Propagator for the time-dependent three-dimensional charged harmonic oscillator in a time-varying magnetic field

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We evaluate the exact propagator for the time-dependent three-dimensional charged harmonic oscillator in a time-varying magnetic field. We show that such a propagator can be obtained from that for a charged particle in a constant magnetic field.

Despite the vast range of the operational versatility of Feynman's path integration, the evaluation of the propagator for certain time-dependent systems, if carried out in a straightforward manner, can become much more difficult than to obtain the solution to the corresponding Schrödinger equation.

As an example of the state of the art, we point out the recent, formidable calculation of the propagator for the time-dependent forced harmonic oscillator with damping by Cheng,[1] via a generalized version of a method introduced by Montroll.[2] In contrast, the exact solution to the corresponding Schrödinger equation can be found in a much simpler way.[3] In another illustrative example, the exact evaluation of the propagator for a charged particle in a time-varying electromagnetic field was possible to be carried out *only* for the case of a constant cyclotron frequency.[4]

It would be, therefore, somewhat discouraging to proceed further on applying the aforementioned path-integration techniques for other more elaborated time-dependent problems. Rather, they appeal for alternative, versatile methods for the evaluation of propagators without undergoing to tedious and lengthy calculations, in such a way to make Feynman's path integration aesthetically more attractive.[5-8]

In this paper, by generalizing an earlier work,[5] we set forth an alternative protocol which makes it possible to evaluate the *exact* propagator for the time-dependent three-dimensional charged harmonic oscillator in a time-varying magnetic field. Overall, we show that such a propagator can be obtained from that for a charged particle in a constant magnetic field.

We begin by writing the Hamiltonian of our system as

$$H(\mathbf{p},\mathbf{x},t)=\frac{1}{2m(t)}\left[\mathbf{p}+\frac{q}{c}\mathbf{A}(t)\right]^2 + \tfrac{1}{2}m(t)\omega^2(t)(x^2+y^2+z^2)\ , \tag{1}$$

where the time-varying magnetic field $B(t)$ is applied along the z axis and the gauge is chosen such that the vector potential $\mathbf{A}$ is given by $[\frac{1}{2}B(t)y,\ -\frac{1}{2}B(t)x,0]$. Then, the corresponding Schrödinger equation reads

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m(t)}\nabla^2\psi+\frac{\hbar\omega_c(t)}{2i}\left[y\frac{\partial\psi}{\partial x}-x\frac{\partial\psi}{\partial y}\right] + \tfrac{1}{2}m(t)[\Omega^2(t)(x^2+y^2)+\omega^2(t)z^2]\psi\ , \tag{2}$$

where $\Omega^2(t)\equiv\omega^2(t)+\frac{1}{2}\omega_c^2(t)$, with $\omega(t)$ and $\omega_c(t)$ $[=qB(t)/m(t)c]$ being the harmonic and cyclotron frequencies, respectively.

Let a special solution of Eq. (2) be of the form[9]

$$\psi(\mathbf{x},t)=K(\mathbf{x},t;\mathbf{x}_0,t_0)\ , \tag{3}$$

subject to

$$\lim_{t\to t_0}K(\mathbf{x},t;\mathbf{x}_0,t_0)=\delta(\mathbf{x}-\mathbf{x}_0)\ , \tag{4}$$

with

$$\mathbf{x}\equiv(x,y,z)\ \text{ and }\ \mathbf{x}_0\equiv(x_0,y_0,z_0)\ .$$

This K (the kernel or propagator) gives us the solution for any arbitrary initial state, $\psi(\mathbf{x}_0,t_0)$:

$$\psi(\mathbf{x},t)=\int K(\mathbf{x},t;\mathbf{x}_0,t_0)\psi(\mathbf{x}_0,t_0)dx_0dy_0dz_0\ . \tag{5}$$

