RAPID COMMUNICATIONS

PHYSICAL REVIEW A VOLUME 34, NUMBER 3 SEPTEMBER 1986

Universal transition between Hamiltonian and dissipative chaos

G. Chen, G. Györgyi,* and G. Schmidt
Department of Physics and Engineering Physics, Stevens Institute of Technology, Hoboken, New Jersey 07030
(Received 25 June 1986)

The inverse bifurcation sequence in one-dimensional maps is modified in two-dimensional ones in a universal way. Two-piece, four-piece, etc., strange attractors are eliminated one by one as the Jacobian approaches one. Renormalization theory around the Hamiltonian T^* leads to universal behavior including universal strange attractors.

Chaos arising out of period-doubling bifurcations in one-dimensional maps is well understood. As the strength parameter K is increased, period-doubling bifurcations lead to a set of 2^n period attracting orbits.[1] There are the universal numbers δ_1 and α_1 associated with this sequence, as $n \to \infty$, $K \to K_\infty$. When $K > K_\infty$, an inverse bifurcation sequence results, where the attractors are chaotic bands rather than points.[2] This inverse bifurcation sequence leads to a single band, interrupted by periodic windows. Universal properties of the above scenario can be attributed to the universal function $g(x)$, corresponding to the K_∞ state and satisfying the fixed point relation $g(x) = -\alpha g[g(x/\alpha)]$, and to its unstable invariant manifold globally attracting with respect to the renormalization transformation.[1,3]

Hamiltonian systems with three phase-space dimensions are characterized by area-preserving maps of the plane. Hamiltonian chaos often arises out of a period-doubling-bifurcation sequence qualitatively similar to that of one-dimensional maps. There is a universal number δ_2 associated with the change of the parameter K between bifurcations as $K \to K_\infty$, and there are two universal scaling constants α_2, β_2 characterizing rescaling in two perpendicular directions of the map.[4] The universal map T^* is the solution of the renormalization fixed point equation $T^* = BT^*T^*B^{-1}$, where

$$B = \begin{pmatrix} \alpha_2 & 0 \\ 0 & \beta_2 \end{pmatrix}$$

is the rescaling matrix.[4] The inverse bifurcation sequence is missing; at K_∞ the system is fully chaotic in the neighborhood of the original fixed point whose bifurcation sequence produced local chaos. In fact, an inverse bifurcation sequence would imply the existence of local Kolmogorov-Arnold-Moser surfaces, which we know cannot exist.

Between the Hamiltonian system with Jacobian $J = 1$ and the one-dimensional map with $J = 0$, are dissipative systems with $0 < J < 1$, which are less well understood. Chaos there is usually characterized by strange attractors. Several investigators studied two-parameter maps $T(J,K)$ to establish contact between these different systems.[5,6] Within the framework of renormalization theory the bifurcation sequences for intermediate Jacobians have also been described.[7,8] Here we study the relationship between Hamiltonian and dissipative chaos, and find universal strange attractors.

Figure 1 shows the parameter space J,K for the fixed point of a map of the type

$$\begin{aligned} x' &= -Jy + f(K,x) \ , \\ y' &= x - f(K,x') \ , \end{aligned} \tag{1}$$

for $K > K_\infty$. When the Jacobian $J = 1$ it is a deVogelaere map,[3] when $J = 0$ it is the one-dimensional map $x' = f(K,x)$. On the right the inverse bifurcation sequence is shown producing 2^n piece bands (strange attractors), starting at K_∞ and ending up at K_1, where the one band first appears.[9] As J increases the K_∞ and K_1 lines approach and meet at $J = 1$. The K_2 line designates the first appearance of two-piece strange attractors from four-piece ones, etc. Regions 1, 2, and 4 are areas of parameter space where one-, two-, and four-piece strange attractors live. The two-piece attractor exists only for $J < J_1$, the 2^n piece only for $J < J_n$.

