\exp[-i(\omega_{+}t+\phi)]\}\,,
\end{aligned} \tag{16}$$

where

$$\rho=\frac{\omega_{+}\mathscr{U}_{\infty}^{2}-1}{\omega_{+}\mathscr{U}_{\infty}^{2}+1} \tag{17}$$

plays the role of a reflection coefficient. Accordingly, we have

$$\begin{aligned}
&w(t)\underset{t\rightarrow\infty}{\rightarrow}\frac{1-\rho}{1+\rho}\tan(\omega_{+}t+\phi)\,,\\
&v(t)\underset{t\rightarrow\infty}{\rightarrow}-\frac{1+\rho}{2(1-\rho)}\sin[2(\omega_{+}t+\phi)]\ .
\end{aligned} \tag{18}$$

Another interesting case, encountered in several problems of quantum mechanics, involves a frequency $\omega(t)$ undergoing small variations with respect to an average value. In order to tackle this specific problem, we rehandle conveniently Eq. (13) by means of the Prüfer substitution

$$y=r\sin\theta,\quad \dot{y}=r\cos\theta\,, \tag{19}$$

where r and θ satisfy the equations of motion

$$\begin{aligned}
&\dot{\theta}=(\omega^2-1)\sin^2\theta+1\,,\\
&\dot{r}=-\frac{r}{2}(\omega^2-1)\sin2\theta\,,
\end{aligned} \tag{20}$$

with the initial conditions $r(t_0)=1$, $\theta_0\equiv\theta(t_0)=\pi/2$ for $\mathscr{U}$ and $r(t_0)=1$, $\theta_0=0$ for $\mathscr{W}$. Then, assuming without loss of generality $\omega^2\cong1$ and introducing the function $\delta(t)$, according to

$$\theta(t)=\theta_0+t-t_0+\frac{\delta(t)}{2}\,, \tag{21}$$

we can replace Eqs. (20) with

$$\begin{aligned}
\dot{\delta}&=2\left\langle(\omega^2-1)\sin^2\left[\frac{\delta}{2}+t-t_0+\theta_0\right]\right\rangle_t\\
&=A-B\cos\delta+C\sin\delta\,,\\
2\frac{\dot{r}}{r}&=-B\sin\delta-C\cos\delta\,,
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
&A=\langle(\omega^2-1)\rangle_t\,,\\
&B=\langle(\omega^2-1)\cos[2(t-t_0+\theta_0)]\rangle_t\,,\\
&C=\langle(\omega^2-1)\sin[2(t-t_0+\theta_0)]\rangle_t\,,
\end{aligned} \tag{23}$$

the symbol $\langle\cdots\rangle_t$ denoting the average over an interval, during which θ and r remain almost constant. When A, B, and C are independent of t, the above system can be integrated by quadrature. In particular, for $A=C=0$

$$\begin{aligned}
&r^2(\delta)\cos\delta=1\,,\\
&\delta=2\arctan\{\exp[B(t-t_0)]\}-\frac{\pi}{2}\ .
\end{aligned} \tag{24}$$

Accordingly, we obtain for $\mathscr{U}$ and $\mathscr{W}$

$$\begin{aligned}
&\mathscr{U}=\cosh^{1/2}[B(t-t_0)]\cos(t-t_0+\delta/2)\,,\\
&\mathscr{W}=\cosh^{1/2}[B(t-t_0)]\sin(t-t_0+\delta/2)\,,\\
&\mathscr{U}^2+\mathscr{W}^2=\cosh[B(t-t_0)]\,,
\end{aligned} \tag{25}$$

that for $t\rightarrow\infty$ reduce to

$$\begin{aligned}
&\mathscr{U}\underset{t\rightarrow\infty}{\rightarrow}\frac{1}{\sqrt{2}}\exp[\tfrac{1}{2}|B|(t-t_0)]\cos\left[t-t_0-\frac{\pi}{4}\operatorname{sgn}(B)\right],\\
&\mathscr{W}\underset{t\rightarrow\infty}{\rightarrow}\frac{1}{\sqrt{2}}\exp[\tfrac{1}{2}|B|(t-t_0)]\sin\left[t-t_0-\frac{\pi}{4}\operatorname{sgn}(B)\right].
\end{aligned} \tag{26}$$

The above analysis shows that the problem of the time-dependent quantum harmonic oscillator can be treated within the framework of an algebraic approach. The above solutions of the CDE, even though limited to particular cases, show that useful information about the dynamical behavior of the system can be inferred quite straightforwardly.