For $t>t_0$, K is defined as the amplitude to go from $(\mathbf{x}_0,t_0)$ to $(\mathbf{x},t)$, and for $t<t_0$ K is zero. So, we may write

$$\left[i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m(t)}\nabla^2-\frac{\hbar\omega_c(t)}{2i}\left[y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right] - \tfrac{1}{2}m(t)[\Omega^2(t)(x^2+y^2)+\omega^2(t)z^2]\right]K=0\ . \tag{6}$$

Following Feynman[10] and others[5,11] we seek the propagator (for quadratic potentials) with the structure

$$K(\mathbf{x},t;\mathbf{x}_0,t_0)=\phi(t,t_0)\exp\left[\frac{i}{\hbar}S(\mathbf{x},t;\mathbf{x}_0,t_0)\right]\ , \tag{7}$$

where $S(\mathbf{x},t;\mathbf{x}_0,t_0)$ is Hamilton's two-point characteristic function defined as the action along a real path connecting $(\mathbf{x}_0,t_0)$ to $(\mathbf{x},t)$, and the preexponential modulating function $\phi(t,t_0)$, depending only on time, is determined such that K satisfies Eq. (4).

After the substitution of Eq. (7) into Eq. (6) and some rearrangements, we obtain from its real and imaginary parts, respectively,[5]

$$\frac{\partial S}{\partial t}+\frac{1}{2m(t)}\left[\left[\frac{\partial S}{\partial x}+\frac{m(t)\omega_c(t)}{2}y\right]^2+\left[\frac{\partial S}{\partial y}-\frac{m(t)\omega_c(t)}{2}x\right]^2+\left[\frac{\partial S}{\partial z}\right]^2\right]+\tfrac{1}{2}m(t)\omega^2(t)[x^2+y^2+z^2]=0\ , \tag{8a}$$

or

$$\frac{dS}{dt}=L[\mathbf{x}(t),\dot{\mathbf{x}}(t),t] \tag{8b}$$

and

$$\frac{1}{\phi}\frac{\partial\phi}{\partial t}=-\frac{1}{2m(t)}\nabla^2 S\ , \tag{9}$$

where we have defined

$$L[\mathbf{x}(t),\dot{\mathbf{x}}(t),t]=\tfrac{1}{2}m(t)\{\dot{x}^2(t)+\dot{y}^2(t)+\dot{z}^2(t)+\omega_c(t)[x(t)\dot{y}(t)-y(t)\dot{x}(t)]-\omega^2(t)[x^2(t)+y^2(t)+z^2(t)]\}\ , \tag{10}$$

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\ , \tag{11a}$$

$$\mathbf{v}=\frac{1}{m(t)}\left[\frac{\partial S}{\partial x}+\frac{m(t)\omega_c(t)}{2}y,\ \frac{\partial S}{\partial y}-\frac{m(t)\omega_c(t)}{2}x,\ \frac{\partial S}{\partial z}\right]. \tag{11b}$$

The path of the particle

$$\mathscr{x}(t^*)\equiv[\mathscr{x}(t^*),\mathscr{y}(t^*),\mathscr{z}(t^*)] \tag{12a}$$

is assumed to obey

$$\mathscr{x}(t^*=t)\equiv\mathbf{x}=(x,y,z) \tag{12b}$$

with

$$\mathscr{x}(t^*=t_0)\equiv\mathbf{x}_0=(x_0,y_0,z_0) \tag{12c}$$

and

$$\mathscr{x}(t^*=t)\equiv\mathbf{x}=(x,y,z) \tag{12d}$$

representing the initial and end points, respectively. The canonical momentum, in turn, is denoted as $\mathbf{p}=\nabla S$.

Integration of Eq. (8b) along the path $\mathscr{x}(t^*)$ of the particle yields

$$S(\mathbf{x},t;\mathbf{x}_0,t_0)=S[\mathscr{x}(t^*)]=\int_{\mathbf{x}_0,t_0}^{\mathbf{x},t}dt^*L[\mathscr{x}(t^*),\dot{\mathscr{x}}(t^*),t^*]\ . \tag{13}$$

In turn, Eq. (9) can be integrated in an ordinary way to give

$$\phi(t,t_0)=\phi_0(t_0)\exp\left[-\tfrac{1}{2}\int_{t_0}^{t}dt^*\frac{1}{m(t^*)}\nabla^2 S\right]\ , \tag{14}$$

where the constant $\phi_0(t_0)$ is introduced in Eq. (14) in order to match the initial condition Eq. (3).