Such diagrams have been constructed for the quadratic map $f(x) = -Kx + (1+K)x^2$, $f(x) = K \sin \pi x$, and earlier for the dissipative standard map showing similar behavior.[6] The diagram has been obtained by studying the behavior of stable and unstable invariant manifolds of the

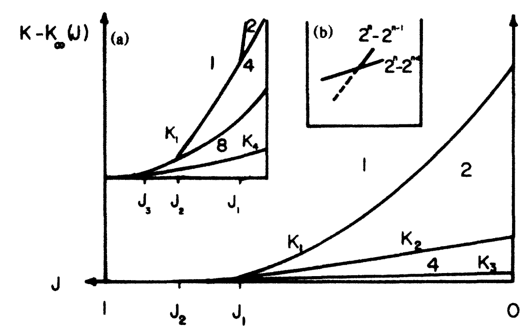

FIG. 1. Typical phase diagram in the J, $K - K_\infty(J)$ space for inverse bifurcations computed from Eq. (1). Inset (a) shows details of the region $0.6 < J < 1$. Inset (b) displays the neighborhood of a general point at J_n with heteroclinic tangency lines between the 2^{n-1}, 2^n and 2^n, 2^{n+1} piece manifolds, respectively. The tangency indicated by the dotted line has no influence on the structure of the attractor.

periodic orbits of the map. The lines on the diagram represent heteroclinic crises,[10] where the unstable manifolds of the 2^{n+1} orbit first intersect the stable manifolds of the 2^n orbit. These lines intersect in critical points at J_n, where the heteroclinic crisis between manifolds of period 2^{n+1}, 2^n and 2^n, 2^{n-1} occur simultaneously.

In phase space, points are attracted toward a stable manifold and flow toward, and out along the unstable manifold. The latter may or may not be part of a strange attractor depending on its linkages with other manifolds. Figure 2 gives a diagrammatic representation of the situation. It appears[11] that the unstable 2^n is always linked to the stable 2^{n+1}, so there is a flow downward, on Fig. 2. As K is increased heteroclinic intersections are established one by one between the lower and higher manifolds. Where these linkages upward on Fig. 2 are established, the flow can go either way forming a strange attractor. This is evidently a composite entity made up of all manifolds of longer periodic orbits.

To the left of the critical point at J_1 there is an upward link between the period two and the period one manifolds forming a composite. Close to J_1, however, when $K < K_1$, period four manifold (containing 8, 16, etc.) is not yet linked upwards to this composite, thus the latter is not part of the attractor, since it is drained to the attractor, but not fed by it. Inset (b) in Fig. 1 shows the general mechanism. The dotted line shows where 2^n links with 2^{n-1}, which is linked to 2^{n-2}, etc. all the way to one. Since 2^{n+1} is not yet linked to 2^n the composite $2^n \to 2^{n-1} \to \cdots 1$ is not part of the strange attractor $\infty \to \cdots \to 2^{n+1}$. At the line K_1 these two composites together form the large strange attractor $\infty \to \cdots \to 2^n \to \cdots 1$, when the "switch," the dotted line on Fig. 2, is closed.

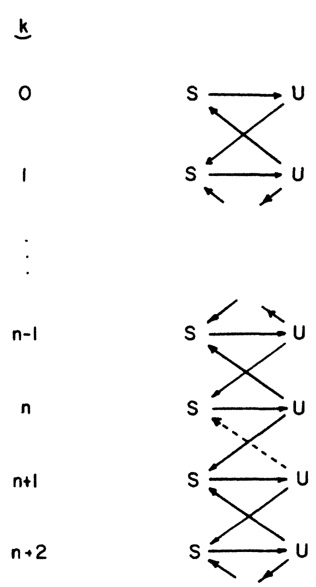

FIG. 2. Diagram explaining the dynamics within a strange attractor. The arrows indicate flows between stable (S) and unstable (U) manifolds of periodic points of period 2^n, $k = 0,1,2,\ldots$. Horizontal arrows show an omnipresent flow from the stable manifold to the unstable one of a periodic point. Oblique arrows are consequences of heteroclinic intersections.

