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Stability and dynamics of a noise-induced stationary state

Charles R. Doering
*Department of Physics, Center for Relativity and Department of Astronomy, The University of Texas at Austin, Austin, Texas 78712 and Theoretical Division and Center for Nonlinear Studies, MS B258, Los Alamos National Laboratory, Los Alamos, New Mexico 87545**

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The dynamics of several model systems which display a noise-induced transition to bistability are studied in detail, focusing in particular on the question of the stability of the noise-induced stationary states. The examples exhibit a variety of phenomena in the high-noise limit deep in the bistable regime. The noise-induced states can have an infinite lifetime, a finite lifetime, or a vanishingly small lifetime. Some general results concerning the high-noise limit are reported, most notably the destablizing effect of small correlations in the noise.

The influence of external fluctuations on nonlinear systems has been a topic of growing interest in recent years.[1-3] Among the new phenomena arising from the interplay of nonlinearities and fluctuating nonequilibrium constraints are noise-induced transitions (NIT's) where the qualitative state of a system changes from its deterministic or low-noise behavior when it is subjected to relatively strong broad-band perturbations.[4] In particular, NIT's can consist of a transition to bistable behavior in a deterministically monostable system, and these are the systems which will concern us here. It is of great interest to determine the stability of these noise-induced states which have no deterministic counterpart. Are they long lived? And if so, on what time scale? As shown below, the answer depends not only on the details of the dynamics of the system in question, but also on the detailed characteristics of the noise and its coupling to the system. We will discuss two systems which illustrate some general features of the high-noise regime, but leave the full treatment to a report to be published elsewhere.[5]

To review briefly the concept of a NIT to bistability, consider the one-variable dynamical system

$$dx/dt = f(x) + \mu g(x) \ ,$$

where for each fixed value of the external constraint μ there is a unique stable stationary state. When the environment fluctuates rapidly like a Gaussian white noise, i.e., $\mu = \sigma\xi(t)$ with

$$\langle \xi(t) \rangle = 0, \ \langle \xi(t)\xi(s) \rangle = \delta(t-s) \ ,$$

the dynamical equation becomes a stochastic differential equation (SDE). The probability distribution of the state variable, $\rho(x,t)$, evolves according to a Fokker-Planck equation (FPE) which is not unique.[6] The FPE depends on whether the SDE is a continuous-time version of a certain discrete-time problem (the Ito interpretation) or the white-noise limit of a colored-noise problem (the Stratonovich interpretation).[7] The FPE may be written

$$\partial_t \rho = L_{\mathrm{FP}} \rho = \partial_x [-f + (v-2)(\sigma^2/2) g g' + (\sigma^2/2)\partial_x] \rho \ ,$$

where $v = 2$ is the Ito FPE and $v = 1$ is the Stratonovich FPE, and the stationary probability distribution of the system is

$$\rho_s(x) = N g(x)^{-v} \exp\left[(2/\sigma^2) \int^x f/g^2\right] \ ,$$

where N is a normalization constant.

The qualitative shape of the distribution, i.e., the number and location of its extrema, can depend on the amplitude σ of the noise when g is not constant. Of interest here are the cases when ρ_s is single peaked for small values of σ, but double peaked above a critical intensity σ_c of the noise. Above σ_c the system is most likely found at one of the two peaks of the probability distribution, and these are identified as the noise-induced stationary states. This peak splitting arises from a competition between the drift (f) which tends to drive the system toward the steady state fixed by the average value of the noise, and the diffusion ($\sigma g \xi$) which constantly kicks the system away from the deterministically preferred state. Identifying the amplitude of the external fluctuations with the control parameter and the peak(s) of the distribution with the order parameter, this phenomenon is called a noise-induced phase transition to bistability.

There are several techniques available to study the stability and dynamics of these noise-induced states. One is the mean first-passage time (MFPT) from one state to the other. This is a concept which has been thoroughly studied in the context of tunneling in thermally activated multistable systems, and for one-variable problems all the moments of the first-passage time are calculable (up to quadrature).[8] If we denote by $G(x,x_0)$ the Green's function of the Fokker-Planck operator

$$L_{\mathrm{FP}} G(x,x_0) = -\delta(x-x_0)$$

with vanishing boundary conditions at $a \le x_0 \le b$ then the nth moment of the time it takes to diffuse from x_0 to the exterior of $[a,b]$ is

$$\langle T^n \rangle = n! \int dx_1 G(x_1,x_0) \int dx_2 G(x_2,x_1) \dots \times \int dx_n G(x_n,x_{n-1})$$

(all the integrals are over $[a,b]$). For systems with symmetric double-peaked distributions, the inverse of the

MFPT between a maximum and the minimum is proportional to the rate of switching between the most probable states.

