New approach to the problem of chaotic repellers

P. Szépfalusy
Central Research Institute for Physics, Hungarian Academy of Sciences, P.O. Box 49, H-1525 Budapest 114, Hungary
and Institute for Theoretical Physics, Eötvös University, H-1088 Puskin u.5-7, Budapest, Hungary

T. Tél
Institute for Theoretical Physics, Eötvös University, H-1088 Puskin u.5-7, Budapest, Hungary
(Received 25 February 1986)

It is shown that a smooth stationary distribution can be derived for "coarse-grained" repellers by compensating the local escape. This provides a convenient framework to investigate statistical properties of long chaotic transients. Furthermore, our procedure yields a powerful tool for calculating fractal dimensions.

Chaotic repellers related to long-lasting chaotic transients may play an important role in dissipative dynamical systems (see Refs. 1–6 and references therein). The fractal structure of chaotic repellers can be deduced from finite-resolution measurements which are made in practice by using a fixed grid of phase space. A chaotic repeller specified by those cells of the grid which cover it will be called a coarse-grained repeller (CGR) in the following. The aim of the present paper is to discuss how a stationary distribution can be defined and calculated for the asymptotic dynamics on such CGR's, meaningful in the limit when the resolution becomes infinitely fine. It is important that such a coarse-grained distribution is accessible in a natural way in numerical simulations or in experiments.

Let us first consider chaotic repellers appearing in one-dimensional maps $x'=f(x)$ where f is a noninvertible function. We assume that there are no critical points [i.e., points where $f'(x)=0$] of f in the vicinity of the repeller. A smooth initial distribution does not remain localized on the CGR; probability flows away from it. The probability of finding a point on the repeller specified with a resolution ϵ inside an interval of length Δx ($\gg\epsilon$), which overlaps with the repeller and is much shorter than the diameter of the repeller, is $N(\epsilon)\epsilon/\Delta x$, where $N(\epsilon)$ is the number of bins covering the repeller inside Δx. According to the definition of the fractal dimension,[7] $N(\epsilon)\sim\epsilon^{-d}$. The probability that a point of the image interval $\Delta x' =f'(x)\Delta x$ belongs to the CGR is $N(\epsilon\Delta x/|\Delta x'|)\epsilon/|\Delta x'|$ since the map is locally linear above the CGR. Thus, after one step of the discrete dynamics the probability of being on the CGR has changed by a factor $|\Delta x/\Delta x'|^{1-d}$. Therefore, we introduce a *local* escape rate $\alpha(x)$ by the relation

$$\exp(\alpha(x))=|f'(x)|^{1-d}\,. \tag{1}$$

Note that $\alpha(x)$ has proved to be independent of the resolution.

The flowing out of the probability from a CGR can be compensated by multiplying the probability of being on the CGR in Δx by $\exp(\alpha(x))$. [It is assumed that the local escape rate (1) can be applied to a good approximation even when Δx is finite.] This leads to the iteration procedure for a distribution $\widetilde{P}_n(x,\epsilon)$ on the CGR as follows:

$$P_{n+1}(x',\epsilon)\equiv\sum_{x\in f^{-1}(x')}\frac{\widetilde{P}_n(x,\epsilon)}{|f'(x)|^{d}}\,, \tag{2}$$

where $\widetilde{P}_m(x,\epsilon)=P_m(x,\epsilon)$ if x belongs to the CGR and is zero otherwise. Starting with any smooth initial function $P_0(x)$, it is assumed that $\widetilde{P}(x,\epsilon)=\lim_{n\to\infty}\widetilde{P}_n(x,\epsilon)$ exists and is unique. The dependence of $\widetilde{P}(x,\epsilon)$ on ϵ is weak if the grid is fine enough. In this case, $\widetilde{P}(x,\epsilon)$ can be replaced by $\widetilde{P}(x)$ which is obtainable by a simpler procedure. Namely, $\widetilde{P}(x)=P(x)$ on the CGR and is zero otherwise, where

$$P(x)=\lim_{n\to\infty}P_n(x) \tag{3}$$

with

$$P_{n+1}(x')=\sum_{x\in f^{-1}(x')}\frac{P_n(x)}{|f'(x)|^{d}}\,. \tag{4}$$

For $d=1$, (4) is the Frobenius-Perron equation[8] for one-dimensional maps.