As one suspects, the similarity between the parameter space diagrams is not only qualitative but quantitative as well. Within numerical accuracy we find that $J_n = J_{n+1}^2$. But the effective Jacobian corresponding to a 2^n orbit is $J_{\rm eff} = (J_n)^{2^n}$, so at all J_n values the effective Jacobian is a constant. It is known, that below K_∞ the bifurcation tree has a universal dependence on $J_{\rm eff}$, it appears now that there are universal crises for $K > K_\infty$ that depend on $J_{\rm eff}$ alone.

Near $J = 1$, the $J_n - s$ converge as a geometric series; when $J_n = (1-\varepsilon_n)$, $(J_n - J_{n-1})/(J_{n+1} - J_n) \to 2$, so for large n, $(1-J_n) = A2^{-n}$. This suggests the use of renormalization theory. In fact, if there is self-similarity between the mapping at J_n, $T(J_n)$, and at J_{n+1}, strongly indicated by numerical evidence, there should be some scaling matrix A so that $T(J_n) = AT(J_{n+1})T(J_{n+1})A^{-1}$. Since scaling does not effect the Jacobian, it follows that $J_n = J_{n+1}^2$.

At $J = 1$ there is the universal Hamiltonian map[4] T^*. There should be an unstable eigendirection in function space around T^*, such that application of the renormalization transformation leads to a reduction of the Jacobian with eigenvalue two. Defining the renormalization transformation $RT = BT^2B^{-1}$, where B is the scaling matrix associated with T^*, we have the eigenvalue equation

$$R(T^* + \varepsilon T_1) = T^* + \lambda \varepsilon T_1 \ , \qquad (2)$$

where $RT^* = T^*$, and $J(T^*) = 1$. To order ε the Jacobian of the left-hand side is $1 + 2\varepsilon J_1$, while the right-hand side gives $1 + \lambda\varepsilon J_1$, where J_1 is some number depending on T^* and T_1 only. (We assumed a uniform Jacobian.) It follows immediately that $\lambda = 2$ as expected. Hence, if there is a dissipative eigendirection with constant Jacobian, it must have the eigenvalue two.

In order to demonstrate that such an eigendirection T_1 exists, we have carried out a renormalization calculation. First, T^* was found using the procedure given in Ref. 4. It turned out that calculations on low order polynomials did not lead to significant results in contrast with the period doubling problem (see Appendix of Ref. 4). Eventually, Eq. (2) has been solved for the relevant $\lambda - s$ and $T_1 - s$ using the program REDUCE. The eigenvalues listed in Ref. 4 were recovered with eigenvectors not changing the Jacobian within numerical error. In addition, a nondegenerate $\lambda = 2.000\ldots$ was found belonging to the eigenfunction listed in Table I.

In consequence there exists a universal two-parameter function for the entire K,J plane. There is a universal one-parameter function for the 1D map, corresponding to the right vertical line of Fig. 1, namely, the function growing out of $g(x)$ in the unstable (bifurcating) direction in function space. The left boundary of Fig. 1 has similarly a universal one-parameter family of Hamiltonian functions growing out of T^*.

Consider now the function T_J, that moves T^* in the direction of changing Jacobians, as shown in Table I, and the function T_δ corresponding to the (bifurcating) eigenvalue δ, moving T^* in the direction of Hamiltonian functions. The two-parameter function $T^* + \varepsilon_J T_J + \varepsilon_\delta T_\delta$ upon repeated iteration of the renormalization transformation can be expected to produce functions with arbitrary values

TABLE I. Coefficients in the expansion of the dissipative eigendirection T_J defined as $T_{Jx}=\sum_{ij} t^x_{ij} x^i y^j$ and $T_{Jy}=\sum_{ij} t^y_{ij} x^i y^i$. The normalization condition $t^y_{00}=1$ has been chosen.

