Monte Carlo study of lattice animals in d dimensions

P. M. Lam*
Institut für Theoretische Physik, Universität zu Köln, 5 Köln 41, West Germany
(Received 17 March 1986)

A new Monte Carlo method, the incomplete-enumeration method, is used to calculate both the square radii of gyration $\langle R_N^2 \rangle$ and the number of animals a_N of size N up to $N=30$ in the case $d=3$ and $N=20$ in the cases $d=4$–7. We obtain about 2% accuracy for a_N and 0.2% accuracy for $\langle R_N^2 \rangle$ for the largest sizes N. For $d=9$ longer runs yield values of a_N and $\langle R_N^2 \rangle$ to better than 0.25% and 0.04% accuracy, respectively. Our data are consistent with upper critical dimension $d_c=8$ and with the Flory formula for all $d<d_c$.

I. INTRODUCTION

Lattice animals are models for branched polymers in a dilute solvent.[1] The system is believed to have the upper critical dimension $d_c=8$ at and above which mean-field exponents are exact.[2] The number of animals a_N and the square radius of gyration $\langle R_N^2 \rangle$ of animals of size N behave asymptotically for large N as

$$a_N = A N^{-\theta} \lambda^N, \quad N \to \infty \tag{1}$$

$$\langle R_N^2 \rangle \sim N^{2\nu}, \quad N \to \infty \tag{2}$$

where θ and ν are universal exponents and λ is a nonuniversal constant. By relating the animal problem to the Lee-Yang edge singularity in dimension $d-2$ (Ref. 3) the exact results $\theta=\frac{3}{2}$ and $\nu=\frac{1}{2}$ for $d=3$ and $\theta=1$ for $d=2$ are obtained. A mean-field theory, exact in the limit $d\to\infty$, gives $\theta=\frac{5}{2}$ and $\nu=\frac{1}{4}$.[2-4] A Flory-type theory gives for the radius-of-gyration exponent[5]

$$\nu_F(d) = 5/2(2+d), \quad d \le 8 \tag{3}$$

which coincides with the exact result at $d=3$ and $d=8$. In two dimensions, phenomenological renormalization theory[6] gives the very accurate result $\nu=0.6404\pm0.0001$. This differs slightly from the Flory-type result $\nu_F(d=2)=0.625$. Gaunt *et al.*[7] have made expansions in the variable $1/\sigma$ where $\sigma=2d-1$ for a_N on hypercubic lattices and obtained estimates for θ and λ for d up to 6. At $d=7$, the exact series for a_N was obtained only up to $N=8$. Since their result for θ seemed to approach the mean-field value $\frac{5}{2}$ within error bars (which are rather large), they concluded that $d_c=6$ for the lattice-animal problem as well as for the related percolation problem.[1] There does not seem to be any other calculation, particularly for the square radius of gyration $\langle R_N^2 \rangle$ and therefore for the exponent ν, in higher dimensions. The purpose of this paper is to perform such a calculation using a newly developed method, called the incomplete-enumeration method[8] which is capable of generating unbiased random lattice animals in arbitrary dimensions. We have performed calculations up to $d=7$ for a_N and $\langle R_N^2 \rangle$ and obtained 2% and 0.2% accuracy, respectively, for size N up to 30 in the case $d=3$ and N up to 20 for the higher dimensions. For $d=9$ we have performed extra runs so that the accuracy is now better than 0.25% and 0.04%, respectively, for a_N and $\langle R_N^2 \rangle$. Our data for ν are consistent with $d_c=8$ and with the Flory-formula prediction for all $d \le d_c$. Estimates are also given for λ, θ, and A of Eq. (1). The values obtained for λ and θ are in general agreement with those of Ref. 7.

II. INCOMPLETE-ENUMERATION METHOD

We have previously reported on this method and its application to the lattice-animal problem in two dimensions.[8] It is essentially based on the exact series-expansion method[9] which is a very powerful technique for obtaining physical quantities such as the critical exponents. However, the computational effort needed to calculate the Nth-order term by exact enumeration increases exponentially with N. The situation is even worse in higher dimensions. The exact enumeration method is based on a well-known deterministic algorithm using backtracking.[9] One starts by choosing a rule for designating for each animal configuration one of its sites as the last-added site. Deleting the last-added site from an N-site animal Γ, we get an $(N-1)$-site animal Γ', called the parent of Γ. The animal configurations are then classified into a tree structure according to their parentage. The unique one-site animal forms the root of the tree. The N-site configurations are at a height $N-1$ connected to their parents at height $N-2$ by simple bonds. The exhaustive enumeration of all animals having N sites proceeds by a systematic exploration of the genealogical tree to height $N-1$ (Ref. 9).