III. ANALOGY WITH THE PROPAGATION OF OPTICAL BEAMS

In dealing with the time evolution of quantum states, the search for a suitable representation of the evolution operator is only the first step. The second one involves the study of the transformation induced by the evolution operator on the initial state, represented in the case we are interested in by the eigenfunctions u_n of the harmonic oscillator with constant frequency $\omega=1$,

$$u_n(q)=(n!2^n\sqrt{\pi})^{x-1/2}H_n(q)\exp(-\tfrac{1}{2}q^2)\,, \tag{27}$$

H_n being the Hermite polynomial of degree n. To this end, it is convenient to introduce the function

$$\tilde{u}_n(q;z)=\frac{(n!2^n\sqrt{\pi})^{x-1/2}}{(1+z^2)^{1/4}}H_n\left[\frac{q}{(1+z^2)^{1/2}}\right]\times\exp\left[i(n+\tfrac{1}{2})\arctan z-\frac{i}{2}\frac{q}{z+i}\right], \quad (28)$$

with z real. Now, it can be shown that $\tilde{u}_n(q;z)$ satisfies the parabolic wave equation with wave number equal to unity,[10] that is,

$$\left[\frac{i}{2}\frac{\partial^2}{\partial q^2}+\frac{\partial}{\partial z}\right]\tilde{u}_n(q;z)=0, \quad (29)$$

so that we have

$$\begin{aligned}\exp(-w\hat{K}_-)u_n(q)&=\exp(-w\hat{K}_-)\tilde{u}_n(q;0)\\ &=\exp\left[-w\frac{\partial}{\partial z}\right]\tilde{u}_n(q;z)\,|_{z=0}\\ &=\tilde{u}_n(q;-w)\,. \end{aligned} \quad (30)$$

By interpreting $\tilde{u}_n(q;z)$ as the field associated to a TEM_n Gaussian beam, Eq. (30) states the equivalence of the operator $\exp(-w\hat{K}_-)$ with the propagation of the beam over a distance $-w$. Analogously, it may be immediately shown that $\exp(v\hat{K}_+)$ is equivalent to a lens of focal length $f=1/v$. For what concerns $\hat{K}_0$, the relation

$$\begin{aligned}\exp(2u\hat{K}_0)f(q)&=\exp\left[\frac{u}{2}\right]\exp\left[uq\frac{\partial}{\partial q}\right]f(q)\\ &=\exp\left[\frac{u}{2}\right]f[q\exp(u)]\end{aligned} \quad (31)$$

shows that $\exp(2u\hat{K}_0)$ is equivalent to a beam expander with a transverse magnification $M=\exp(-u)=\mathcal{U}$. Thus, we can assimilate the operator $\hat{U}$ to the optical system of Fig. 1. Correspondingly, the oscillator wave function $u_n(q;t)$ at time t is given by

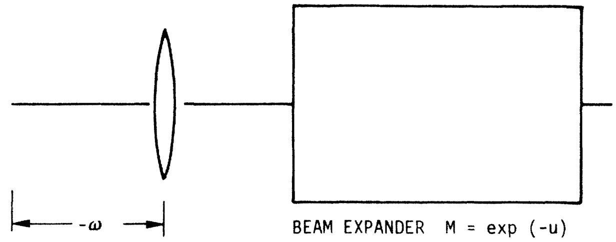

FIG. 1. Optical system equivalent to the evolution operator for a one-dimensional (1D) harmonic oscillator. The free-space section corresponds to the operator $\exp(-\omega\hat{K}_-)$, the lens represents $\exp(v\hat{K}_+)$, while $\exp(2u\hat{K}_0)$ corresponds to the beam expander. For the 2D problem, the cylindrical optical elements are replaced by the spherical ones.