$T_{Jx}=\sum_{ij} t^x_{ij} x^i y^j$

$j\backslash i$	0	1	2	3	4	5	6
0	0.0585	0.433	4.46	0.74	−1.00	0.08	−0.06
1	2.017	0.33	−0.874	0.076	−0.081		
2	−0.19	0.017	−0.036				
3	−0.0055						

$T_{Jy}=\sum_{ij} t^y_{ij} x^i y^j$

$j\backslash i$	0	1	2	3	4	5	6	7	8
0	1	−1.27	8.5	−7.74	−48.6	−10.817	16.2	0.89	−0.77
1	3.75	−3.46	−43.0	−9.6	21.8	1.148	−1.22		
2	−9.5	−2.1	9.75	0.489	−0.716				
3	1.45	0.069	−0.183						
4	−0.017								

of J and K. In other words, for each choice of ε_J, ε_δ we obtain an infinite set of universal functions, situated on some line in the K,J plane. When ε_J and ε_δ are varied, a dense set of lines, a fan emanating from T^* results. The period-doubling set of Refs. 7 and 8 are situated on this fan, and so are the functions corresponding to the critical points at J_n described in this paper. Lines moving up and to the right of T^* pass through a sequence of 2^n-piece strange attractors. These attractors are all identical in their shape, fractal dimension, etc., except for rescaling for the universal map. Hence they exhibit universality in the same sense as the period-doubling sequence. For our experimental maps, which are of course not universal, many of these statements hold. While the attractors are deformed by shear (shearing is one of the unstable eigendirections[4] near T^*), this has no effect on Liapunov exponents, the destruction of windows in the 1D map by boundary crisis, and other gross features. Indeed, we find experimentally that this is the case. Details will be presented in an extended paper. It should be stressed, however, that these findings are a direct consequence of the renormalization theory which is the main result of this work.

In summary, we found numerically that strange attractors in different maps disappear due to heteroclinic crises in a systematic way as the Jacobian is increased. This suggested that this sequence of disappearances is part of a universal scenario, which we proved by using renormalization theory. This theory implies universality not only for the heteroclinic crisis producing the disappearances of attractors but for all other aspects of strange attractors in dissipative systems.

This work was supported in part by U.S. Department of Energy Contract No. DE-AC02-84 ER 13146. One of the authors (G. Gy.) is indebted to P. Szepfalusy for illuminating discussions on universality in dissipative maps.

*Permanent address: Institute of Theoretical Physics, Eotvos University, Budapest, Hungary.

[1]M. J. Feigenbaum, J. Stat. Phys. **19**, 25 (1978); **21**, 669 (1979).

[2]S. Grossman and S. Thomae, Z. Naturforsch. A **32**, 1353 (1977).

[3]E. B. Vul and K. M. Khanin, Russ. Math. Surv. **37**, 200 (1982).

[4]J. M. Greene, R. S. Mackay, F. Vivaldi, and M. J. Feigenbaum, Physica D **3**, 468 (1981).

[5]P. Holmes, Phys. Lett. **104A**, 299 (1984).

[6]G. Schmidt and B. W. Wang, Phys. Rev. A **32**, 2994 (1985).

[7]A. B. Zisook, Phys. Rev. A **24**, 1640 (1981).

[8]G. R. W. Quispel, Phys. Rev. A **31**, 3924 (1985).

[9]Note that certain maps drive the trajectory towards an attractor lying at infinity. This can be prevented by defining the map on the torus instead of the plane.

[10]C. Grebogi, E. Ott, and J. A. Yorke, Physica D **7**, 181 (1983).

[11]P. Holmes and D. Whitley, Philos. Trans. R. Soc. London, Ser. A **311**, 43 (1984).