In the incomplete-enumeration method, we arbitrarily choose a set of $N-1$ real numbers p_i with $0<p_i<1$ and $i=2$ to N. Any configuration with r sites ($2 \le r \le N$) has probability $1-p_r$ of being deleted from the tree (and probability p_r of remaining on the tree) independent of the state of all other configurations. All configurations which are descendants of these deleted configurations are also deleted. We then systematically enumerate the remaining N-site configurations in the genealogical tree. This can be done very efficiently using backtracking. When a particu-

TABLE I. Quantities A, λ, $\tilde{\lambda}$, exponent θ, and trial values θ_T, λ_T for $d=3$ to 9.

d	θ_T	λ_T	θ	A	$\tilde{\lambda}$	λ
3	1.50	8.0	1.5±0.050	0.153±0.016	1.046±0.005	8.368±0.04
4	2.00	13.0	1.95±0.225	0.128±0.069	1.037±0.015	13.481±0.20
5	2.25	19.0	2.25±0.200	0.117±0.043	0.996±0.015	18.924±0.29
6	2.50	25.0	2.45±0.250	0.103±0.041	0.979±0.020	24.475±0.50
7	2.75	30.0	2.675±0.125	0.107±0.028	1.010±0.010	30.300±0.30
9	2.50	40.0	2.50±0.05	0.047±0.004	1.020±0.005	40.800±0.20

TABLE II. Quantities $\langle X_N \rangle$, $\langle R_N^2 \rangle$, and a_N for $d=3$ to 9.

N	$\langle X_N \rangle$	$\langle R_N^2 \rangle$	a_N
$d=3$			
4	1.3429±0.0068	0.7327±0.0004	0.8594×10^{2}
5	1.4525±0.0079	0.9792±0.0006	0.5321×10^{3}
6	1.5586±0.0090	1.2300±0.0010	0.3475×10^{4}
7	1.6622±0.0104	1.4839±0.0014	0.2353×10^{5}
8	1.7596±0.0114	1.7425±0.0016	0.1631×10^{6}
9	1.8584±0.0135	2.0038±0.0021	0.1155×10^{7}
10	1.9535±0.0168	2.2658±0.0025	0.8291×10^{7}
11	2.0530±0.0179	2.5311±0.0030	0.6042×10^{8}
12	2.1495±0.0187	2.7973±0.0037	0.4442×10^{9}
13	2.2445±0.0218	3.0675±0.0044	0.3291×10^{10}
14	2.3448±0.0255	3.3370±0.0051	0.2461×10^{11}
15	2.4602±0.0285	3.6047±0.0054	0.1862×10^{12}
16	2.5766±0.0315	3.8701±0.0056	0.1416×10^{13}
17	2.6953±0.0343	4.1370±0.0066	0.1082×10^{14}
18	2.8248±0.0386	4.4076±0.0073	0.8329×10^{14}
19	2.9633±0.0410	4.6796±0.0081	0.6446×10^{15}
20	3.1044±0.0439	4.9468±0.0084	0.5002×10^{16}
21	3.2525±0.0479	5.2178±0.0094	0.3897×10^{17}
22	3.4149±0.0510	5.4886±0.0097	0.3052×10^{18}
23	3.5745±0.0576	5.7597±0.0112	0.2391×10^{19}
24	3.7391±0.0628	6.0287±0.0122	0.1877×10^{20}
25	3.9164±0.0666	6.2981±0.0132	0.1480×10^{21}
26	4.0975±0.0718	6.5713±0.0148	0.1168×10^{22}
27	4.2749±0.0775	6.8458±0.0163	0.9209×10^{22}
28	4.4670±0.0841	7.1220±0.0163	0.7290×10^{23}
29	4.6714±0.0916	7.3938±0.0165	0.5786×10^{24}
30	4.8952±0.0983	7.6686±0.0165	0.4610×10^{25}
$d=4$			
4	1.7030±0.0053	0.6943±0.0003	0.2338×10^{3}
5	1.8902±0.0065	0.9064±0.0004	0.2159×10^{4}
6	2.0646±0.0083	1.1143±0.0005	0.2129×10^{5}
7	2.2209±0.0087	1.3191±0.0008	0.2188×10^{6}
8	2.3742±0.0111	1.5223±0.0011	0.2328×10^{7}
9	2.5118±0.0130	1.7233±0.0012	0.2530×10^{8}
10	2.6529±0.0150	1.9224±0.0015	0.2813×10^{9}
11	2.7848±0.0164	2.1193±0.0016	0.3173×10^{10}
12	2.9098±0.0178	2.3138±0.0020	0.3621×10^{11}
13	3.0416±0.0201	2.5069±0.0023	0.4193×10^{12}
14	3.1677±0.0223	2.7006±0.0029	0.4895×10^{13}
15	3.2927±0.0246	2.8921±0.0032	0.5762×10^{14}
16	3.4262±0.0274	3.0817±0.0034	0.6850×10^{15}
17	3.5646±0.0302	3.2702±0.0035	0.8208×10^{16}
18	3.7077±0.0329	3.4580±0.0040	0.9899×10^{17}
19	3.8402±0.0338	3.6422±0.0042	0.1196×10^{19}
20	3.9736±0.0352	3.8259±0.0045	0.1452×10^{20}