A stationary distribution $\widetilde{P}(x)$ for small ϵ corresponds to the distribution obtained by observing trajectories remaining for a sufficiently long time on the CGR. This relation representing a kind of ergodicity will be demonstrated later on. Accordingly, the mean value of a certain physical quantity $g(x)$ can be calculated up to an accuracy given by the grid size ϵ as

$$\langle g(x)\rangle_\epsilon=\sum_i\widetilde{P}(x_i)g(x_i)\Big/\sum_i\widetilde{P}(x_i)\,, \tag{5}$$

where the sums run over the bins. It is supposed that both $\widetilde{P}(x)$ and $g(x)$ are smooth on the scale of ϵ and, thus, x_i can be chosen to be a typical point, say the midpoint, of the ith bin.

First, we illustrate how $P(x)$ is achieved from iterating

Eq. (4) in a simple example. We consider the asymmetric tent map defined by $f(x)=1-a_1x$ for $x\geq 0$ and $f(x)=1+a_2x$ for $x<0$, where a_1,a_2 are positive numbers. If the condition $a_1^{-1}+a_2^{-1}<1$ is fulfilled, a chaotic repeller shows up. The evolution of a linear initial distribution $P_0(x)=\alpha_0 x+\beta_0$ governed by (4) can then be followed exactly. The result is $P_n(x)=\alpha_n x+\beta_n$ with $\alpha_{n+1}=\alpha_n(a_2^{-1-d}-a_1^{-1-d})$, $\beta_{n+1}=\beta_n(a_1^{-d}+a_2^{-d})-\alpha_{n+1}$. Since $a_1,a_2>1$, α_n tends always toward zero, but β_n remains finite only for $a_1^{-d}+a_2^{-d}=1$, which is exactly the condition for the fractal dimension of the repeller (see, e.g., Ref. 5). The corresponding stationary solution of (4) is then a finite constant distribution. The stability of this solution in the space of more general smooth distributions can be checked numerically.

This example shows that Eq. (4) can also be considered as an eigenvalue problem for the dimensionality d at which a nontrivial stable stationary solution exists. Furthermore, it also illustrates that the corresponding eigenfunction may be smooth (see also Figs. 2 and 3).

We have investigated in some detail the parabola map given by $f(x)=1-ax^2$, $a>2$. In this case no exact solution of (4) has been found.

In order to determine the stable stationary solution $P(x)$ of (4) with a certain accuracy we used the iterates $P_n(x)$, $n=1,2,\ldots$, of a constant initial distribution. The explicit expression for $P_n(x)$ is easy to write down and can be evaluated numerically. If the value of d is appropriately adjusted, the convergence toward a stationary solution has proved to be as fast as at $a=2$; an accuracy of less than 1% is reached at the fourth iterate. We investigated the first six iterates of $P_0(x)=\text{const}$ as a function of the parameter d at a fixed a. As long as d is too large, $P_n(x)$ monotonously decreases with n at a fixed x_0 $(=0.5)$. At a low value of d a monotonous increase sets in. We used this property to obtain an upper bound for the fractal dimension as the value of d where $P_5(x_0)$ first became less than $P_6(x_0)+1.5\times 10^{-3}$ when decreasing d, and a lower bound as the value where $P_6(x_0)>P_5(x_0)+1.5\times 10^{-3}$ was first realized. For $a=2.05$, for example, we obtained $d=0.904\pm 0.002$. Figure 1 exhibits the plot d versus a in the region $2\leq a\leq 3$. The method sketched here turns out to be a rather efficient one for calculating the fractal dimension of chaotic repellers, the numerics of which can be made on home computers. It is worth noting that this procedure can be applied also for Cantor sets arising in other contexts if, with the aid of the construction rule, an auxiliary map has been found, the repeller of which is the set in question.

