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Class renormalization: Islands around islands

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A "class"-c orbit is one that rotates around a periodic orbit of class $c-1$ with some definite frequency. This contrasts with the "level" of a periodic orbit which is the number of elements in the continued-fraction expansion of its frequency. Level renormalization is conventionally used to study the structure of quasiperiodic orbits. The scaling structure of periodic orbits encircling other periodic orbits in area-preserving maps is discussed here. Renormalization fixed points of p/q bifurcations are found and scaling exponents determined. Fixed points for $q>2$ correspond to self-similar islands around islands. Frequencies of the island boundary circles at the fixed points are obtained. Importance of this scaling for the motion of particles in stochastic regions is emphasized.

I. INTRODUCTION

A typical, two degree-of-freedom Hamiltonian system exhibits motion of such complexity that, despite a hundred years of study, only in the past fifteen have we begun to approach a complete understanding. One aspect of this complexity is the structure of the periodic orbits, which form the framework for the phase portrait. Periodic orbits in two degrees of freedom are either elliptic or hyperbolic (or exceptionally parabolic). Birkhoff showed that in the neighborhood of a generic elliptic periodic orbit there are other periodic orbits which rotate around the given orbit.[1] Some of these "satellite" orbits will typically be elliptic, and will also have satellite orbits. He realized that this structure repeats *ad infinitum:*

"Il est évident que non seulement les solutions e-périodiques quelconques possèdent des solutions e- et h-périodiques voisines, mais aussi qu'en recommencant avec ces solutions e-périodiques voisines qui sont pour ainsi dire des satellites de ces solutions, on peut obtenir d'autres solutions e-périodiques et h-périodiques qui sont des satellites secondaires."[1]

Furthermore, Arnol'd proved that in the neighborhood of a typical elliptic orbit there are invariant tori encircling the orbit.[2] On a surface of section, which is a plane transverse to the orbits, these tori become closed curves or a set of closed curves and have the appearance of a chain of "islands." The satellite elliptic orbits are also encircled by invariant curves, and thus form islands around islands.

In this paper we discuss the scaling of these islands as one looks on ever finer scales. Islands around islands provide an example of the renormalization transformations which have been recently used in dynamics to study the onset of chaotic motion.[3–8]

There have two types of renormalization applied to Hamiltonian systems, which here will be called "level" and "class" renormalization. Level renormalization is used to study the evolution and eventual destruction of an invariant surface as some system parameter is varied.[3,4] Class renormalization similarly described the behavior of periodic orbits.

For a brief description of level renormalization, recall that motion on an invariant torus occurs with a given winding number, or frequency, ω. According to the Kolmogorov-Arnol'd-Moser (KAM) theorem, only tori with sufficiently irrational ω are preserved for any interval of parameter variation. The frequency can be expanded in continued-fraction representation:

$$\omega=a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cdots}}}\equiv[a_0,a_1,a_2,a_3,\ldots]\ , \tag{1}$$

where the a_i are positive integers. Recall that irrationals have infinite continued-fraction expansions. The convergents to ω are those rationals obtained by truncating its continued fraction after some finite number of terms. The number of terms is called the *level* of approximation. As first shown by Greene,[7] the properties of the quasiperiodic orbit can best be obtained by examining the convergent orbits as approximations. Convergent orbits of increasing level look more and more like the limiting irrational frequency orbit. Level renormalization is the operation of rescaling space and time to map one level onto the next.[4–6] Remarkably, one finds a self-similarity in structure under this transformation in special cases. The simplest is that of the golden-mean frequency, $\gamma=(1+\sqrt{5})/2$, for which $a_i=1$ for all i. Here the renormalized Hamiltonian approaches a fixed function: The properties of one level are asymptotically the same as those of the next. When an invariant torus is below the threshold of destruction, the asymptotic Hamiltonian looks integrable. At the threshold for destruction, the asymptotic Hamiltonian has a nontrivial form, corresponding to a system with exactly one invariant circle.

Class renormalization is familiar from the case of period doubling.[8] Here one follows a stable fixed point as a parameter is varied. The fixed point, or *class-0* orbit, typically loses stability by period-doubling bifurcation. The resulting period-2 orbit "rotates" about the fixed point with frequency $\frac{1}{2}$, and we call it a *class-1* orbit. One can again follow the period-2 orbit as the parameter is varied. When it loses stability a period-4, *class-2* orbit is born, and so forth. Self-similarity in this case appears as a repetition of the structure from one class to the next.

We discuss below the generalization of this to bifurcations of higher order, e.g., period q-tupling, and furthermore, to the birth of orbits of any frequency, p/q. These orbits can be organized in classes by analogy with the levels of a continued fraction. Asymptotic self-similarity is found in the cases when p/q is the same for each class. Some of the low-order fixed points have been studied previously (period 3 in Ref. 4, and $1/q$ to $q=6$ in Ref. 9). Approximate class renormalization schemes were discussed in Ref. 10, and improved in Sec. 2.4 of Ref. 3.

Our purpose is firstly to attempt some organization of the fixed points of various frequencies. We show how the scaling parameters vary with p/q, and obtain a simple formula for the scaling of area when $p=1$. We indicate the analogy with the Farey-tree organization of scalings for quasiperiodic orbits of the circle map.[11]

Primarily, however, our interest is to use the results of scaling in studies of the motion of particles in the stochastic regions of phase space.[12] Self-similarity is a useful concept because the smallest spatial scales most strongly affect the statistics of orbits on the longest time scales. In particular, orbits find the neighborhood of invariant surfaces extremely "sticky," due to "cantori," which are quasiperiodic orbits covering only a Cantor set on a torus.[12,13] Thus, to understand motion in chaotic regions one needs to understand the orbits in the neighborhood of the invariant circles which bound the region.