TABLE II. (*Continued*).

N	$\langle X_N \rangle$	$\langle R_N^2 \rangle$	a_N
$d=5$			
4	1.6241±0.0070	0.6736±0.0003	0.4923×10^{3}
5	1.7423±0.0080	0.8688±0.0005	0.6074×10^{4}
6	1.8233±0.0084	1.0571±0.0007	0.8013×10^{5}
7	1.8818±0.0093	1.2389±0.0009	0.1111×10^{7}
8	1.9249±0.0100	1.4162±0.0010	0.1599×10^{8}
9	1.9591±0.0121	1.5890±0.0014	0.2372×10^{9}
10	1.9911±0.0129	1.7574±0.0017	0.3613×10^{10}
11	2.0094±0.0144	1.9231±0.0019	0.5591×10^{11}
12	2.0196±0.0156	2.0851±0.0022	0.8778×10^{12}
13	2.0181±0.0182	2.2450±0.0027	0.1392×10^{14}
14	2.0197±0.0203	2.3989±0.0031	0.2240×10^{15}
15	2.0257±0.0227	2.5526±0.0036	0.3655×10^{16}
16	2.0244±0.0249	2.7056±0.0039	0.6003×10^{17}
17	2.0200±0.0268	2.8563±0.0039	0.9929×10^{18}
18	2.0102±0.0287	3.0050±0.0045	0.1651×10^{20}
19	1.9928±0.0313	3.1527±0.0053	0.2753×10^{21}
20	1.9829±0.0039	3.2992±0.0065	0.4638×10^{22}
$d=6$			
4	1.8556±0.0078	0.6619±0.0003	0.9061×10^{3}
5	1.9904±0.0089	0.8459±0.0004	0.1391×10^{5}
6	2.0912±0.0104	1.0214±0.0006	0.2316×10^{6}
7	2.1577±0.0118	1.1899±0.0008	0.4063×10^{7}
8	2.1963±0.0127	1.3523±0.0011	0.7406×10^{8}
9	2.2277±0.0137	1.5087±0.0011	0.1399×10^{10}
10	2.2487±0.0150	1.6627±0.0011	0.2716×10^{11}
11	2.2570±0.0149	1.8109±0.0013	0.5363×10^{12}
12	2.2507±0.0148	1.9584±0.0017	0.1076×10^{14}
13	2.2341±0.0157	2.1007±0.0021	0.2185×10^{15}
14	2.2044±0.0181	2.2414±0.0022	0.4479×10^{16}
15	2.1730±0.0196	2.3785±0.0028	0.9290×10^{17}
16	2.1426±0.0211	2.5145±0.0033	0.1949×10^{19}
17	2.1084±0.0234	2.6476±0.0041	0.4120×10^{20}
18	2.0651±0.0239	2.7742±0.0042	0.8744×10^{21}
19	2.0271±0.0246	2.9011±0.0048	0.1875×10^{23}
20	1.9772±0.0259	3.0272±0.0058	0.4021×10^{24}
$d=7$			
4	2.4955±0.0137	0.6537±0.0004	0.1489×10^{4}
5	2.8277±0.0171	0.8315±0.0007	0.2740×10^{5}
6	3.1082±0.0235	1.0003±0.0009	0.5473×10^{6}
7	3.3536±0.0268	1.1591±0.0010	0.1159×10^{8}
8	3.5803±0.0340	1.3124±0.0014	0.2572×10^{9}
9	3.7442±0.0406	1.4600±0.0017	0.5837×10^{10}
10	3.8900±0.0451	1.6023±0.0018	0.1362×10^{12}
11	4.0226±0.0508	1.7388±0.0019	0.3250×10^{13}
12	4.1531±0.0557	1.8728±0.0025	0.7924×10^{14}
13	4.2513±0.0600	2.0005±0.0028	0.1953×10^{16}
14	4.3620±0.0624	2.1271±0.0031	0.4902×10^{17}
15	4.4580±0.0691	2.2503±0.0034	0.1243×10^{19}
16	4.5206±0.0725	2.3709±0.0041	0.3167×10^{20}
17	4.5924±0.0775	2.4917±0.0046	0.8171×10^{21}
18	4.6388±0.0848	2.6075±0.0052	0.2116×10^{23}
19	4.6862±0.0911	2.7207±0.0052	0.5562×10^{24}
20	4.7426±0.0930	2.8322±0.0055	0.1457×10^{26}