Another tool at our disposal is the analysis of the low-lying spectrum of the FPE.[9] Often the spectrum is discrete and the eigenfunctions form a complete set with

$$0 \le \lambda_0 \le \lambda_1 \le \lambda_2 \le \dots .$$

The stationary state corresponds to $\lambda_0 = 0$, while the next two eigenvalues yield information about the stability and dynamics of a bistable system. The first eigenvalue (λ_1) controls the long-term "approach to equilibrium" and can be identified with the rate at which the system switches from one peak of the distribution to another. The second eigenvalue (λ_2) determines the time scale on which initial distributions concentrated between the peaks decay to one peak or the other. In general, the spectrum is not exactly known, but it may be approximated by variational techniques. Variational principles usually give only upper bounds on the eigenvalues, but as has been recently pointed out,[9] Weinstein's intermediate theorem[10] can be applied to FPE's to obtain lower bounds. I have extended the recently discovered supersymmetry in the FPE[11] to systems with multiplicative noise and made great use of this to simplify the calculation of bounds on the eigenvalues.

The first example system we consider, called the genetic model, is given by the SDE

$$dx/dt = -\gamma x + (1-x^2)\sigma\xi, \quad x \in [-1,1] .$$

This equation is applicable as a dynamic model of genetic selection and it can be related to a nonlinear chemical reaction.[4] This system has a NIT to bistability at $\sigma_c^2 = \gamma/2$ (Ito) or $\sigma_c^2 = \gamma$ (Stratonovich), and as the amplitude of the noise is increased all the probability becomes concentrated at the end points of the state space ($x = \pm 1$).

Figure 1 is a plot of the low-lying spectrum and inverse MFPT from one peak of the probability distribution to the center of the state space ($\langle T \rangle^{-1}$) for the Ito interpretation of the SDE. The first eigenvalue is exactly computable ($\lambda_1 = \gamma$) and the inverse MFPT approaches γ as σ^2 diverges. Hence the noise-induced states have a finite lifetime in the large-noise limit. The full first passage time distribution can be computed in the large-noise limit, and we find an exponential probability density. The second eigenvalue diverges so that the system falls immediately into one of the noise-induced states when $\sigma^2 = \infty$. The large-noise limit of this system, a two-level system with an exponentially distributed waiting time in each state, is known as a dichotomous Markov process. The finite and exponentially distributed lifetime of the noise-induced states clearly indicates that they are *metastable* states.

On the other hand, λ_1 for this model in the Stratonovich interpretation has been computed in Ref. 9 and found to increase monotonically as the noise is increased. The inverse of the MFPT diverges along with λ_1 in the high noise limit indicating that the system degenerates into a two-level white noise as the amplitude of the noise increases. The distinction between the Ito and Stratonovich versions of this model exhibit the general large-noise behavior of systems which undergo a noise-induced transition to bistability on a bounded state space with entrance boundaries:

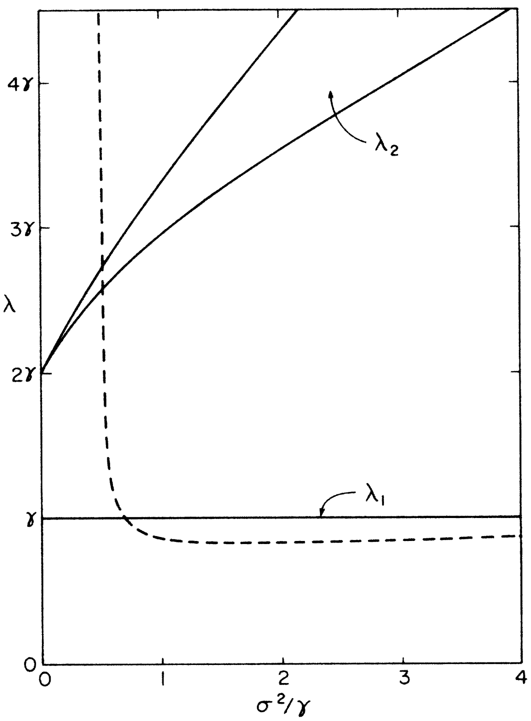

FIG. 1. Spectrum for the Ito interpretation of the genetic model. Upper and lower bounds are given for λ_2, and the dashed line is $\langle T \rangle^{-1}$.