We show that the renormalization structure of p/q bifurcations, with $q>2$, is that of islands around islands. Each island is a set of class-$(c+1)$ invariant circles surrounding an orbit of class c. The outermost of these is the boundary of the island and forms a surface near to which a chaotic orbit can stick. At the fixed point of a p/q bifurcation, there is a universal structure to this boundary, which can be used to estimate its "stickiness."[14]

Furthermore, we give some evidence below that class sequences which do not consist of orbits of the same p/q, also have some remnant of the self-similarity of the simple case. This is analogous to generalized scaling found for levels of boundary circles by Greene *et al.*[15]

The results of this paper are primarily computational, and thus no mathematical rigor is pretended. We use the work "typical" for situations in which we observe a phenomena computationally, but for which there could well be counterexamples. Thus, for example, one can say that typically period doublings accumulate with an increase of some parameter representing the strength of the nonlinearity, though one can easily reparametrized a map so that this is not true.

II. REVERSIBLE, AREA-PRESERVING TWIST MAPS

Instead of directly studying two degree-of-freedom Hamiltonians, the computations in this paper are for area-preserving maps of the plane. As is well known, Hamiltonian flow can be reduced to a map by the surface of a section. We will consider a general class of maps: reversible, twist maps.[4,16] We explain these terms in the context of an example: the quadratic map,

$$T:\ \begin{cases} y'=x \\ x'=-y+2(k-x^2)\ . \end{cases} \tag{2}$$

This map has fixed points at $x=y=\frac{1}{2}(1\pm\sqrt{1+4k}\,)$, with the upper sign representing an elliptic orbit when k is in the range $-\frac{1}{4}<k<\frac{3}{4}$. Denote this point by $x_e=y_e$. The lower sign gives a hyperbolic orbit for $-\frac{1}{4}<k$.

First, an *area-preserving* map has unit Jacobian, $\det(DT)=1$. Here DT is the 2×2 matrix of partial derivatives of x' and y' with respect to x and y.

Second, a *reversible* map can be written as the product of two involutions, $T=(TS)S$ [involutions satisfy $(TS)^2=S^2=1$]. Such a decomposition represents time-reversal symmetry since conjugation with the symmetry transformation inverts the map: $STS=T^{-1}$. For the quadratic map, symmetry operations are

$$S:\ \begin{cases} y'=x \\ x'=y\ , \end{cases}$$

$$TS:\ \begin{cases} y'=y \\ x'=-x+2(k-y^2)\ . \end{cases}$$

Fixed sets of the involutions are the symmetry sets of the map. In general, for maps of the plane, each involution has a fixed set which is a curve going to infinity[4] (these are called *symmetry lines*):

$$\begin{aligned} &\mathrm{Fix}(TS)\equiv\mathscr{R}=\{(x,y)\,|\,x=k-y^2\} \\ &\mathrm{Fix}(S)\equiv\mathscr{S}=\{(x,y)\,|\,x=y\}\ . \end{aligned} \tag{3}$$

They intersect at the fixed points. We are interested in the structure of the map near the elliptic fixed point, and so we divide the two lines into the four *half-lines* which meet at (x_e,y_e). These lines are assigned an orientation, pointing away from the fixed point. Positive orientation, denoted by the subscript "$+$," corresponds to pointing in the positive y direction; similarly, negative orientation is denoted "$-$."

Finally, a *twist* map is a map that has rotational shear: There exist coordinates $(\theta(x,y),J(x,y))$ where θ is an angle, $d\theta\, dJ=dx\,dy$, and $\partial\theta'/\partial J\,|_\theta<0$. In terms of θ, which we can think of as the angle about the elliptic fixed point, the *rotation number* of an orbit is defined as the average number of rotations per iteration of the map

$$\omega=\lim_{n\to\infty}(\theta_n/2\pi n)\ . \tag{4}$$

The twist property is not satisfied for the quadratic map (2) in polar coordinates. However, it is possible to find coordinates such that the twist property is satisfied in the neighborhood of a typical (nondegenerate) elliptic point.[1,4]

A twist map can always be obtained from a generating function of the type $F(\theta,\theta')$ where $J'=\partial F/\partial\theta'$ and $J=-\partial F/\partial\theta$. For the quadratic map there is a generating function of this form given by

$$F(x,x')=xx'-2x(k-x^2/3)\ . \tag{5}$$

The total *action* of an orbit $\{x_0,x_1,\ldots,x_q\}$ is defined by

$$W(\{x_i\})=\sum_{i=0}^{q-1} F(x_i,x_{i+1}) . \tag{6}$$

The orbits are stationary points of W, holding x_0 and x_q fixed.

III. CLASSES

Periodic orbits are fixed points of T^q. The stability of such orbits is determined by the linearization of T along the orbit: This gives a matrix defined as

$$M=(DT)^q=DT(\mathbf{z}_q)DT(\mathbf{z}_{q-1})\cdots DT(\mathbf{z}_1) .$$

Area preservation implies that $\det(M)=1$, so there is only one parameter of M, its trace, which determines stability. It is convenient to parametrize the trace using Greene's *residue:*

$$R=\tfrac{1}{4}[2-\mathrm{Tr}(M)] . \tag{7}$$

When $R<0$ the orbit is hyperbolic, when $0<R<1$ it is elliptic, and when $R>1$ it is hyperbolic with reflection (eigenvalues negative). For example the residue of (x_e,y_e) is $\sqrt{1+4k}$. For an elliptic orbit, the residue also determines the rate at which nearby points rotate around the given orbit according to

$$R=\sin^2(\pi\omega_0) . \tag{8}$$

Here ω_0 is the central rotation frequency about a point on the orbit. Typically, the rotation frequency decreases with increasing distance from the elliptic point, implying that coordinates (θ,J) can be found in the neighborhood of this point such that the twist condition is satisfied for T^q.