TABLE II. (*Continued*).

N	$\langle X_N \rangle$	$\langle R_N^2 \rangle$	a_N
$d=9$			
4	1.6567±0.0016	0.64250±0.0000	0.3313×10^{4}
5	1.7936±0.0027	0.81193±0.0000	0.8214×10^{5}
6	1.9069±0.0046	0.96980±0.0000	0.2214×10^{7}
7	2.0074±0.0057	1.11855±0.0000	0.6342×10^{8}
8	2.1010±0.0075	1.26014±0.0000	0.1902×10^{10}
9	2.1855±0.0094	1.39603±0.0000	0.5984×10^{11}
10	2.2618±0.0118	1.52614±0.0001	0.1875×10^{13}
11	2.3300±0.0145	1.65158±0.0001	0.6088×10^{14}
12	2.3927±0.0170	1.77217±0.0001	0.2012×10^{16}
13	2.4605±0.0195	1.88941±0.0002	0.6775×10^{17}
14	2.5247±0.0235	2.00300±0.0002	0.2310×10^{19}
15	2.5860±0.0275	2.11404±0.0002	0.7966×10^{20}
16	2.6391±0.0312	2.22184±0.0003	0.2767×10^{22}
17	2.6901±0.0360	2.32777±0.0004	0.9696×10^{23}
18	2.7385±0.0416	2.43074±0.0004	0.3423×10^{25}
19	2.7881±0.0475	2.52995±0.0005	0.1218×10^{27}
20	2.8364±0.0544	2.62771±0.0006	0.4358×10^{28}

lar r-site configuration is first generated, one chooses to ignore it and all its descendants in the enumeration with probability $1-p_r$.

The probability that a particular r-site animal will be enumerated in a given trial is $p_2p_3\cdots p_r=P_r$, and is the same for all configurations with the same r. The algorithm thus generates an unbiased sample of configurations. The number of N-site animals generated in one trial is a random variable X_N. Averaging over many trails, we can estimate $\langle X_N \rangle$. Since the probability P_r that a particular r-site animal will be generated in a given trial is the same for all configurations with the same r, we have $\langle X_N \rangle = P_N a_N$. Thus our method gives a direct estimate of the number of configurations, hence of entropy, a difficult quantity to estimate by conventional Monte Carlo algorithms.[10] Furthermore, compared with conventional methods, as developed by Metropolis *et al.*,[11] which relies on the construction of a Markov process whose time averages replace the ensemble average, our approach is free from problems of strong time correlations in samples and slow approach to equilibrium due to critical slowing. Having retained the backtracking aspect in our method, a step that would lead to rejection makes the algorithm backtrack to attempt another allowed completion, and not to a rejection of the full configuration. Of course our algorithm is not completely free of bias. Different configurations produced in a single trial are correlated with each other, being more likely to share a common lineage. This, however, can be overcome by averaging over many trials. There is also a small anticorrelation as a configuration can occur at most once in a single trial (sampling without replacement). By judiciously choosing p_i, we can get $\langle X_N \rangle$ to be of order 1. Since the number of configurations of animals of size N is very much larger than one for large N, this bias caused by anticorrelation is very small.