Histograms at different values of a were constructed by starting iterations from N_0 points distributed uniformly on the interval $x^*<x<-x^*$, $x^*=-(1+\sqrt{1+4a})/(2a)$, i.e., between the two "end points" of the repeller. The numerical procedure we applied was practically that of Kantz and Grassberger.[5] The first ten iterations were omitted and the simulation was stopped when $|x|$ became larger than $|x^*|$. The last 30 steps before that were omitted, too. N_0 is to be chosen in such a way that the content of nonempty bins in the resulting histogram is much larger than unity. The histogram obtained at $a=2.05$ with $N_0=2\times 10^5$ is shown in Fig. 2, together with the stable stationary solution of Eq. (4) restricted to the CGR at the same value of a and d.

As a case where the fractal dimension was determined independently, we have investigated the chaotic repeller coexisting with the superstable period-3 cycle in the parabola map. This situation occurring at $a=1.75488$ was first studied in Ref. 5 and $d=0.9335\pm 0.0002$ was obtained. Starting again with a constant initial distribution and substituting the value of d given above, the iteration prescribed by Eq. (4) has been found to converge fast numerically. The distribution $\tilde{P}(x)$ on the CGR deduced from the sixth iterate and the histogram are displayed in Fig. 3.

In order to give evidence also for the quantitative agreement between histogram and $\tilde{P}(x)$ we list here the Lyapunov exponents computed numerically from the histogram and that obtained via Eqs. (3)–(5). For $a=2.05$, $\lambda_{\text{hist}}=0.752$, $\lambda_{\text{anal}}=0.759$ (± 0.01), whereas at $a=1.75488$, $\lambda_{\text{hist}}=0.490$, $\lambda_{\text{anal}}=0.481$ (± 0.01).

The existence of Eq. (4) may be of importance also for general theoretical purposes. It is easy to show from (4) that for maps defined by a symmetric $f(x)$ and having a symmetric repeller (such as in Fig. 2) a symmetric $P(x)$ is exceptional since a necessary condition for its existence is $|f'(x^*)|^d=2$. A consequence of this generic asymmetry is that the metric entropy[9] of the map is less than log2.

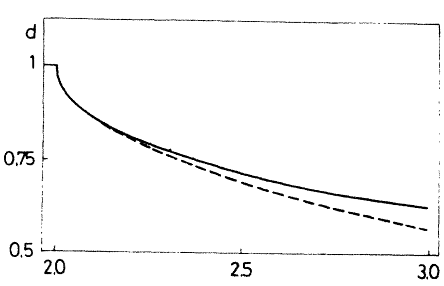

FIG. 1. Fractal dimension d obtained as eigenvalue of Eq. (4) for the map $x'=1-ax^2$ in the interval $2\leq a\leq 3$. The dashed line corresponds to $1-d=0.44(a-2)^{1/2}$ which has the scaling form valid near $a=2$.

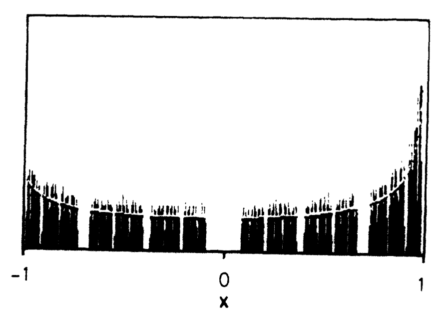

FIG. 2. Histogram of the stationary distribution on the coarse-grained repeller of the map $x'=1-2.05x^2$ obtained in a numerical simulation. Bin size, $\epsilon=\frac{1}{160}$; number of bins, 320. $N_0=2\times 10^5$, the histogram consists of $N=149\,149$ events. The curve drawn through the histogram in inverted color is the stationary solution $\tilde{P}(x)$ restricted to the CGR with $d=0.904$. The same normalization was used for both histogram and curve.