Along with the residue and period, one can further classify periodic orbits in terms of their origin as the parameter is varied. For example, suppose we increase k for the quadratic map beginning at some negative value. The fixed points are born at $k=-\frac{1}{4}$ in a saddle-node bifurcation. We call the fixed points class-0 orbits. The elliptic class-0 orbit is the father of an entire family of orbits which bifurcate from it as k is increased further.[4,17] A periodic orbit encircling the class-0 orbit is born whenever ω_0 reaches a rational value, p_1/q_1. These orbits we call class 1; they rotate p_1 times around the fixed point in q_1 iterations of the map. Furthermore each class-1 periodic orbit is a fixed point of the map T^{q_1} and, if it is elliptic, has its own central rotation frequency, ω_1, Eq. (8). As k is increased, ω_1 typically increases from zero at birth. New orbits are emitted from the class-1 orbit whenever $\omega_1=p_2/q_2$, and form chains of q_2 islands surrounding each point on the orbit. These new orbits are class 2; they rotate p_2 times around a point on the class-1 orbit in q_2 iterations of T^{q_1}. Similarly, the frequency, ω_2, about the class-2 orbit is computed using the linearization of $T^{q_1q_2}$ which has as fixed points the q_1q_2 points of the class-2 orbit. This construction can be repeated, yielding islands around islands of all classes.

There are of course periodic orbits which do not fit into the above tree. They can appear by saddle-node bifurcation, and generate their own trees of orbits. We will study only the progeny of a single elliptic parent.

At a bifurcation point, $\omega=p/q$, an elliptic orbit typically maintains its stability, and in its neighborhood are invariant circles.[2,17] This can be shown for sufficiently differentiable maps providing $q>4$. The quadrupling and tripling bifurcations usually result in the loss of stability at the bifurcation point, but stability is regained as the parameter is changed so that ω is no longer equal to the low-order rational ($\frac{1}{4}$ or $\frac{1}{3}$). However, when the residue of the elliptic orbit reaches the point $R=1$ where $\omega=\frac{1}{2}$, it usually permanently loses stability by period-doubling bifurcation,[8] creating an elliptic orbit of double period, and higher class.

IV. USING SYMMETRY TO FIND PERIODIC ORBITS

The importance of symmetry lines becomes clear when one tries to find periodic orbits.[7,18] Note that if (x_i,y_i) is an orbit of T, then $S(x_i,y_i)$ is an orbit of T^{-1}. A symmetric orbit is an orbit which is invariant under S: such an orbit is its own time reversal. It is easy to see that a symmetric orbit must have two points on the symmetry lines. Thus to find symmetric periodic orbits one need only look along these one-dimensional curves.

There are many orbits of T which are not symmetric; however, the symmetric orbits are of great importance. Aubry has shown that an area-preserving map with the twist property has two symmetric periodic orbits for each rational ω in the range of the twist.[19] One of these orbits minimizes the action (6), and the other is a minimax point. Since this is a generalization of the Poincaré-Birkhoff theorem,[17] these are called the Birkhoff orbits. These are the orbits which result from the breakup of the rational invariant circles of an integrable twist map. The Poincaré-Birkhoff theorem implies that the two orbits have opposite indices, which correspond to one with positive residue and one with negative residue. This property can be shown more generally using Aubry's existence proof (e.g., orbits which minimize the action must have negative residue, while minimax orbits have positive residue[20]).

A remarkable, but not understood, fact is that positive residue, class-1, periodic orbits all tend to have a point on *one* of the four symmetry half-lines, called the dominant line, and denoted E_1.[4] For T, the dominant line is $\mathscr{R}_+$. Assuming this, one can easily determine the rules for which points of the symmetric orbits lie on which symmetry lines. This depends on whether p and/or q are even or odd, as shown in Table I.[4] We denote the second elliptic line by E_2, and the two hyperbolic lines by H_1 and H_2.

We have found that the higher-class orbits also have a dominant symmetry line, determined by a simple rule depending on the parent orbit. Suppose we consider a class-c orbit with rotation number p_c/q_c. Since the class-$(c+1)$ orbit encircles the elliptic parent, the four symmetry lines for class c are obtained from E_1 and E_2 of the parent by dividing each in half at where it crosses the parent orbit. Each half-line is oriented to point away from the parent orbit. The orientation is designated positive if the line points in the *same* direction as that of the parent line. Thus E_1^c splits into the two half-lines E_{1+}^c

and E^c_{1-} where the plus designates the half-line pointing in the same direction as E^c_1. We find that the dominant symmetry line for class $c+1$ is always E^c_{2+}. We can write this as the equation

$$E_1^{c+1}=E^c_{2+} \; . \tag{9}$$

All the positive-residue, class-$(c+1)$ orbits have a point on this new dominant symmetry line.

This rule seems to hold for any reversible, area-preserving twist map, though so far as we know, there are no mathematical results to this effect. In particular for the "standard map" ($y'=y-k\sin x$, $x'=x+y'$) one finds that the simplest factorization gives the dominant line for the primary-class periodic orbits (those that encircle the cylinder) as the $x=0$ line.[4] The other symmetry lines are $x=\pi$, and $y=2x$ and $y=2(x-\pi)$. Orientation is such that all lines point in the *negative* y direction. The assignments E_1, etc., follow as in Table I. With this orientation the rule for finding the dominant symmetry line of higher-class orbits is (9).

To find a class-c orbit we use a secant method to search for a point that begins on one symmetry line, say H^c_1, and rotates around the class-$(c-1)$ elliptic point p_c times after q_c iterations of the $(\prod_{j=1}^{c-1} q_j)$th power of T, and then returns exactly to ${H_1}^c$. It is easy to see that these conditions do give a periodic orbit. One can save a factor of 2 in time by requiring that the orbit ends up on the second line, ${H_2}^c$, after half of the period [for odd periods the number of iterations from $\mathcal{R}$ to $\mathcal{S}$ is (period + 1)/2]. This technique works quite well so long as the magnitude of the residue of the orbit is $\leq O(1)$.