The method may be viewed as constructing a site-percolation process on the genealogical tree, with the deleted configurations as the blocked nodes on the tree. The enumerated configurations form the connected part of the tree.

III. INCOMPLETE-ENUMERATION METHOD IN d DIMENSIONS

Contrary to the $(1/\sigma)$ expansion of Ref. 7, our method can be applied without much trouble to any type of lattice in arbitrary dimensions. For convenience of comparison, however, we study lattice animals on a d-dimensional hypercubic lattice $d=3$–9. The result for $d=2$ had been reported before.[8] We choose $p_r(d)$ to be of the form

$$p_r(d)=[r/(r-1)]^{\theta_T(d)}/\lambda_T(d), \quad r>2 \tag{4}$$

where λ_T and θ_T are trial values for λ and θ in (1). The values of λ_T and θ_T are given in Table I. The average number of animals generated per trial $\langle X_N \rangle$ is related to the number of animals a_N of size N by

$$\langle X_N \rangle = \prod_{r=2}^{N} p_r a_N$$

$$= \lambda_T^{-(N-1)} N^{\theta} a_N \ . \tag{5}$$

Equation (5) follows from the fact that the probability that a particular animal configuration of size N be generated in a given trial is the same for all configurations with the same N and is given by $P_N=\lambda_T^{-(N-1)}N^{\theta}$.

Using (1), (4), and (5), we have

$$\langle X_N(d) \rangle = \lambda_T(d) A(d) N^{\Delta\theta(d)} \tilde{\lambda}(d)^N \ , \tag{6}$$

where $\Delta\theta(d)=\theta_T(d)-\theta(d)$ and $\tilde{\lambda}(d)=\lambda(d)/\lambda_T(d)$. Taking the Nth root of (6) gives

TABLE III. Exact result for $\langle R_N^2 \rangle$ and a_N.

N	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=9$
			Exact result for $\langle R_N^2 \rangle$			
4	0.732 558	0.694 44	0.674 242	0.661 765	0.653 302	0.64257
5	0.979 326	0.906 716	0.869 237	0.846 510	0.831 295	0.81224
6	1.230 944	1.115 269	1.056 871	1.022 035	0.998 953	
7	1.486 388	1.320 835				
8	1.744 884	1.523 948				
9	2.005 720					
10	2.268 396					
11	2.532 565					
12	2.797 982					
			Exact result for a_N			
4	86	234	495	901	1 484	3309
5	534	2 162	6 095	13 881	27 468	81 837
6	3 481	21 272	80 616	231 008	551 313	
7	23 502	218 740	1 121 075	4 057 660	11 710 328	
8	162 913	2 323 730	16 177 405	74 174 927	259 379 101	
9	1 152 870	25 314 097	240 196 280	1 398 295 989	5 933 702 467	
10	8 294 738	281 345 096	3 648 115 531			
11	60 494 540	3 178 474 308				
12	446 205 905					
13	3 322 769 129					

$$\langle X_N \rangle^{1/N} = (\lambda_T A N^{\Delta\theta})^{1/N} \tilde{\lambda} \ . \tag{7}$$

Equation (7) offers a way for estimating $\tilde{\lambda}$ since the first factor approaches unity asymptotically linearly in $1/N$ for large N. The values of $\langle X_N \rangle$ and $\langle R_N^2 \rangle$ are shown in Table II, together with the values of a_N obtained using (5). We have used 80 000 trials for $d=3$ and 6, 160 000 trials for $d=4$, 120 000 trials for $d=5$, 30 000 trials for $d=7$, and 240 000 trials for $d=9$. The standard error of estimate shown was obtained by grouping the data into about 40 equal parts and calculating fluctuations about the mean value. The calculation for each of the dimensions from 3 to 7 requires about 2 h of CPU time on a Cyber 76 computer. For the calculation at $d=9$, 16 h were used. For comparison, results of exact series expansions for these quantities are shown in Table III. The results of the radius of gyration in $d=3$ for $N=10$ to $N=12$ have been recently calculated by the present author[12] using an improved algorithm of Peters *et al.*[13] Comparing the values a_N obtained by our Monte Carlo method with those obtained by exact series, we verify that they agree up to Monte Carlo accuracy. We have therefore confirmed numerically Eq. (5).