$$\begin{aligned}u_n(q;t)&=\frac{1}{\mathcal{U}^{1/2}}\tilde{u}_n\left[\frac{q}{\mathcal{U}};-w\right]\exp\left[\frac{1}{2\mathcal{U}}\dot{\mathcal{U}}q^2\right]\\ &=\exp(i\Theta)\frac{(\mathcal{U}^2+\mathcal{W}^2)^{-1/4}}{(n!2^n\sqrt{\pi})^{1/2}}H_n\left[\frac{q}{(\mathcal{U}^2+\mathcal{W}^2)^{1/2}}\right]\\ &\quad\times\exp\left[\frac{-q^2}{2(\mathcal{U}^2+\mathcal{W}^2)}\right],\end{aligned} \quad (32)$$

where

$$\Theta(t)=\frac{q^2}{2\mathcal{U}}\left[\dot{\mathcal{U}}+\frac{\mathcal{W}}{\mathcal{U}^2+\mathcal{W}^2}\right]-(n+\tfrac{1}{2})\arctan\omega(z)\,. \quad (33)$$

The above-stated analogy allows us to easily define the evolution of a two-dimensional oscillator, ruled by the Hamiltonian

$$\hat{H}(t)=\tfrac{1}{2}(\hat{p}_1^2+\hat{p}_2^2)+\tfrac{1}{2}\omega^2(t)(\hat{q}_1^2+\hat{q}_2^2)\,. \quad (34)$$

Indeed, the time evolution of the simultaneous eigenfunctions of the Hamiltonian (34) and of the angular momentum $\hat{J}=\hat{p}_1\hat{q}_2-\hat{p}_2\hat{q}_1$ is equivalent to the propagation of Gauss-Laguerre modes through the system of Fig. 1, in which the cylindrical lens and beam expander used for the one-dimensional problem have been replaced by the spherical ones. Therefore, for a two-dimensional oscillator the wave function at time t can be immediately derived from the expression (32), by replacing the Hermite polynomial with the Laguerre one L_n^m and modifying slightly Θ,

$$\begin{aligned}u_n^m(q\cos\phi,q\sin\phi;t)&=\left[\frac{2n!}{\pi(n+m)!}\right]^{1/2}(\mathcal{U}^2+\mathcal{W}^2)^{-1/2}\left[\frac{q^2}{\mathcal{U}^2+\mathcal{W}^2}\right]^{m/2}\exp(im\phi)L_n^m\left[\frac{q^2}{\mathcal{U}^2+\mathcal{W}^2}\right]\\ &\quad\times\exp\left[-\frac{1}{2}\frac{q^2}{\mathcal{U}^2+\mathcal{W}^2}\right]\exp\left\{i\left[\frac{q^2}{2\mathcal{U}}\left[\dot{\mathcal{U}}+\frac{\mathcal{W}}{\mathcal{U}^2+\mathcal{W}^2}\right]+(2n+m+1)\arctan\omega(z)\right]\right\},\end{aligned} \quad (35)$$

where $q^2=q_1^2+q_2^2$.

Finally, in the context of the discussed analogy, we want to stress the equivalence of a longitudinally nonuniform optical fiber with a parabolic profile of the refractive index with the optical system of Fig. 1, whose parameters are defined by the law of variation of the refractive index of the fiber.