V. CLASS RENORMALIZATION FIXED POINTS

At any typical value of the parameter there exist class-1 orbits encircling the elliptic fixed point with all frequencies in the range ω_0 to zero. For each elliptic class-1 orbit there will be class-2 orbits of all frequencies in the range ω_1 to zero, around it. As is by now a familiar question, one would like to know how the structure of the class-2 orbits is related to that of the class-1 orbit they surround. The most well-known case of this is when one considers the period-2 orbits.[8] A period-2, class-1 orbit is born when the residue, R_0, reaches 1. As k is further increased a class-2 orbit of period 4 and frequency $\omega_1=\frac{1}{2}$ is born when $R_1=1$. Interestingly, the parameter values for period doublings accumulate geometrically at some value k^*:

$$k\rightarrow k^*+A\delta^{-c} \; , \tag{10}$$

TABLE I. Symmetry lines for periodic orbits. Symmetry lines $\mathcal{R}$ and $\mathcal{S}$ are given by Eq. (3). Subscript " + " denotes the half-line with $y>y_e$, and "$-$" denotes $y<y_e$. Class-1, symmetric orbits with positive residue have points on the lines labeled E, and negative residue on those labeled H (Ref. 4).

p/q	$\mathcal{R}_+$	$\mathcal{R}_-$	$\mathcal{S}_+$	$\mathcal{S}_-$
even/odd	E_1	H_1	E_2	H_2
odd/even	E_1	E_2	H_1	H_2
odd/odd	E_1	H_1	H_2	E_2

where the exponent is $\delta=8.721$.

At the accumulation point the map is asymptotically self-similar. Orbits of all periods 2^c, $c=0,1,2,\ldots$, coexist, all with $R>1$. In fact as $c\rightarrow\infty$, the residue of the period 2^c orbit approaches the value $R^*=1.1359$. Define a coordinate transformation, B, which maps the minimax class-$(c+1)$ point on the dominant symmetry line to the corresponding class-c point on its dominant symmetry line, and rescales areas. Self-similarity is expressed in terms of a renormalization transformation

$$N_q(T)=BT^qB^{-1} \tag{11}$$

by the relation

$$N_2(N_2(\cdots(T)))\rightarrow T^*=N_2(T^*)$$

at the parameter value k^*. It turns out that the determinant of the coordinate transformation B, which represents the area scaling factor, is 65.751.

It is quite straightforward to generalize this to other self-similar p/q sequences.[4,9] To find the birth of a p/q orbit of class $(c+1)$, we search for a parameter value for which the class-c, frequency p/q orbit has residue $R_c=\sin^2(\pi p/q)$. The values, k^*, of the accumulation points of some of the p/q sequences are shown in Table II. Values of the scaling parameters, $\delta(p,q)$ and $\xi(p,q)=\det(B)$ are given in Table III, as is the value of the residue, R^*, of the p/q orbit at the fixed point.

To determine area scaling we do not use special symme-

TABLE II. p/q fixed points. Parameter values for which the map (2) has a self-similar sequence of p/q periodic orbits. Typical uncertainties in k^* are ± 3 in the last significant figure. Values for $q=2$ are taken from Ref. 8.

p/q	k^*
$\frac{1}{2}$	1.034 041 700 98
$\frac{1}{3}$	0.295 392 355 88
$\frac{1}{4}$	0.035 680 866 41
$\frac{1}{5}$	−0.087 242 966 99
$\frac{1}{6}$	−0.139 855 132 32
$\frac{1}{7}$	−0.171 979 979 40
$\frac{1}{8}$	−0.190 945 228 18
$\frac{1}{9}$	−0.202 915 280 61
$\frac{1}{10}$	−0.211 261 983
$\frac{1}{11}$	−0.217 110 4
$\frac{2}{5}$	0.704 932 226 95
$\frac{2}{7}$	0.172 819 459 88
$\frac{2}{9}$	−0.277 201 870
$\frac{2}{11}$	−0.114 935 951
$\frac{3}{8}$	0.604 093 625 13
$\frac{3}{10}$	0.212 491 926 3

try coordinates but rely on the action, (6), which has the units of area. Computing the difference in action between the positive and negative residue class-c orbits, ΔW, gives a measure of the area of the class-c island ($q>2$). In fact, this is the area of the "turnstile" through that periodic orbit.[12] Thus ξ represents the rate at which ΔW approaches zero,

$$\Delta W_c \rightarrow A\xi^{-c} . \tag{12}$$

Previous works considered the eigenvalues of the matrix B, which give length scaling coefficients; the accumulation rate along the symmetry line is called α, and across the symmetry line, β.[4,8,9] Unfortunately, these coefficients do not always approach fixed values: They oscillate with period 2 when q is odd.[4,9] This leads one to suppose that N_q does not approach a fixed map, T^*, for odd q. However, we have seen that if one looks only at area scaling then there appears to be a fixed point. Expressed in terms of α and β, we find that $\alpha\beta=\xi$ is a constant. Note that ξ is a canonical invariant (all maps canonically related will have the same coefficient), while α and β depend on the coordinate system. It remains to be seen if a coordinate-free version of (11) can be formulated.

In Table III we see that fixed points with $p>1$ typically have quite large values for the area-scaling coefficient: $\xi>230$. This implies that high-class islands at these fixed points are very small, and probably do not strongly affect motion in the stochastic region.

Table III shows that R^* for all $q>2$ is less than one. This implies that the elliptic orbits at the fixed point are linearly stable. When $\omega \neq p/q$ for some $q \le 4$, the KAM theorem guarantees that these orbits are surrounded by islands.[2] Thus at these fixed points one truly has a self-similar island around island structure. Figure 1 shows four classes of orbits at the $\frac{1}{8}$ fixed point for the quadratic map. We do not use the proper symmetry coordinates, so the shape of the islands changes with class, however the coordinate independent properties of the islands (residue and area) do scale geometrically.

