Differential-equation approach to functional equations: Exact solutions for intermittency

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A differential-equation method to obtain the exact solutions to the functional renormalization-group fixed-point equation, the deterministic eigenvalue equation, and the stochastic eigenvalue equation for intermittency is introduced. Application of the method to the one-dimensional case is explained in detail, and the generalization to two and higher dimensions is immediate.

I. INTRODUCTION

In recent years there has been much interest in the study of the transition to chaos in dynamical systems. Various scenarios have been proposed. In particular, the discovery of universality in the period-doubling route to chaos has attracted a great deal of attention.[1] The formulation of the universality theory in terms of functional renormalization-group equations has become an important and standard method of gleaning useful information at the onset of chaos.

Unfortunately, not much is known about the structure of these functional equations or their solutions. However, in the case of intermittency,[2-4] a complete set of exact solutions[5] was found. Two methods were employed to obtain the exact solutions: resummation of series expansions and reformulation of the functional equations into implicit recursion relations topologically equivalent to translations.[5,6] However, neither of these two methods seemed to lend themselves readily to the solution of intermittency in higher dimensions. To overcome this obstacle, a third method of solution was discovered. This method is based on a differential-equation formulation.[5] Using this method, we obtained the exact solution to the fixed-point equation in two dimensions.[5] In this paper we would like to show how exact solutions to the deterministic and stochastic eigenvalue equations can also be obtained in this way. The differential-equation method is readily applicable to higher dimensions.

The paper is organized as follows. In Sec. II we will illustrate in detail how the differential-equation method works in the one-dimensional case. In Sec. III we indicate how the method can be applied to obtain the exact solutions in the two-dimensional case. Some concluding remarks are given in Sec. IV.

II. EXACT SOLUTION IN ONE DIMENSION

To illustrate how the differential-equation method works, we will rederive the exact solutions to the three one-dimensional functional equations: the fixed-point equation, the deterministic eigenvalue equation, and the stochastic eigenvalue equation.

A. The fixed-point equation

The functional renormalization-group equation governing the universal fixed-point function $f^*(x)$ for intermittency is

$$f^*(f^*(x))=\frac{1}{\alpha}f^*(\alpha x)\ , \tag{1}$$

where α is the universal rescaling factor. The boundary condition corresponding to tangency is $f^*(0)=0$ and $f^{*\prime}(0)=1$. At this point, the saddle-point map can be expanded as follows:

$$f(x)=x+ax^z+\cdots\ , \tag{2}$$

where the exponent z determines different universality classes.

If we consider an infinitesimal change, $a\rightarrow dt$, and neglect all the higher-order terms in Eq. (2), then we obtain a simple differential equation

$$\frac{dx}{dt}=x^z\ . \tag{3}$$

We will first show how to find the rescaling factor α of Eq. (1). Define

$$w(\tau)=\alpha x(t(\tau))\ , \tag{4}$$

where

$$t(\tau)=2\tau\ . \tag{5}$$

The equation for w is

$$\frac{dw}{d\tau}=(2\alpha^{1-z})w\ . \tag{6}$$

The invariant nature of the fixed-point equation dictates that

$$2\alpha^{1-z}=1 \tag{7}$$

so that the equations for $x(t)$ and $w(\tau)$ are the same under rescaling. We have therefore

$$\alpha=2^{1/(z-1)}\ . \tag{8}$$

We proceed to find the fixed-point function $f^*(x)$. Integrating Eq. (3)

$$\int_x^{x'} x^{-z}dx = \int_0^a dt \ , \tag{9}$$

we obtain immediately the fixed-point function

$$x' = f^*(x) = [x^{-(z-1)} - (z-1)a]^{-1/(z-1)} \ . \tag{10}$$

That the global solution can be obtained from the local differential equation is of course well known in the theory of Lie groups. Since intermittency is based on the tangent bifurcation, the transition is continuous. The underlying invariance group is simply the one-parameter translation group. The solutions to the functional equation, which expresses the recursion relation between $x(t+\tau)$ and $x(t)$, can therefore be simply obtained by integrating the differential equation.

Now that we have obtained the exact solution to Eq. (1), it is relatively straightforward to find the exact solutions to the eigenvalue equations, as will be shown in Secs. II B and II C.