The Ito process always goes to a dichotomous Markov process with

$$\lambda_1 \le [f(x_l) - f(x_r)]/(x_r - x_l) ,$$

where x_r and x_l are the end points so the dynamics are governed by the local *deterministic* time scales at the end points of the state space, where all the probability is concentrated. In the Stratonovich interpretation of such an SDE however, the MFPT always vanishes in the high-noise limit so that the white-noise behavior is generic.

This result does not hold for systems on an unbounded state space. For example, we consider Hongler's model[12]

$$dx/dt = -\gamma \tanh(x) + \sigma\xi/\cosh(x), \quad x \in [-\infty,\infty] .$$

A noise-induced transition to bistability occurs at $\sigma_c^2 = 2 \times \gamma$ (Stratonovich) or $\sigma_c^2 = \gamma$ (Ito). The peaks of the bistable distribution are uniformly exponentially localized and move away from each other proportionally to $\log(\sigma)$. As a Stratonovich equation this model is exactly soluble. A change of variables to $y = \sinh(x)$ yields a linear equation for y, i.e., y is a Gaussian Ornstein-Uhlenbeck process. Since the spectrum is unaffected by such a change in variables, the eigenvalues are simply

$$\lambda_n = n\gamma, \quad n = 0,1,2, \dots .$$

The inverse MFPT decreases from ∞ at $\sigma^2 = 2$ to 1.11γ as σ^2 diverges. Since λ_1 and $\langle T \rangle^{-1}$ are finite in the large-noise limit, this model displays a metastability as in the genetic model in the Ito interpretation.

Honlger's model behaves quite differently as an Ito equation. Figure 2 is a plot of the low-lying spectrum and inverse MFPT as a function of the ($\log_{10}$ of the) noise amplitude. Both λ_1 and $\langle T\rangle^{-1}$ vanish in the large noise limit indicating that the noise-induced states become *absolutely stable.* The second eigenvalue remains finite so that it still takes a finite amount of time for the system to diffuse into one of the two degenerate "ground states" in the large noise limit.

These examples illustrate the general behavior in the large-noise limit: Identical systems will evolve faster in the Stratonovich interpretation. Noise-induced states which are absolutely stable or metastable for Ito equations may be merely metastable or completely unstable as Stratonovich equations, so the question of modeling is crucial. This is hinted at by the presence of the noise-induced drift [the coefficient of $(v-2)$] in the Stratonovich FPE. The deterministic evolution time scales controlled by $f(x)$ can be overwhelmed by the large term $(\sigma^2/2)g(x)g'(x)$.

Since the Stratonovich FPE results from a white-noise limit of a colored noise problem, this suggests that even vanishingly small correlations in the driving noise have a severe destabilizing effect on the noise-induced states. This is in contrast to the situation for a deterministically bistable system subjected to a small amount of additive noise where small correlations increase the lifetime of the metastable states.[13] It is straightforward to use a recently developed effective FPE for colored noise problems[14] to verify the destablizing effect of small correlations directly in Hongler's model in the Stratonovich interpretation. This destablization should be kept in mind in the numerical simulation of Ito equations with multiplicative noise in the large-noise limit. If there are small correlations in a discrete time simulation, the continuous time limit is a Stratonovich equation and this fact will be reflected in the faster large noise dynamics.

The important physical point of this result is that the small correlations can enhance the noise absorption of a nonlinear system when it is considered as a filter. The genetic model degenerates into a bounded white noise in the Stratonovich interpretation so the system's fluctuations contain no power. The limit for the Ito equation, the dichotomous Markov process, has a nonvanishing power spectrum and can strongly influence other systems to which it is coupled.[15]