At a period-tripling or -quadrupling bifurcation point the period 3^c or 4^c orbit, while linearly stable, is generically unstable.[2] The tripling fixed point, however, occurs at a parameter value below that of each bifurcation point: the central frequency of each minimax orbit is 1/3.054. Thus there are invariant curves surrounding each orbit. The quadrupling fixed point is at a parameter value slightly greater than that of each quadrupling bifurcation (as are each of the $q>4$ fixed points). At k^* the central frequency is 1/3.91, and so the positive residue orbits are indeed stable.

TABLE III. Universal parmeters. Universal numbers at the p/q fixed points. δ is the unstable eigenvalue of N_q, ξ is the area-scaling factor, and R^* is the residue of the positive residue p/q orbits. The last significant figure is typically uncertain by ± 3.

p/q	δ	$\xi=\alpha\beta$	$\zeta=\frac{\ln(\xi)}{\ln(q)}$	R^*
$\frac{1}{2}$	8.721 09	65.751	6.039	1.135 88
$\frac{1}{3}$	20.185	90.616	4.102	0.733 71
$\frac{1}{4}$	24.448	80.108	3.162	0.517 81
$\frac{1}{5}$	20.048	57.232	2.515	0.389 15
$\frac{1}{6}$	13.875	51.997	2.205	0.312 95
$\frac{1}{7}$	10.808	70.632	2.187	0.265 80
$\frac{1}{8}$	9.4397	104.53	2.236	0.233 02
$\frac{1}{9}$	9.0841	161.92	2.315	0.206 08
$\frac{1}{10}$	8.5708	221	2.34	0.189 49
$\frac{1}{11}$	8.45	360	2.45	0.178
$\frac{2}{5}$	30.257	506	3.87	0.986 22
$\frac{2}{7}$	39.279	322	2.97	0.656 50
$\frac{2}{9}$	32.264	234	2.48	0.473 50
$\frac{2}{11}$	24.331	305	2.39	0.372 06
$\frac{3}{8}$	45.758	1490	3.5	0.935 37
$\frac{3}{10}$	46.3	570	2.8	0.708

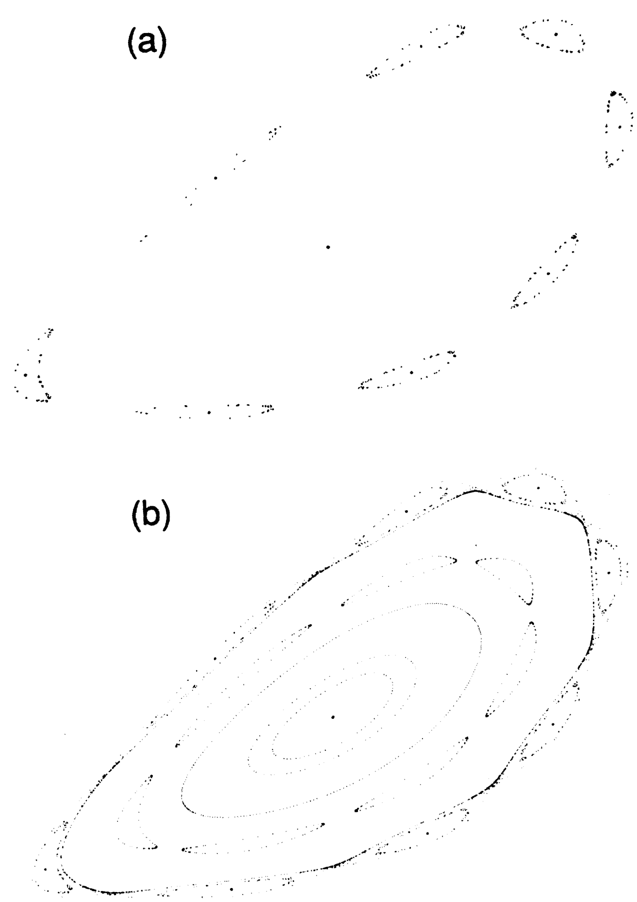

FIG. 1. Phase space at the $\frac{1}{8}$ fixed point. (a) An enlargement of the region near a point on the elliptic, class-1 orbit showing the elliptic class 2, 3, and 4 orbits, i.e., up to period 8^4. (b) The same as (a), with additional orbits, including the $\frac{1}{7}$ class-1 chain, the "separatrix" of the $\frac{1}{8}$ class-1 chain and the class-1 boundary circle (close to the inner separatrix of the $\frac{1}{8}$ orbit).

VI. BOUNDARY CIRCLES

The region occupied by the invariant circles surrounding an elliptic periodic orbit is confined by the stable and unstable manifolds emanating from its hyperbolic partner. Since these manifolds typically intersect transversally, they are embedded in a region of chaotic behavior. The outermost class-$(c+1)$ invariant circle forms the boundary between the chaotic region and the island, and hence is called a boundary circle. At the accumulation point of p/q bifurcations each island has its boundary circle, and since the map approaches T^* on fine scales, these boundary circles will asymptotically approach a universal circle with a universal rotation number depending only on p and q.

To determine the rotation number of a boundary circle we use a method developed by Greene, MacKay, and Stark,[15] which is based on the residue. Greene showed that periodic orbits in the neighborhood of an invariant circle have residues which approach a value smaller than 1.[7] Thus if the residue of an orbit is significantly larger than 1, it is probable that there are no invariant circles nearby. This criterion can be refined as follows.[4,15]

Consider two class-c periodic orbits, p_1/q_1 and p_2/q_2 with $q_2>q_1$, which are neighboring, in the sense that $p_1q_2-p_2q_1=\pm 1$. The residue criterion states that there is no class-c, invariant circle with ω between these orbits if the mean residue

$$\overline{R}=(R_1+\gamma R_2)/(1+\gamma)>R_t\ , \tag{13}$$

where R_t can be taken to be of order 0.3. This is a conjecture, based on the renormalization theory for "levels."