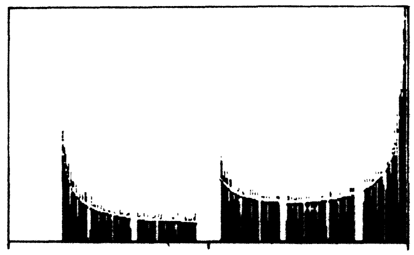

FIG. 3. Same as Fig. 2 for $x' = 1 - ax^2$, with $a = 1.754\,88$, $n_0 = 3 \times 10^4$ initial points in $(1-a, 1)$, $N = 238\,521$ events for the histogram, $d = 0.9335$.

We now turn to a short discussion of how to extend the ideas discussed above to higher-dimensional systems. We illustrate it by considering invertible maps of the plane $\mathbf{x}' = T(\mathbf{x})$. At nearly all points of chaotic trajectories a stable and an unstable direction exists.[10,11] Let $\lambda_i(\mathbf{x})$, $i = 1,2$, denote the local coefficients of expansion[12] along unstable ($i = 1$) and stable ($i = 2$) directions.

The local rate describing escape along the unstable direction should be defined in analogy with (1) as

$$e^{\alpha(\mathbf{x})} = |\lambda_1(\mathbf{x})|^{1-d_1} , \tag{6}$$

where d_1 is the partial fractal dimension[11] along the unstable direction. For chaotic attractors, of course, $d_1 = 1$, while a value d_1 in the region $0 < d_1 < 1$ characterizes a repeller (sometimes called semiattractor[5]). The procedure, how a finite stationary distribution can be found in this case, is most conveniently formulated in terms of a certain measure, analogous with the natural invariant measure[12,13] of chaotic attractors.

Let us follow the development of the probability μ_j that a point is in a tiny area around $\mathbf{x}_j$ overlapping with the CGR. To compensate the escape from the CGR, μ_j is to be multiplied by $e^{\alpha(\mathbf{x}_j)}$ prior to the map being applied. Thus, after one step,

$$(\mu_j)' = T(\mu_j e^{\alpha(\mathbf{x}_j)}) , \tag{7}$$

where the induced map in the space of measures has been denoted by the same symbol as the map itself. This is then to be repeated after dividing the support of the new measure into regions within which $\alpha(\mathbf{x})$ can be regarded as constant. When the number of iterations becomes large the distribution gets to be concentrated along narrow strips. The situation is similar to the one-dimensional problem which leads one to assume the existence of a limiting measure which is absolutely continuous along the unstable direction. In practice, the thickness of the strips are fixed by the finite resolution. The distribution after restricting it to the CGR can be compared with the results of numerical simulations considering trajectories staying close to the repeller for asymptotically long times.

Alternatively, if the fractal dimension d_1 is not known, the condition for the existence of the limiting distribution in the sense above yields its value. Note that the determination of d_1 by this way does not require the knowledge of the geometrical structure of the repeller.[14]

To illustrate this method with an example, we consider an extension of the well-known baker transformation[10] in a region of parameters where a chaotic repeller exists. The dynamics is then defined by $x' = ax$, $y' = sy$ for $y < c$ and $x' = bx + \frac{1}{2}$, $y' = 1 - t(1-y)$ for $y > c$, where $a,b,c < \frac{1}{2}$ and $sc, t(1-c) > 1$. Starting with the Lebesgue measure on the unit square, an application of (7) leads to a measure $\sigma = s^{-d_1}$ and $\tau = t^{-d_1}$ on the strips $0 < x < a$, $0 < y < 1$ and $\frac{1}{2} < x < \frac{1}{2} + b$, $0 < y < 1$, respectively, since $\lambda_1(\mathbf{x}) = \partial y' / \partial y$ in this case. The total measure on the unit square is conserved if $\sigma + \tau = 1$ which yields the condition determining d_1. After n steps the measure on strips of width $a^m b^{n-m}$ will be $\sigma^m \tau^{n-m}$. The limiting distribution obtained for $n \to \infty$ has a smooth density along the unstable direction. On CGR we have a fractal measure in the spirit of Ref. 13, i.e., the fractal and pointwise dimensions do not agree. There is an interesting change also in the behavior of dimensions along the stable direction. The fractal dimension d_2 is determined by $a^{d_2} + b^{d_2} = 1$ both in the region of the attractor treated in Ref. 10 [where $sc = t(1-c) = 1$] and in the region of a repeller. On the other hand, the pointwise dimension, obtained as $d_{2p} = (\sigma \log\sigma + \tau \log\tau)/(\sigma \log a + \tau \log b)$, depends heavily on the fractal structure of the unstable direction.