B. The deterministic eigenvalue equation

Under a small perturbation to the fixed-point equation, $f^*(x) \to f^*(x) + \epsilon h(x)$, the eigenfunctions $h(x)$ satisfy the functional eigenvalue equation

$$f^{*\prime}(f^*(x))h(x) + h(f^*(x)) = (\lambda_\epsilon/\alpha)h(\alpha x) \ , \tag{11}$$

where λ_ϵ denotes a spectrum of eigenvalues.

To apply the differential-equation method to the solution of this eigenvalue equation, consider a perturbation to Eq. (3):

$$\frac{dx}{dt} = x^z + \epsilon x^n \ . \tag{12}$$

We will again first find the eigenvalues λ_ϵ. In terms of the variable $w(\tau)$, Eq. (12) reads

$$\frac{dw}{d\tau} = w^z + \epsilon(\alpha^{z-n})w^n \ . \tag{13}$$

It has exactly the same form as Eq. (12) except for a factor of α^{z-n}. The eigenvalues of Eq. (11) are therefore

$$\lambda_\epsilon = 2^{(z-n)/(z-1)} \ . \tag{14}$$

We proceed to find the eigenfunctions $h(x)$. Under an infinitesimal change $x \to x + \epsilon y$, the variation y satisfies the differential equation

$$\frac{dy}{dt} = zx^{z-1}y + x^n \ . \tag{15}$$

Letting $y = x^z v$, we obtain the differential equation for v,

$$\frac{dv}{dt} = x^{n-z} \ . \tag{16}$$

Integrating this equation

$$\int_0^{v'} dv = \int_0^a dt\, x^{n-z} = \int_x^{x'} dx\, x^{-z}x^{n-z} \ , \tag{17}$$

we get

$$v' = \frac{1}{2z-n-1}\{[x^{-(z-1)} - (z-1)a]^{(2z-n-1)/(z-1)} - x^{-(2z-n-1)}\} \ . \tag{18}$$

Writing $y' = x'^z v'$, we obtain the eigenfunctions $h(x) = y'$ of the deterministic eigenvalue equation:

$$h(x) = \frac{1}{2z-n-1}[x^{-(z-1)} - (z-1)a]^{-z/(z-1)} \times \{x^{-(2z-n-1)} - [x^{-(z-1)} - (z-1)a]^{(2z-n-1)/(z-1)}\} \ . \tag{19}$$

C. The stochastic eigenvalue equation

To study the effect of external noise,[7,8] consider a small stochastic perturbation to the fixed-point function, $f^*(x) \to f^*(x) + \xi g(x)$, where ξ is a Gaussian random variable of unit width. Under this perturbation the eigenfunctions $g(x)$ obey the following stochastic eigenvalue equation:

$$f^{*\prime 2}(f^*(x))g^2(x) + g^2(f^*(x)) = (\lambda_\xi/\alpha)^2 g^2(\alpha x) \ , \tag{20}$$

where λ_ξ denotes the stochastic eigenvalues.

To solve this eigenvalue equation by the differential-equation method, let us add a stochastic perturbation to Eq. (3):

$$\frac{dx}{dt} = x^z + \epsilon x^n \xi \ , \tag{21}$$

where $\xi(t)$ is a Gaussian random variable satisfying

$$\langle \xi(t)\xi(t') \rangle = \delta(t-t') \ . \tag{22}$$

Due to the singular nature of this random variable, we have to exercise some care in trying to find the stochastic eigenvalues λ_ξ. It will be clearer to rewrite the stochastic differential Eq. (21) as a finite-difference equation:

$$\frac{x(t+\Delta) - x(t)}{\Delta} = x^z + \epsilon x^n \xi_\Delta \ , \tag{23}$$

where ξ_Δ is a random variable uncorrelated from time increment to time increment. Furthermore,

$$\langle \xi_\Delta^2 \rangle = \frac{1}{\Delta} \ . \tag{24}$$

This ensures a finite cumulative effect of the stochastic contribution to the equations of motion.