IV. EVOLUTION OPERATOR FOR AN ADIABATICALLY VARYING BASIS

In this section we want to approach the problem from another point of view, which will allow us to insert the problem of the time evolution of the harmonic oscillator into the more general context of nonlinear quantum optics. To this end, let us introduce the eigenstate $|n,t\rangle$ of the Hamiltonian (1)

$$\hat{H}(t)|n,t\rangle=E_n(t)|n,t\rangle \; . \tag{36}$$

Consequently, by expanding the state $|\psi\rangle$ at the time t in the series

$$|\psi\rangle=\sum_n a_n(t)\exp[-i(n+\tfrac{1}{2})\psi(t,t_0)]\,|n,t\rangle\,, \tag{37}$$

with

$$\phi(t,t_0)=\int_{t_0}^{t}\omega(t')dt'\,, \tag{38}$$

the Schrödinger equation turns into the set of differential difference equations

$$\dot{a}_n(t)=\frac{\dot{\omega}}{4\omega}[\sqrt{n(n-1)}\exp(2i\phi)a_{n-2} -\sqrt{(n+1)(n+2)}\exp(-2i\phi)a_{n+2}]\,, \tag{39}$$

whose initial conditions are specified by the initial state of the oscillator.

Equations of the type (39) have come into widespread use in physics since the late thirties, when Raman and Nath proposed a similar system for describing the amplitudes of the diffracted beams produced by the scattering of light by ultrasound.[11] A comprehensive analysis of the Raman-Nath (RN)-type equations together with their relevance to physical problems has been put forward in Refs. 4, 5, and 12. For the interested reader we recall that the RN equations can be classified in accordance with the algebraic structure of the Hamiltonian (*if any*) from which they can be derived. For example, a free-electron laser,[13] a one-dimensional random chain diffusion,[14] or multiphoton processes[15] are ruled by a Hamiltonian associated with the "shift group." Similarly, the time evolution of Glauber states[16] or the interaction of a multilevel system with electromagnetic radiation[17] can be associated with the Weyl-Heisenberg group structure, which in turn characterizes the harmonic RN equation[18] governing the evolution of the eigenstate amplitudes. Finally, the evolution of two-level systems[19,20] can be associated with Hamiltonians with SU(2) symmetry and the equations of motion of the eigenstate amplitudes for these systems are accordingly called spherical RN equations.[4]

The group structure underlying Eqs. (39) can be immediately found by introducing the auxiliary Hamiltonian

$$\hat{H}(t)=i\frac{\dot{\omega}}{2\omega}[\exp(2i\phi)\hat{K}'_{+}-\exp(-2i\phi)\hat{K}'_{-}]\,, \tag{40}$$

and as basis the set of states $|n,k\rangle$, which diagonalize the compact generator $\hat{K}'_0$, i.e., $\hat{K}'_0|n,k\rangle=(n+k)|n,k\rangle$. Thus, from the Schrödinger equation and the properties of the raising and lowering operators $\hat{K}'_{\pm}$, one easily obtains the system (39). Consequently, the algebraic structure underlying the RN equation (39) is that of the SU(1,1) group, which can be recognized as a dynamical group for the parametric amplifier[8] for the laser-light—plasma interaction,[21] and for the particle motion in a rapidly oscillating field,[22] just to mention a few important cases.

The system (39) can be solved by the intermediary of an evolution operator $\hat{U}$ obeying the equation of motion

$$\frac{d}{dt}\hat{U}=\frac{\dot{\omega}}{2\omega}[\exp(2i\phi)\hat{K}'_{+}-\exp(-2i\phi)\hat{K}'_{-}]\hat{U}\,, \tag{41}$$

with the initial condition $\hat{U}(t_0,t_0)=\hat{I}$.

Equation (41) can be integrated by following the WN procedure illustrated in Sec. II, which gives

$$\hat{U}(t,t_0)=\exp(2h\hat{K}'_0)\exp(g\hat{K}'_{+})\exp(-f\hat{K}'_{-})\,, \tag{42}$$

where

$$\begin{aligned}
&\dot{h}=-\Omega g\exp(2h)\,,\\
&\dot{g}=\Omega^*\exp(-2h)-g\dot{h}\,,\\
&\dot{f}=\Omega\exp(2h)\,,\\
&\Omega(t)\equiv\frac{\dot{\omega}}{2\omega}\exp(-2i\phi)\,,
\end{aligned} \tag{43}$$

with the initial conditions $h(t_0)=g(t_0)=f(t_0)=0$.