To find the boundary circle for an island, we begin by calculating the residues of the orbits with frequencies $1/n$. For the smallest n such that $1/n<\omega_0$, we compute $\overline{R}$ for $1/n$ and $1/(n+1)$. If it is smaller than R_t we increase n by 1 and try again. This corresponds to moving radially away from the center of the island. Eventually the residues begin to increase; they do so dramatically as the chaotic region near the separatrix is approached. Take the last n for which $\overline{R}<R_t$; we expect there is an invariant circle between n and $n+1$, but not between $n+1$ and $n+2$. Find the daughter orbit with the mediant frequency $(1+1)/[n+(n+1)]$ between the two parents. Check the value of $\overline{R}$ for the daughter and the outer parent, $1/(n+1)$. If it is smaller than R_t there is probably a boundary circle between these two orbits, otherwise, we check $\overline{R}$ for the inner parent. Construct a new daughter between the old daughter (now mother) and outermost of the parents with $\overline{R}<R_t$ (the father) by adding numerators and denominators of these two frequencies. Repeat the procedure by checking $\overline{R}$ for the daughter and its outer parent.

In this way we obtain the outermost frequency for which the residue stays bounded, and hence the frequency of the outermost invariant circle. This procedure sometimes proceeds down dead-end paths, in which, though $\overline{R}$ is smaller than R_t at one generation, all the daughter orbits at some later generation have arbitrarily large $\overline{R}$. If a larger value for R_t is used then more dead ends are attempted. Upon reaching a dead end, it is a simple matter to move back a generation and test for a boundary circle towards the other parent.

TABLE IV. Universal boundary circles. Continued-fraction expansions for boundary circle frequencies at the $1/q$ fixed points. Computed using $R_t=0.3$, and the class shown.

p/q	Class	ω
$\frac{1}{5}$	4	[0,4,1,14,1,1,1,2,. . .]
$\frac{1}{6}$	3	[0,5,1,6,1,1,1,2,1,3,1,2,. . .]
$\frac{1}{7}$	3	[0,6,1,4,1,6,2,1,1,2,1,. . .]
$\frac{1}{8}$	3	[0,7,1,4,1,5,1,5,1,. . .]
$\frac{1}{9}$	2	[0,8,1,5,1,1,1,1,1,3,1,1,1,. . .]
$\frac{1}{10}$	2	[0,9,1,5,1,6,1,2,1,. . .]
$\frac{1}{11}$	1	[0,11,2,1,1,1,1,2,1,1,1,2,1,. . .]

The resulting frequency is expressed in terms of its continued-fraction expansion (1). In fact, the procedure of generating frequencies by constructing the mediant (the Farey tree procedure) directly gives the continued-fraction expansion.[15] As in Ref. 15, we find that boundary circles have continued-fraction expansions of a special form:

$$[\ldots,1,a_{2l},1,a_{2(l+1)},1,\ldots]\ , \tag{14}$$

where the odd elements are usually 1's (10% of the time there is a 2), and the even elements are small integers, apparently less than seven.

The boundary circle frequencies for some of the universal p/q islands are given in Table IV. For the $1/q$ fixed points with $5\le q\le 10$, the universal island is the first outer convergent to the boundary circle. These islands sit

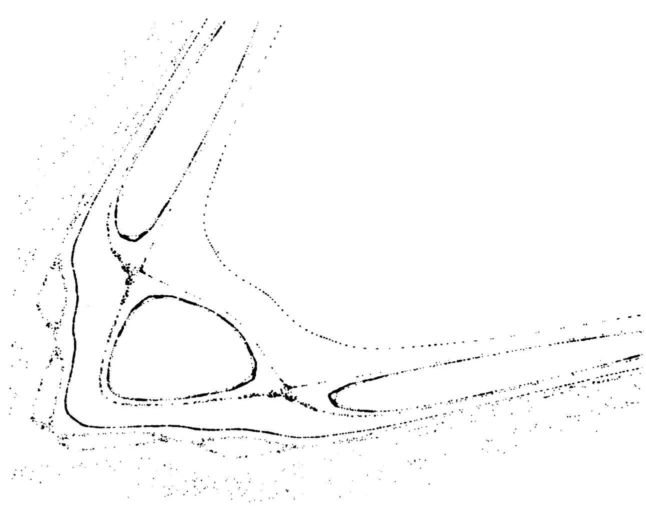

FIG. 2. Phase space at the fixed point of the $\frac{1}{11}$-bifurcation sequence. Shown are the "separatrix" surrounding three of the $\frac{1}{11}$ class-1 islands, and that of six of the islands in a $\frac{2}{13}$ class-1 chain. The class-1 boundary circle is between these; it has a rotation frequency smaller than $\frac{1}{11}$. The $\frac{1}{11}$ class-2 chain and, just outside it, the class-2 boundary circle are also shown.

out in the stochastic sea, just outside the boundary circle. Since T^* is self-similar, all the classes of $1/q$ islands are in the same situation. An orbit in the stochastic region near the separatrix of the class-0 fixed point, can get arbitrarily close to boundary circles of all classes of islands.

When $q > 10$, however, the islands are inside the boundary circle. They are inaccessible to orbits in the connected stochastic region. An example of this is shown in Fig. 2, for the $\frac{1}{11}$ accumulation point. In this case, the islands will not affect orbits in the outer stochastic region.

VII. RENORMALIZATION FOR BOUNDARY ISLANDS

At an arbitrary parameter value, one is not necessarily at the fixed point of any p/q bifurcation. However, it is still possible to define a class sequence in terms of the outermost convergents of boundary circles. For example, at $k=-0.175$ the class-1 boundary circle has frequency $[0,6,1,6,1,6,1,4,1,\ldots]$. The outermost convergent (e.g., the convergent with the lowest frequency) is $[0,6,1]=\frac{1}{7}$. Since the residue of this orbit is less than one ($R=0.204$), it forms an island; its class-2 boundary circle has the frequency $[0,8,1,1,1,2,1,2,1,1,1,\ldots]$. The outermost convergent of this circle is the class-2, $\frac{1}{9}$ orbit, with residue 0.375. The class-3 boundary circle of this orbit has frequency $[0,5,2,6,2,2,1,2,1,\ldots]$, which implies that the $\frac{2}{11}$ orbit (with residue 0.207) is the outermost convergent. Its outermost class-4 convergent has frequency $\frac{1}{9}$ and residue 0.299, etc. In most cases the residue of the outermost convergent is smaller than 1 and it forms an island; in the exceptional case we take the next outer convergent.