In terms of the variable w, Eq. (23) becomes

$$2\left[\frac{w(\tau+\Delta') - w(\tau)}{\Delta}\right] = w^z + \epsilon w^n \alpha^{z-n}\xi_\Delta \ . \tag{25}$$

It can be seen easily that Δ and ξ_Δ are related to Δ' and $\xi_{\Delta'}$ by

$$\Delta = 2\Delta' \ , \tag{26}$$

$$\xi_\Delta = 2^{-1/2}\xi_{\Delta'} \ . \tag{27}$$

Eq. (25) therefore becomes

$$\frac{w(\tau+\Delta')-w(\tau)}{\Delta'}=w^z+\epsilon(\alpha^{z-n}2^{-1/2})w^n\xi_{\Delta'} . \tag{28}$$

This stochastic difference equation for $w(\tau)$ is the same as that for $x(t)$ except for the factor $\alpha^{z-n}2^{-1/2}$, which gives the stochastic eigenvalues

$$\lambda_\xi=2^{(z-2n+1)/(z-1)} . \tag{29}$$

We now proceed to find the eigenfunctions of the stochastic eigenvalue equation. Under an infinitesimal change, $x\rightarrow\epsilon y$,

$$\frac{dy}{dt}=zx^{z-1}y+x^n\xi . \tag{30}$$

Writing $y=x^z v$, we obtain the stochastic differential equation for v:

$$\frac{dv}{dt}=x^{n-z}\xi . \tag{31}$$

The solution to this stochastic differential equation is

$$v'=\left[\int_0^a x^{2(n-z)}dt\right]^{1/2}\xi' , \tag{32}$$

where ξ' is another random variable. Working out the integral, we get

$$v'=\frac{1}{(3z-2n-1)^{1/2}}\times\{x^{-(3z-2n-1)}-[x^{-(z-1)}-(z-1)a]^{(3z-2n-1)/(z-1)}\}^{1/2}\xi' . \tag{33}$$

Writing $y'=x'^z v'$, we obtain the eigenfunctions, $g^2(x)=y'^2$, of the stochastic eigenvalue equation,

$$g^2(x)=\frac{1}{3z-2n-1}[x^{-(z-1)}-(z-1)a]^{-2z/(z-1)}\times\{x^{-(3z-2n-1)}-[x^{-(z-1)}-(z-1)a]^{(3z-2n-1)/(z-1)}\} . \tag{34}$$

This completes the derivation of the exact solutions to the functional equation of intermittency in the one-dimensional case.

III. EXACT SOLUTIONS IN TWO DIMENSIONS

We will now indicate how the differential-equation method can be easily applied to obtain the exact solutions to the two-dimensional functional equations for intermittency. Since it is a simple extension of the one-dimensional case, we will be brief in our presentations.

A. The fixed-point equation

The functional renormalization-group equation for the fixed-point function in two dimensions read[9,10] ($i=1,2$)

$$f_i^*(f_1^*(x_1,x_2),f_2^*(x_1,x_2))=(1/\alpha_i)f_i^*(\alpha_1 x_1,\alpha_2 x_2) . \tag{35}$$

Since the saddle-point map at tangency has the form

$$f_1(x_1,x_2)=x_1+ax_2^{z_1}+\cdots ,$$
$$f_2(x_1,x_2)=x_2+ax_1^{z_2}+\cdots , \tag{36}$$

the differential equation for the fixed-point function is simply

$$\frac{dx_1}{dt}=x_2^{z_1} ,$$
$$\frac{dx_2}{dt}=x_1^{z_2} ; \tag{37}$$

or, in abbreviation,

$$\frac{dx_i}{dt}=x_j^{z_i} . \tag{38}$$

We will first try to calculate the rescaling factors α_i. Using the same method as applied to the one-dimensional case, we define

$$w_i(\alpha)=\alpha_i x_i(t(\tau)) , \tag{39}$$

where $t(\tau)=2\tau$, as previously. The equations for w_i are then

$$\frac{dw_i}{d\tau}=(2\alpha_i\alpha_j^{-z_i})w_i . \tag{40}$$

Therefore,

$$\alpha_i=2^{(z_i+1)/(z_1z_2-1)} , \tag{41}$$

which agree with those obtained by series expansion.[10]

We now proceed to find the fixed-point function. Integrating Eq. (38), get

$$\int_0^a dt=\int_{x_i}^{x_i'}dx_i\left[(z_i+1)\left[\frac{x_i^{z_j+1}}{x_j+1}-C\right]\right]^{-z_i/(z_i+1)} ,$$

where (43)

$$C=\frac{x_1^{z_2+1}}{z_2+1}-\frac{x_2^{z_1+1}}{z_1+1} .$$

The problem of finding the fixed-point function has thus been reduced to one of evaluating integrals (they generally give elliptic or hyperelliptic functions). For the case $z_1=1$ and $z_2=2$, explicit form of the fixed-point function has been obtained.[5]