Next, by introducing the auxiliary functions $\mathscr{H}=\exp(-h)$ and $\mathscr{F}=f\exp(-h)$, from Eqs. (43) we derive that $\mathscr{H}$ and $\mathscr{F}$ are independent integrals of the differential equation

$$\ddot{y}+p(t)\dot{y}-qy=0\,, \tag{44}$$

with

$$\begin{aligned}
&p(t)=-\frac{\dot{\Omega}}{\Omega}\,,\\
&q(t)=|\Omega(t)|^2\,,
\end{aligned} \tag{45}$$

satisfying the initial conditions $\dot{\mathscr{H}}(t_0)=\mathscr{F}(t_0)=0$, $\mathscr{H}(t_0)=1$, and $\dot{\mathscr{F}}(t_0)=\Omega(t_0)$. Next, let us introduce the function $\mathscr{G}=g\exp(h)$, which can be shown to satisfy Eq. (44) with p replaced by p^* and the initial conditions $\mathscr{G}(t_0)=0$, $\dot{\mathscr{G}}(t_0)=\Omega^*(t_0)$. Consequently, $\mathscr{G}$ coincides with $\mathscr{F}^*$ and the system (43) is superseded by

$$\begin{aligned}
&\dot{\mathscr{H}}=\Omega\mathscr{F}^*\,,\\
&\dot{\mathscr{F}}=\Omega\mathscr{H}^*\,,\\
&\mathscr{H}\dot{\mathscr{F}}-\mathscr{F}\dot{\mathscr{H}}=\Omega\,,
\end{aligned} \tag{46}$$

from which it follows the integral of motion

$$|\mathscr{H}|^2-|\mathscr{F}|^2=1 . \tag{47}$$

If we define the two vectors

$$\mathbf{b}^{(1/4)}(t)=\begin{bmatrix} a_0(t) \\ a_2(t) \\ \vdots \\ a_{2n}(t) \\ \vdots \end{bmatrix} \equiv \begin{bmatrix} b_0^{(1/4)} \\ b_1^{(1/4)} \\ \vdots \\ b_n^{(1/4)} \end{bmatrix}, \mathbf{b}^{(3/4)}(t)=\begin{bmatrix} a_1(t) \\ a_3(t) \\ \vdots \\ a_{2n+1}^{(t)} \end{bmatrix} \equiv \begin{bmatrix} b_0^{(3/4)} \\ b_1^{(3/4)} \\ \vdots \\ b_n^{(3/4)} \end{bmatrix}, \tag{48}$$

Eq. (42) yields

$$\mathbf{b}^{(k)}(t)=\underline{U}^{(k)}(t,t_0)\mathbf{b}^{(k)}(t_0) , \tag{49}$$

where the elements of the matrix $\underline{U}^{(k)}(t,t_0)$ are defined by

$$U_{n,m}^{(k)}(t,t_0)=\langle n,k \mid \hat{U}(t,t_0) \mid m,k \rangle . \tag{50}$$

From Eq. (9), with the help of the relation (47), we obtain

$$\begin{aligned} U_{n,m}^{(k)}(t,t_0)&=(n'!m'!)^{1/2}\mathscr{H}^{-(n'+m'+1)/2}\left[\frac{|\mathscr{F}|}{2}\right]^{|n-m|}\exp[+i(n-m)X] \\ &\quad\times[\operatorname{sgn}(m-n)]^{n-m}\sum_{l=0}^{n_<}\frac{(-1)^l|\mathscr{F}|^{2l}}{2^{2l}l!(n'_<-2l)!(|n-m|+l)!} \\ &=\left[\frac{n'_>!}{n'_<!}\right]^{1/2}\left[\frac{\Gamma(n'_>+2k)}{\Gamma(n'_<+2k)}\right]^{1/2}[2\operatorname{sgn}(m-n)]^{|n-m|}\exp\{-i[(m-n)X+(n+m+2k)\arg(\mathscr{H})]\} \\ &\quad\times\frac{1}{|\mathscr{H}|^{1/2}}P_{(m'+n')/2}^{-|m'-n'|/2}\left[\frac{1}{|\mathscr{H}|}\right], \end{aligned} \tag{51}$$