By differentiating Eq. (8) with respect to $\mathbf{x}$ and using definitions (11) and (12) [or by applying the variational method on Eq. (13)], one obtains

$$\ddot{\mathscr{x}}+(\dot{m}/m)\dot{\mathscr{x}}+\omega^2\mathscr{x}=\omega_c\dot{\mathscr{y}}+[\dot{\omega}_c+\omega_c(\dot{m}/m)]\mathscr{y}/2\ , \tag{15a}$$

$$\ddot{\mathscr{y}}+(\dot{m}/m)\dot{\mathscr{y}}+\omega^2\mathscr{y}=-\omega_c\dot{\mathscr{x}}-[\dot{\omega}_c+\omega_c(\dot{m}/m)]\mathscr{x}/2\ , \tag{15b}$$

$$\ddot{\mathscr{z}}+(\dot{m}/m)\dot{\mathscr{z}}+\omega^2\mathscr{z}=0\ . \tag{15c}$$

Let the following space-time transformations

$$\mathscr{x}(t)=\mathscr{s}(t)\mathscr{p}(\tau)\ , \tag{16a}$$

$$\mathscr{y}(t)=\mathscr{s}(t)\mathscr{q}(\tau)\ , \tag{16b}$$

$$\mathscr{z}(t)=\mathscr{s}(t)\mathscr{r}(\tau)\ , \tag{16c}$$

where τ is related to t through

$$\tau(t)=\int^t\mu(\lambda)d\lambda\quad[d\tau=\mu(t)dt]\ .$$

The functions $\mathscr{s}(t),\mu(t)$ are chosen in such a way to reduce the set of Eqs. (15) to a desired simpler form.

Analogously to Eqs. (12), the following notations are understood:

$$p\equiv\mathscr{p}(\tau^*=\tau),\quad p_0\equiv\mathscr{p}(\tau^*=\tau_0)\ ,$$

$$s\equiv\mathscr{s}(t^*=t),\quad s_0\equiv\mathscr{s}(t^*=t_0)\ ,$$

$$\mathscr{p}'(\tau)\equiv d\mathscr{p}(\tau^*)/d\tau^*|_{\tau^*=\tau}\ ,$$

$$\mathscr{p}''(\tau)\equiv d^2\mathscr{p}(\tau^*)/d\tau^{*2}|_{\tau^*=\tau}\ ,$$

$$\dot{\mathscr{s}}(t)\equiv d\mathscr{s}(t^*)/dt^*|_{t^*=t}\ ,$$

$$\ddot{\mathscr{s}}(t)\equiv d^2\mathscr{s}(t^*)/dt^{*2}|_{t^*=t}\ ,$$

$$\dot{s}\equiv\dot{\mathscr{s}}(t),\quad \dot{s}_0\equiv\dot{\mathscr{s}}(t_0)\ ,$$

and so on.

After substituting Eqs. (16) into Eqs. (15), we obtain[5]

$$\mathscr{p}''=\omega_{0c}\mathscr{q}'\ , \tag{17a}$$

$$\mathscr{q}''=-\omega_{0c}\mathscr{p}'\ , \tag{17b}$$

$$\mathscr{r}''=0\ , \tag{17c}$$

where we have made

$$\frac{2\dot{\mathscr{s}}}{\mathscr{s}}+\frac{\dot{m}}{m}+\frac{\dot{\mu}}{\mu}=0\Longrightarrow\mu=c_1/\mathscr{s}^2m\ , \tag{18a}$$

$$\frac{2\dot{\mathscr{s}}}{\mathscr{s}}+\frac{\dot{m}}{m}+\frac{\dot{\omega}_c}{\omega_c}=0\Longrightarrow\omega_c=c_2/\mathscr{s}^2m\ , \tag{18b}$$

$$\ddot{\mathscr{s}}+\frac{\dot{m}}{m}\dot{\mathscr{s}}+\omega^2(t)\mathscr{s}=0 \; . \tag{18c}$$

The constants of integration are relabeled as $c_1=\overline{m}_0$ and $c_2=\overline{m}_0\omega_{0c}$ ($\overline{m}_0$ and ω_{0c} are also arbitrary constants), implying that $\omega_c(t)=\omega_{0c}\mu(t)$.