We will now show that the solutions to the differential equations actually satisfy the functional equations. The recursion relations between the solutions $x_i(t+\lambda)$ and $x_i(t)$, obtained by integrating the flow equations, will be of the form

$$x_i(t+\lambda)=f_i(x_1(t),x_2(t),\lambda) \tag{44}$$

due to the translation invariance of the difference equations. Since the $w_i(\tau)$'s satisfy the same equations,

$$w_i(\tau+\lambda)=f_i(w_1(\tau),w_2(\tau),\lambda) . \tag{45}$$

As $w_i(\tau)=\alpha_i x_i(t(\tau))$ and $w_i(\tau+\lambda)=\alpha_i x_i(t(\tau)+2\lambda)$,

$$\alpha_i x_i(t+2\lambda)=f_i(\alpha_1 x_1(t),\alpha_2 x_2(t),\lambda) . \tag{46}$$

Thus the τ variable has been eliminated. Now

$$x_i(t+2\lambda)=f_i(x_1(t+\lambda),x_2(t+\lambda),\lambda)$$
$$=f_i(f_1(x_1(t),x_2(t),\lambda),f_2(x_1(t),x_2(t),\lambda),\lambda) . \quad (47)$$

This implies that the f_i's satisfy the fixed-point functional equation (35).

B. The deterministic eigenvalue equation

We apply a perturbation to Eq. (37),

$$\frac{dx_1}{dt}=x_2^{z_1}+\epsilon x_2^{n_1} ,\quad \frac{dx_2}{dt}=x_1^{z_2} . \quad (48)$$

Let $x_i(t)$ be the solution to Eq. (38), and write the perturbed solution in the form $x_i(t)+\epsilon y_i(t)$. Then

$$\frac{dy_1}{dt}=z_1x_2^{z_1-1}y_2+x_2^{n_1} ,\quad \frac{dy_2}{dt}=z_2x_1^{z_2-1}y_1 , \quad (49)$$

or,

$$x_1^{z_2}\frac{dy_1}{dt}=z_1x_2^{z_1-1}x_1^{z_2}y_2+x_2^{n_1}x_1^{z_2} ,\quad x_2^{z_1}\frac{dy_2}{dt}=z_2x_1^{z_2-1}x_2^{z_1}y_1 . \quad (50)$$

Hence,

$$\frac{d}{dt}(x_1^{z_2}y_1)=z_1x_2^{z_1-1}x_1^{z_2}y_2+z_2x_1^{z_2-1}x_2^{z_1}y_1+x_2^{n_1}x_1^{z_2} ,\quad \frac{d}{dt}(x_2^{z_1}y_2)=z_2x_1^{z_2-1}x_2^{z_1}y_1+z_1x_2^{z_1-1}x_1^{z_2}y_2 . \quad (51)$$

Since

$$\frac{d}{dt}(x_1^{z_2}y_1-x_2^{z_1}y_2)=x_2^{n_1}x_1^{z_2} , \quad (52)$$

$$x_1^{z_2}(t)y_1(t)-x_2^{z_1}(t)y_2(t)=x_1^{z_2}(t_0)y_1(t_0)-x_2^{z_1}(t_0)y_2(t_0)+\int_{t_0}^{t}dt'\,x_2^{n_1}(t')x_1^{z_2}(t') . \quad (53)$$

Therefore,

$$x_1^{z_2}(t)y_1(t)-x_2^{z_1}(t)y_2(t)=x_1^{z_2}(t_0)y_1(t_0)-x_2^{z_1}(t_0)y_2(t_0)+\frac{1}{n_1+1}[x_2^{n_1+1}(t)-x_2^{n_1+1}(t_0)] . \quad (54)$$

Going back to our original Eq. (49), we have

$$\frac{dy_1(t)}{dt}=\frac{z_1x_2^{z_1-1}(t)}{x_2^{z_1}(t)}\left[x_1^{z_2}(t)y_1(t)-x_1^{z_2}(t_0)y_1(t_0)+x_2^{z_1}(t_0)y_2(t_0)-\frac{1}{n+1}[x_2^{n_1+1}(t)-x_2^{n_1+1}(t_0)]\right]+x_2^{n_1}(t) . \quad (55)$$