where $p'=2p+2(k-\frac{1}{4})$, $X=\arg(\mathscr{F})$, $n_>=\max(m,n)$, and $n_<=\min(m,n)$, while P_ν^μ is the associated Legendre function and Γ is the gamma function.

V. THE SLOWLY VARYING CASE AND THE ADIABATIC APPROXIMATION

The technique we have developed so far leads to an exact treatment of the problem. The possibility of a "global" exact solution depends on whether Eq. (44) may be solved exactly or not. Albeit that exact solutions can be found in a limited number of cases only,[23] asymptotic analysis may give useful information.

In this section we will discuss the case in which the frequency is a slowly varying function of the time. In this hypothesis an asymptotically small term naturally arises in Eq. (44) rewritten as

$$y''(\tau)+\left[-\frac{\eta'(\tau)}{\eta(\tau)}+2i\omega(\tau)\Delta t\right]y'(\tau)-\epsilon^2\eta^2(\tau)y(\tau)=0, \tag{52}$$

where the prime means derivative with respect to the dimensionless time $\tau=t/\Delta t$, Δt being a typical time. According to the assumption of slowly varying frequency, we have defined

$$\frac{\omega'}{2\omega}\equiv\epsilon\eta(\tau), \tag{53}$$

ϵ being the slowness parameter and $\eta(t)$ a time-dependent nonsingular function.

We can express the integral of Eq. (52) as an asymptotic series in the smallness parameter ϵ,

$$y(\tau,\epsilon)\sim\sum_{n=0}^{\infty}\epsilon^n y_n(\tau), \tag{54}$$

which once inserted in Eq. (52) yields the set of recurrence relations

$$\begin{gathered} y_n''(\tau)+\left[-\frac{\eta'(\tau)}{\eta(\tau)}+2i\omega(\tau)\Delta t\right]y_n'(\tau)-\eta^2(\tau)y_{n-2}(\tau)=0, \\ y_{-n}(\tau)=0 . \end{gathered} \tag{55}$$

The initial conditions for $\mathscr{H}_n$ and $\mathscr{F}_n$, specified by those relevant to $\mathscr{H}$ and $\mathscr{F}$, read

$$\begin{gathered} \mathscr{H}_n(\tau_0)=\delta_{n,0},\quad \dot{\mathscr{H}}_n(\tau_0)=0, \\ \mathscr{F}_n(\tau_0)=0,\quad \dot{\mathscr{F}}_n(\tau_0)=\eta(\tau_0)\delta_{n,1} . \end{gathered} \tag{56}$$

Accordingly, Eq. (54) specializes into

$$\begin{gathered} \mathscr{H}(\tau)\sim\sum_{n=0}^{\infty}\epsilon^{2n}\mathscr{H}_{2n}(\tau), \\ \mathscr{F}(\tau)\sim\sum_{n=0}^{\infty}\epsilon^{2n+1}\mathscr{F}_{2n+1}(\tau) . \end{gathered} \tag{57}$$