From the set of Eqs. (17) one may readily infer their associated Hamilton-Jacobi equation

$$\frac{\partial \overline{S}}{\partial \tau}+\frac{1}{2}\left[\left[\frac{\partial \overline{S}}{\partial p}+\frac{\omega_{0c}}{2}q\right]^2+\left[\frac{\partial \overline{S}}{\partial q}-\frac{\omega_{0c}}{2}p\right]^2 +\left[\frac{\partial \overline{S}}{\partial r}\right]^2\right]=0 \; , \tag{19a}$$

and the associated classical action

$$\overline{S}=\int_{\mathbf{p}_0,\tau_0}^{\mathbf{p},\tau} d\tau^*\overline{L}[\mathscr{p}(\tau^*),\mathscr{p}'(\tau^*)] \; , \tag{19b}$$

where

$$\overline{L}[\mathscr{p}(\tau),\mathscr{p}'(\tau)]=\frac{\overline{m}_0}{2}\{\mathscr{p}'^2(\tau)+\mathscr{q}'^2(\tau)+\mathscr{r}'^2(\tau) +\omega_{0c}[\mathscr{p}(\tau)\mathscr{q}'(\tau)-\mathscr{q}(\tau)\mathscr{p}'(\tau)]\} \; , \tag{19c}$$

with

$$\mathscr{p}(\tau^*)=[\mathscr{p}(\tau^*),\mathscr{q}(\tau^*),\mathscr{r}(\tau^*)] \; , \tag{19d}$$

and

$$\mathscr{p}(\tau^*=\tau_0)\equiv \mathbf{p}_0=(p_0,q_0,r_0) \tag{19e}$$

and

$$\mathscr{p}(\tau^*=\tau)\equiv \mathbf{p}=(p,q,r) \; . \tag{19f}$$

Now, since the variations $\delta S=0$ and $\delta\overline{S}=0$ are equivalent [that is, Eqs. (15) and (17) are equivalent], their corresponding Lagrangians [Eqs. (10) and (19c)] differ only by an additive total time derivative of some function f. This may be seen through

$$S=\overline{S}+[f(\mathbf{x},t)-f(\mathbf{x}_0,t_0)] \tag{20a}$$

($\delta f\equiv 0$, since f is a function of the initial and end points only) or

$$\int dt^*L=\int \mu(t^*)dt^*\overline{L}-\int dt^*(df/dt^*) \; , \tag{20b}$$

which implies

$$L(\mathscr{x},\dot{\mathscr{x}},t^*)=\mu(t^*)\overline{L}(\mathscr{p},\mathscr{p}')\,|_{\mathscr{p}=\mathscr{x}/\mathscr{s}}+\frac{df}{dt^*} \; . \tag{20c}$$

Thus, by substituting Eqs. (10), (19c), and (16) into Eq. (20c), and after some manipulations, we arrive at

$$\frac{df}{dt^*}=\frac{d}{dt^*}\left[\frac{m\dot{\mathscr{s}}}{2\mathscr{s}}(\mathscr{x}^2+\mathscr{y}^2+\mathscr{z}^2)\right] \; , \tag{20d}$$

such that we have

$$S=\overline{S}+\frac{m\dot{s}}{2s}(x^2+y^2+z^2)-\frac{m_0\dot{s}_0}{2s_0}(x_0^2+y_0^2+z_0^2) \; . \tag{20e}$$