This yields an infinite class sequence of boundary islands around boundary islands analogous to the level sequence of convergents for a boundary circle. We designate such class sequences by $(p_1/q_1):(p_2/q_2):\ldots$, e.g., $\frac{1}{7}:\frac{1}{9}:\frac{2}{11}:\frac{1}{9}:\ldots$. The boundary island is not necessarily the outermost island: orbits which are not convergents (in the above example the class-3, $\frac{1}{6}$ orbit or the class-4, $\frac{1}{10}$ orbit) can also form islands; it is not precisely clear which islands are most important for transport in the general case.

The property of most interest for determining the stickiness of a region of phase space, is area scaling. It is easy to compute ΔW for each orbit in the sequence of boundary islands. We find that a version of the geometric scaling still survives:

$$\Delta W_c \approx \Delta W_{c-1} q^{-\zeta} , \qquad (15)$$

with $\zeta \approx 2.2$. and where p/q is the frequency of the class-c orbit. This scaling seems to hold within a few percent for any parameter value. This is not too surprising since:

(1) This is approximately the same scaling which occurs for all the $1/q$ fixed points with q between 5 and 11. This is shown in Table III as the ratio $\ln(\xi)/\ln(q)$.

(2) Since the outermost convergent to the boundary circle is typically of the form $[0,a_1,1]$, a p/q island chain for $p>1$ is not usually such a convergent. However 10% of the odd continued-fraction elements of a boundary circle are 2's, and so the outermost convergent will occasionally have the form $[0,a_1,2]=2/(2a_1+1)$.[15]

(3) Finally, the residue of a convergent to a boundary circle should be near the critical value R_t. For the case of noble circles this is 0.25008. Table III shows that the residues of the $\frac{1}{7}$ and $\frac{1}{8}$ fixed points are closest to this critical value, and so they might naturally be expected to be the most common values for the island frequencies.

We have not yet investigated the statistics of the occurrence of p/q's in the island around island sequence. This could yield interesting results, analogous to those for the continued-fraction sequences of boundary circles.

VIII. CONCLUSIONS AND IMPLICATIONS

We have shown that area-preserving maps exhibit a self-similar structure corresponding to islands around islands. Of course, this is a very old idea and perhaps even the very first case of renormalization in dynamics: it was probably envisioned by Birkhoff[1] and perhaps even by Poincaré when he discovered homoclinic orbits.

We label orbits by class and frequency. An orbit of class c encircles a periodic orbit of class $(c-1)$. Frequency is measured relative to the orbit encircled.

The simplest form of self-similarity corresponds to the case of a sequence of islands each with frequency p/q. For a typical one-parameter family of maps, the parameter can be adjusted $[k \to k^*(p/q)]$ so that scaling of the p/q family is asymptotically geometric, and corresponds to the fixed point of a renormalization operator. The parameter value k^* seems to be a monotonic function of the frequency p/q. We suspect that as $q \to \infty$ (for p finite) k^* approaches the class-0, saddle-node bifurcation point $[-0.25$ for Eq. (2)].

There appears to be essentially[4] one unstable eigenvector, $\delta(p,q)$, of the renormalization operator at k^*. This eigenvalue gives the rate of approach of the parameter values for a bifurcation to k^*. Geometric scaling is determined by the ratio of the area of p/q island to that of the p/q island surrounding it; this ratio is called $\xi(p,q)$. Both of these scaling parameters depend on the values of p and q as shown in Table III: in fact, they are probably most properly though of as functions on a Farey tree, as in the circle-map case.[11]

Consider class sequences with a frequency of the form $1/q$. For $q \geq 4$, the eigenvalue $\delta(1,q)$ monotonically decreases with q, while $\xi(1,q)$ monotonically increases. Area scaling obeys the simple relation (15) with $\zeta \approx 2.2$ when q is in the range 6–10. One could have expected that for $q \leq 4$ the fixed points would behave somewhat differently since at such a bifurcation, linear stability of a p/q orbit does not necessarily imply actual stability. For $q > 10$, we find that the class c, $1/q$ islands are trapped inside invariant circles of all classes $\leq c$, and hence are not accessible from the connected stochastic region outside the primary, class-1 island. We expect the areas of these islands to decrease more rapidly with class than those in the stochastic region; this is confirmed by the larger values of ξ (and also ζ) as q increases. For a given class, the largest island is that of type $\frac{1}{6}$, since ξ is smallest.

Results we have obtained indicate that the fixed point for a frequency which is a Farey daughter of two rationals, occurs at a parameter value intermediate to those of the parents. Similarly the residue at daughter fixed points is between those of the parents. Furthermore, the size of islands seems to decrease rapidly, with relation (15) being roughly valid for an exponent $\zeta \approx 2-3$. Perhaps ζ settles down to some particular value for large Farey generations.

More interesting is the possibility of scaling behavior for arbitrary class sequences. At most parameter values, an elliptic fixed point will have an outermost invariant circle, with a frequency of the form $[0,a_1,1,\ldots]$. The outermost convergent to this irrational is the class-1 island of frequency $1/q$ with $q=a_1+1$. This island will have an outermost class-2 invariant circle with a similar frequency, $[0,a'_1,1,\ldots]$. This sequence of islands around islands appears to proceed indefinitely because the typical residue of orbits in the neighborhood of a boundary circle is of order 0.25.[7,15] In fact this implies, from Table III, that the q_j should be most often near 7 or 8. In such a situation the generalized scaling relation (15) should hold.