In particular, $\mathscr{H}_{0,2}(\tau)$ and $\mathscr{F}_1(\tau)$ read

$$\begin{aligned} &\mathscr{H}_0(\tau)=1, \\ &\mathscr{H}_2(\tau)=\int_{\tau_0}^{\tau}d\tau'\left[\int_{\tau_0}^{\tau'}\eta(\tau'')\exp\left[2i\Delta t\int_{\tau_0}^{\tau''}\omega(\tau''')d\tau'''\right]d\tau''\right]\eta(\tau')\exp\left[-2i\Delta t\int_{\tau_0}^{\tau'}\omega(\tau'')d\tau''\right], \\ &\mathscr{F}_1(\tau)=\int_{\tau_0}^{\tau}\eta(\tau')\exp\left[-2i\Delta t\int_{\tau_0}^{\tau'}\omega(\tau'')d\tau''\right]d\tau' . \end{aligned} \tag{58}$$

The above expressions specify the asymptotic expression of the matrix elements (51) up to the second order in ϵ as

$$U^{(k)}_{n,m}(\tau,\tau_0)=\begin{cases}\left[1-\frac{\epsilon^2}{2}(1+2n')\mathcal{H}_2-\frac{\epsilon^2}{4}n'(n'-1)|\mathcal{F}_1|^2\right]\delta_{n,m} \\ \quad +\frac{\epsilon}{2}[(n'+1)(n'+2)]^{1/2}\mathcal{F}_1^*\delta_{n,m-1}+\frac{\epsilon^2}{8}[(n'+1)\cdots(n'+4)]^{1/2}(\mathcal{F}_1^*)^2\delta_{n,m-2}, & n\le m \\ \left[1-\frac{\epsilon^2}{2}(1+2n')\mathcal{H}_2-\frac{\epsilon^2}{4}n'(n'-1)|\mathcal{F}_1|^2\right]\delta_{n,m} \\ \quad +\frac{\epsilon}{2}\frac{\mathcal{F}_1}{[n'(n'-1)]^{1/2}}\delta_{n,m+1}+\frac{\epsilon^2}{8}\frac{\mathcal{F}_1^2}{[n'(n'-1)\cdots(n'-3)]^{1/2}}\delta_{n,m+2}, & n\ge m\ .\end{cases} \tag{59}$$

It is evident from Eq. (59) that, in accordance with the adiabatic theorem,[7] at the lowest order in ϵ no transitions are induced.

For an initially coherent state (in the Glauber sense), driven by the Hamiltonian (1), a direct application of the above results yields after some algebra

$$\begin{aligned}|\alpha;t\rangle=\exp(-\tfrac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\Bigg[&\left[1-\frac{\epsilon^2}{2}(1+2n)\mathcal{H}_2(\tau)-\frac{\epsilon^2}{4}n(n-1)|\mathcal{F}_1|^2\right]\\&+\frac{\epsilon}{2}\left[n(n-1)\alpha^{-2}\mathcal{F}_1^*-\frac{\alpha^2}{(n+1)(n+2)}\mathcal{F}_1\right]\\&+\frac{\epsilon^2}{8}\left[n(n-1)\cdots(n-3)\alpha^{-4}\mathcal{F}_1^{*2}-\frac{\alpha^4\mathcal{F}_1^2}{(n+1)\cdots(n+4)}\right]\Bigg]\exp[-i(n+\tfrac{1}{2})\phi]\,|n;t\rangle\ .\end{aligned} \tag{60}$$

The expression (60) indicates that, in the adiabatic approximation the state remains coherent at any time, while higher-order corrections destroy the initial coherence.

VI. CONCLUSIONS

In this paper we have established the importance of the WN algebraic method for understanding the dynamics of a time-dependent frequency harmonic oscillator. In particular, we have obtained an expression of the evolution operator which, once specialized to the case of a slowly varying frequency, has allowed us to go beyond the Born and Fock adiabatic theorem and to analyze the evolution of an initially coherent state.

In addition, we have discussed a number of optical problems by establishing some results concerning the propagation through a nonuniform self-focusing fiber. In particular, we have shown that a self-focusing fiber, whose quadratic transverse profile of the refractive index is a function of the longitudinal coordinate, is equivalent to a combination of a lens and a beam expander. The unifying element of these seemingly unrelated problems is provided by the underlying group structure, which has been recognized to be SU(1,1).

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