Next, from (14) and (20e) we obtain

$$\phi(t,t_0)=\overline{\phi}(\tau,\tau_0)/(s)^{3/2} \; , \tag{21a}$$

where

$$\overline{\phi}(\tau,\tau_0)\equiv\phi_0(\tau_0)\exp\left[-\frac{1}{2\overline{m}_0}\int_{\tau_0}^{\tau}d\tau^*\overline{\nabla}^2\overline{S}\right] \; , \tag{21b}$$

with

$$\overline{\nabla}^2\equiv\frac{\partial^2}{\partial p^2}+\frac{\partial^2}{\partial q^2}+\frac{\partial^2}{\partial r^2} \; .$$

Then, from Eq. (7) the full propagator K can be obtained from $\overline{K}$

$$K=\frac{\overline{K}}{(s)^{3/2}}\exp\left[\frac{i}{\hbar}\left[\frac{m\dot{s}}{2s}(x^2+y^2+z^2) -\frac{m_0\dot{s}_0}{2s_0}(x_0^2+y_0^2+z_0^2)\right]\right] \; , \tag{22a}$$

where

$$\overline{K}=\overline{\phi}(\tau,\tau_0)\exp\left[\frac{i}{\hbar}\overline{S}\right] \tag{22b}$$

is *the propagator for a charged particle in a constant magnetic field,* whose path is governed by Eqs. (17).

With the help of the solution of the set of Eqs. (17), for $\mathscr{p}(\tau^*=\tau)=\mathbf{p}$ and $\mathscr{p}(\tau^*=\tau_0)=\mathbf{p}_0$, one may find that

$$\overline{S}=\tfrac{1}{2}\overline{m}_0\omega_{0c}(\tfrac{1}{2}\{\cot[\omega_{0c}(\tau-\tau_0)/2]\} \times[(q-q_0)^2+(p-p_0)^2]+(p_0q-q_0p)) +\frac{\overline{m}_0(r-r_0)^2}{2(\tau-\tau_0)} \; . \tag{23}$$

From Eqs. (21b) and (22b) we get

$$\overline{K}=\frac{2\phi_0(\tau_0)}{\omega_{0c}(\tau-\tau_0)^{3/2}}\left[\frac{\omega_{0c}(\tau-\tau_0)/2}{\sin[\omega_{0c}(\tau-\tau_0)/2]}\right]\exp\left[\frac{i}{\hbar}\overline{S}\right] \; . \tag{24}$$

In turn, the constant factor $\phi_0(\tau_0)$ can be obtained by imposing the boundary condition (4) on (22). This yields $\phi_0=(\omega_{0c}/2)(\overline{m}_0/2\pi i\hbar s_0)^{3/2}$.

Hence, the full propagator reads off

$$K(\mathbf{x},t;\mathbf{x}_0,t_0)=\left[\frac{\overline{m}_0}{2\pi i\hbar s s_0(\tau-\tau_0)}\right]^{3/2}\left[\frac{\omega_{0c}(\tau-\tau_0)/2}{\sin[\omega_{0c}(\tau-\tau_0)/2]}\right]\exp\left[\frac{i\overline{m}_0(r-r_0)^2}{2\hbar(\tau-\tau_0)}\right]$$
$$\times\exp\left[\frac{i}{\hbar}\left[\frac{m\dot{s}}{2s}(x^2+y^2+z^2)-\frac{m_0\dot{s}_0}{2s_0}(x_0^2+y_0^2+z_0^2)\right]\right]$$
$$\times\exp\left[\frac{i\overline{m}_0\omega_{0c}}{2\hbar}(\tfrac{1}{2}\{\cot[\omega_{0c}(\tau-\tau_0)/2]\}[(q-q_0)^2+(p-p_0)^2]+(p_0q-q_0p))\right], \quad (25)$$

where p, q, r, s, and τ are related to the original variables through Eqs. (16) and (18). In conclusion, we point out that, through simple calculations, one can reduce our final result [Eq. (25)] to some related *particular* cases found in the literature.[5,6,8,12–15]

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