Finally, a general comment is in order. An averaged escape rate α can be defined for both one- and two-dimensional maps by

$$e^{\alpha} = \langle e^{\alpha(\mathbf{x})} \rangle , \tag{8}$$

where the brackets denote averaging with the stationary distribution on the CGR. From a cumulant expansion of $\langle e^{\alpha(\mathbf{x})} \rangle = \langle e^{(1-d_1)\log|\lambda_1(\mathbf{x})|} \rangle$ one obtains

$$\alpha = \sum_{n=1}^{\infty} (1-d_1)^n \frac{1}{n!} Q_n , \tag{9}$$

where Q_n is the nth cumulant of $\log|\lambda_1(\mathbf{x})|$. In particular, $Q_1 \equiv \lambda_1$ is the largest Lyapunov exponent. Relation (9) which has been conjectured and numerically verified by Kantz and Grassberger[5] appears now as an immediate consequence of our consideration.

We would like to acknowledge several valuable discussions with P. Grassberger and M. Misiurewicz. We also thank Z. Kaufmann for useful remarks.

[1]J. A. Yorke and E. D. Yorke, J. Stat. Phys. **21**, 263 (1979).
[2]C. Grebogi, E. Ott, and J. A. Yorke, Physica **7D**, 181 (1983).
[3]M. Widom, D. Bensimon, L. P. Kadanoff, and S. J. Shenker, J. Stat. Phys. **32**, 443 (1983).
[4]L. P. Kadanoff and C. Tang, Proc. Natl. Acad. Sci. U.S.A. **81**, 1276 (1984).
[5]H. Kantz and P. Grassberger, Physica **17D**, 75 (1985).
[6]J.-P. Eckmann and D. Ruelle, Rev. Mod. Phys. **57**, 617 (1985).
[7]B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, San Francisco, 1982).
[8]P. Collet and J.-P. Eckmann, *Iterated Maps on the Interval as Dynamical Systems* (Birkhauser, Cambridge, MA, 1980).

[9]P. Billingsley, *Ergodic Theory and Information* (Wiley, New York, 1965).

[10]D. Farmer, E. Ott, and J. A. Yorke, Physica **7D**, 153 (1983).

[11]P. Grassberger and I. Procaccia, Physica **13D**, 34 (1984).

[12]Ya. G. Sinai, Selecta Matematica Sovietica 1, 100 (1981).

[13]D. Farmer, Z. Naturforsch. **37a**, 1304 (1982).

[14]After finishing the manuscript, M. Misiurewicz called our attention to an article in the mathematical literature [H. McCluskey and A. Manning, Ergod. Theory Dynam. Sys. **3**, 251 (1983)] where a condition is given for the Hausdorff dimension of horseshoes. The connection between this condition and that given in our work needs further investigation. A step in this direction would be the following modification of our treatment. Instead of starting with a smooth measure on the plane, one should consider an initial distribution concentrated along the unstable manifold of a point of the repeller and follow its development in a similar way as in the case of attractors (see, e.g., Ref. 12), but replacing the coefficient of expansion by $[\lambda_1(\mathbf{x})]^{d_1}$. This procedure, however, seems to be rather difficult to apply for practical calculations.