It is worthwhile to compare the spectra obtained above to that of a system exhibiting a deterministic transition to bistability. Since the NIT's are inherently *nonlinear* and *stochastic,* we consider the stochastic Landau-Ginsberg (SLG) model

$$dx/dt = \mu x - x^3 + \sigma\xi \ ,$$

where the amplitude of the noise is kept fixed and μ is the control parameter. This model has a transition to bistability as μ is increased through $\mu = 0$. Figure 3 is a plot of $\langle T\rangle^{-1}$ and upper bounds for λ_1 and λ_2 computed according to the techniques of Ref. 11 for this model with $\sigma = 1$. Lower bounds are also computed in Ref. 9 and these ensure that our plots give the qualitative behavior of the eigenvalues. Both λ_1 and $\langle T\rangle^{-1}$ vanish as the control parameter μ diverges. The two states at $\pm\mu^{1/2}$ become absolutely stable deep in the bistable regime. Note that the spectrum displays no anomolous behavior near the transi-

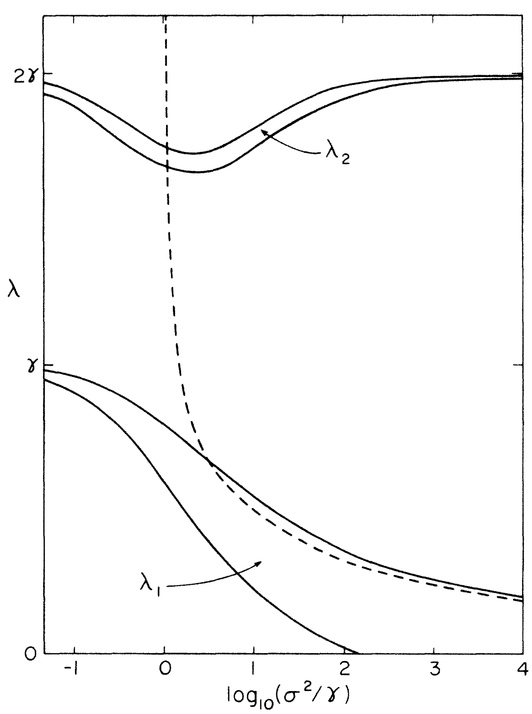

FIG. 2. Spectrum for the Ito interpretation of Hongler's model. Upper and lower bounds are given for λ_1 and λ_2, and the dashed line is $\langle T\rangle^{-1}$.

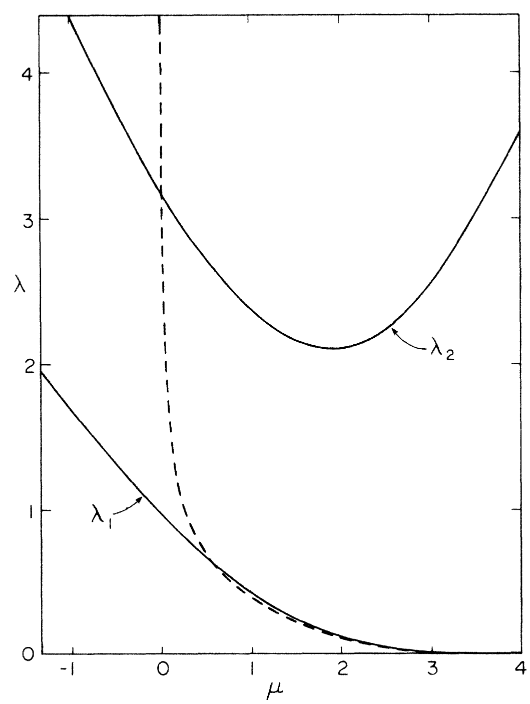

FIG. 3. Upper bounds for λ_1 and λ_2 for the stochastic Landau-Ginsberg model, indicating their qualitative behavior. The dashed line is $\langle T\rangle^{-1}$.

tion point: Using the supersymmetry of the FPE it is easy to show that when $\mu = 0$,

$$0.56\sigma \leq \lambda_1 \leq 0.98\sigma \ .$$

Hence there is *no critical slowing down in terms of the spectra in the SLG model.* The usual concept of critical slowing down in this model comes from considering either a *linearized* or *deterministic* system. The NIT and bistable region in the Ito version of Hongler's model behave just like the SLG model (up to the divergence of λ_2 deep in the bistable regime for the SLG model[9]). There is even a minimum of λ_2 just above the transition as for the SLG model indicating a relative sluggishness of the decay into one of the bistable states. As pointed out in Ref. 4, the correct quantity to consider when looking for a critical slowing down in a stochastic transition to bistability (in few variables) is the time it takes to develop a double-peaked probability distribution starting from a single-peaked one. The result above shows that NIT's to bistability can be completely analogous to a deterministic transition subjected to external fluctuations.

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*Present address.

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