Letting $y_1(t)=x_2^{z_1}(t)v_1(t)$, we obtain

$$x_2^{z_1}(t)\frac{dv_1(t)}{dt}=\frac{z_1}{x_2(t)}\left[C-\frac{1}{n_1+1}[x_2^{n_1+1}(t)-x_2^{n_1+1}(t_0)]\right]+x_2^{n_1}(t) ; \quad (56)$$

or,

$$v_1(t)=\int_{t_0}^{t}\left\{\frac{z_1}{x_2^{1+z_1}(t')}\left[\left[C-\frac{1}{n_1+1}x_2^{n_1+1}(t')-x_2^{n_1+1}(t_0)\right]\right]+x_2^{n_1-z_1}(t')\right\}dt' . \quad (57)$$

The final integral can therefore be set up. After the integration is performed, we will obtain the eigenfunctions, which again consist of in general hyperelliptic or hypergeometric functions and their inverses.

C. The stochastic eigenvalue equation

Let us now apply a stochastic perturbation to Eq. (37),

$$\frac{dx_1(t)}{dt}=x_2^{z_1}(t)+\epsilon x_2^{n_1}(t)\xi(t) ,\quad \frac{dx_2(t)}{dt}=x_1^{z_2}(t) . \quad (58)$$

Again, it is more convenient to rewrite the equations as finite difference equations,

$$\frac{x_1(t+\Delta)-x_1(t)}{\Delta}=x_2^{z_1}+\epsilon x_2^{n_1}\xi_\Delta ,\quad \frac{x_2(t+\Delta)-x_2(t)}{\Delta}=x_1^{z_2} . \quad (59)$$

Introducing the new variables as before, we get

$$2\left[\frac{w_1(\tau+\Delta')-w_1(\tau)}{\Delta'}\right]=w_2^{z_1}+\epsilon w_2^{n_1}\alpha_2^{z_1-n_1}\xi_\Delta ,\quad 2\left[\frac{w_2(\tau+\Delta')-w_2(\tau)}{\Delta'}\right]=w_1^{z_2} . \quad (60)$$

Proceeding in exactly the same way as in the one-dimensional can, we obtain

$$\frac{w_1(\tau+\Delta')-w_1(\tau)}{\Delta'}=w_2^{z_1}+\epsilon(\alpha_2^{z_1-n_1}2^{-1/2})w_2^{n_1}\xi_{\Delta'}\,,$$
$$\frac{w_2(\tau+\Delta')-w_2(\tau)}{\Delta'}=w_1^{z_2}\,. \tag{61}$$

The stochastic eigenvalues can thus be identified

$$\lambda_\xi=2^{[z_1(z_2+2)-2n_1(z_2+1)+1]/2(z_1z_2-1)}\,, \tag{62}$$

which indeed agree with those obtained by series expansion.[10] The eigenfunctions can also be obtained in a way exactly analogous to the one-dimensional case.

IV. CONCLUDING REMARKS

The functional renormalization-group equations provide a powerful approach to the study of the universal scaling properties of dynamical systems at the onset of chaos. Unfortunately, very little is known about the structure of these equations or the methods of solving them. It was indeed very satisfying that a complete set of exact solutions, at least in the case of intermittency, could be found. The differential-equation method provides a simple and elegant approach to the solution of functional equations. However, the reason that intermittency is amenable to exact solutions and that the differential-equation method works is due to the fact that intermittency is a continuous transition and there exists a one-parameter translation group. In this sense, the method is probably not applicable to discrete transitions such as period doubling.

Finally, it should be noted that the exact solutions to the functional equations for intermittency are not just of mathematical interest. As recent studies have shown, they are relevant to a host of interesting physical problems such as $1/f$ noise,[11] superconducting quantum interference device (SQUID),[12] Josephson junctions,[13,14] etc. Moreover, experimental observation of the intermittency route to chaos has also been found in a variety of experiments including Rayleigh-Bénard convection,[15] nonlinear oscillator,[16,14] and Belousov-Zabotinsky reactions.[17]

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