Unfortunately, with the brute-force methods of this paper, fixed points with q and p large are computationally difficult to achieve, and the universal behavior of ξ and δ cannot be ascertained. This is due to the rapid increase of ξ and the period of an orbit, $(q)^c$, with q. The maximum period we can find numerically is around 10^5. A clearly better way to proceed is to use the renormalization equation (11) directly, treating the map as a polynomial of some large order. In this fashion fixed points for many frequencies have been investigated for the circle map.[11]

The implications of class renormalization for transport have been treated in Ref. 14. In the simplest type of transport theory which attempts to include the effects of boundary circles,[12,13] the stochastic region is divided into states which are the stochastic regions outside an island chain of given class and bounded by "cantori." Motion in the state is assumed to be random. This implies that the probability for a transition from one state to a neighbor can be obtained from the ratio of the flux through a bounding cantorus to the area of the state. Fluxes scale as the total area of a state divided by a trapping time in the state; this is equivalent to saying that the flux is proportional to the area of a single island in the state. Thus fluxes decrease at the rate $\xi=w^{-1}$. Time scales according to the period of the island chain in the state, which increases at a rate $q=\epsilon^{-1}$.

There are two kinds of scaling which are important for this model, that of levels, which defines the "stickiness" of a particular boundary circle, and that of classes, which defines the islands around which are orbit is likely to be trapped. Assuming that one could be at a renormalization fixed point corresponding to both kinds of scaling (an unlikely case) gives a model in which motion occurs on a self-similar tree. Moving up branches of the tree corresponds to either increasing level, or class.

The stickiness of the tree can be measured by the first return distribution, which is the probability that given at $t=0$ a particle is put in a state at the base of the tree (first level, and lowest class), it first escapes from the tree at time t. We find that this distribution decays as[14]

$$R(t)\to t^{-(1+z)}\,, \tag{16}$$

where the exponent z obeys the relation $w_c\epsilon_c^{-z}+w_L\epsilon_L^{-z}=1$. Here the subscript refers either to class scaling or level scaling. Using generalized scaling relations like (15), this becomes

$$\epsilon_c^{\zeta_c-z}+\epsilon_L^{\zeta_L-z}=1\ . \tag{17}$$

For general boundary circles, the exponent $\zeta_L=3.05$,[15] while for class scaling we can take $\zeta_c=2.2$. For the golden mean the period scaling, ϵ_L, is $\gamma^{-2}=0.382$, while for class scaling $\epsilon_c=q^{-1}$. Using the $q=7$ we obtain $z=1.96$. The value for z will fluctuate depending on the ϵ's, which in turn vary with the frequencies of the boundary circles and islands. We expect, however, that these fluctuations should have a mean roughly given by the values we have used.

Thus the scaling theory implies that the first return distribution should decay roughly as t^{-3}. Numerical experiments seem to give a decay closer to $t^{-2.5}$. For more discussion of this result we refer to Ref. 14.

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[1] G. D. Birkhoff, "Nouvelles Recherches sur Les Systèmes Dynamiques," Memoire Accad. Pontifica de Nuovi Lincei **1**, 85 (1935), reprinted in *Collected Mathematical Papers* (American Mathematical Society, New York, 1950), Vol. 2, p. 589. Translation of quote: "It is clear that not only do general elliptic periodic solutions possess neighboring elliptic and hyperbolic periodic solutions, but also, beginning again with the neighboring elliptic solutions, who are, as it were, satellites of these solutions, one can obtain other elliptic and hyperbolic solutions which are secondary satellites."

[2] V. I. Arnol'd, Dokl. Akad. Nauk **137**, 255 (1961) [Sov. Math. Dokl. **2**, 247 (1961)].

[3] D. F. Escande, Phys. Rep. **121**, 165 (1985).

[4] R. S. MacKay, Ph.D. thesis (University Microfilms, Ann Arbor, 1982).

[5] R. S. MacKay, Physica **7D**, 283 (1983).

[6] L. P. Kadanoff, Phys. Rev. Lett. **47**, 1641 (1981); S. Shenkar and L. P. Kadanoff, J. Stat. Phys. **27**, 631 (1982).

[7] J. M. Greene, J. Math. Phys. **20**, 1183 (1979).

[8] J. M. Greene, R. S. MacKay, F. Vivaldi, and M. J. Feigenbaum, Physica **3D**, 468 (1981).

[9] K. C. Lee, S. Y. Kim, and D. I. Choi, Phys. Lett. **103A**, 225

(1984) (unpublished).
[10]A. J. Lichtenberg, Physica **14D**, 387 (1985).
[11]P. Cvitanović and J. Myrheim (unpublished); S. Ostlund and S. Kim, Phys. Scr. **T19**, 193 (1985).
[12]R. S. MacKay, J. D. Meiss, and I. C. Percival, Physica **13D**, 55 (1984).
[13]J. D. Hanson, J. R. Cary, and J. D. Meiss, J. Stat. Phys. **39**, 327 (1985).
[14]J. D. Meiss and E. Ott, Phys. Rev. Lett. **55**, 2741 (1985); J. D. Meiss and E. Ott, Physica D (to be published).
[15]J. M. Greene, R. S. MacKay, and J. Stark, Physica D (to be published).
[16]J. Moser, *Stable and Random Motions in Dynamical Systems* (Princeton University, Princeton, N.J., 1973), pp. 34 and 50.
[17]V. I. Arnol'd, *Mathematical Methods of Classical Mechanics* (Springer-Verlag, New York, 1978), Appendixes 7 and 8.
[18]R. DeVogelaere, in *Contributions to the Theory of Nonlinear Oscillations,* edited by S. Lefschetz (Princeton University, Princeton, N.J., 1958), Vol. IV, p. 53.
[19]S. Aubry, Physica **7D**, 240 (1983); S. Aubry and P. Y. Le Daeron, *ibid.* **8D**, 381 (1983).
[20]R. S. MacKay and J. D. Meiss, Phys. Lett. **98A**, 92 (1983).