To determine $\tilde{\lambda}$, we plot $\langle X_N \rangle^{1/N}$ versus $1/N$. These curves are shown in Fig. 1. The values of $\tilde{\lambda}$ which are given by the intercepts with the $\langle X_N \rangle^{1/N}$ axis are shown in Table I, together with values of $\lambda = \tilde{\lambda}\lambda_T$. The errors are obtained by taking reasonable estimates from the shapes of the curves in Fig. 1. To determine A and θ, we rewrite (6) in the form

$$\ln(\langle X_N \rangle N^{-\Delta\theta}) = \ln(\lambda_T A) + N \ln\tilde{\lambda} \ . \tag{8}$$

Now we can plot the quantity $\ln(\langle X_N \rangle N^x)$ vs N for trial values of x until a value x_0 is found such that the curve obtained using this value of x is a straight line at large N with slope given by $\ln\tilde{\lambda}$. We can now identify x_0 with $-\Delta\theta = \theta - \theta_T$ and the intercept of the straight line with the $\ln(\langle X_N \rangle N^x)$ axis as $\ln(\lambda_T A)$. Figure 2 shows such a plot for the case $d=3$. The values of θ, A, and λ determined this way are also shown in Table I. The error estimates in these quantities result from the error estimate in $\tilde{\lambda}$. Our results for θ and λ are consistent with those of Ref. 7. In Fig. 3 we show the exponents for the radii of

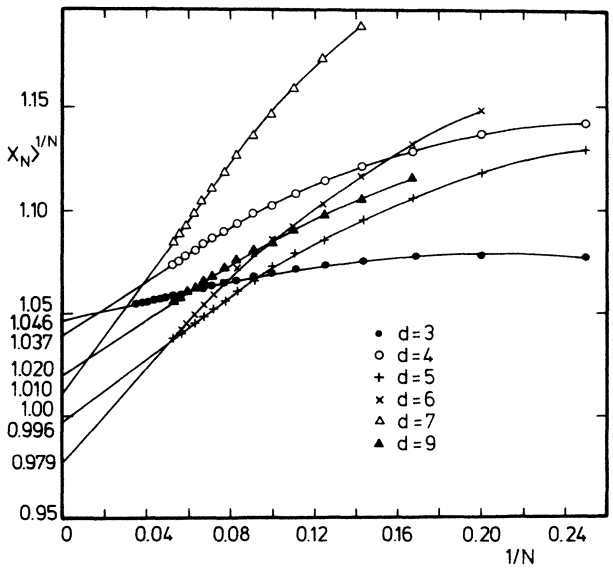

FIG. 1. Quantity $\langle X_N \rangle^{1/N}$ vs $1/N$ for $d=3$ to 9. Also shown are the values of the intercepts on the vertical axis which give values for $\tilde{\lambda}$.

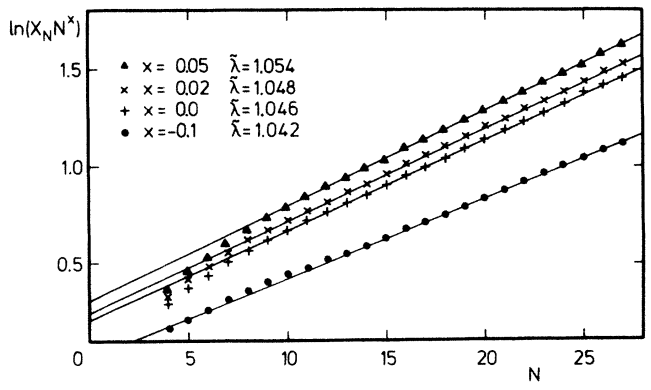

FIG. 2. Quantity $\ln(X_N N^x)$ vs N for $d=3$ and for different values of x. The slopes of these curves should give $\ln\tilde{\lambda}$ which in this case should be 1.046. The intercept of the straight line with the vertical axis then gives $\ln(\lambda_T A)$.

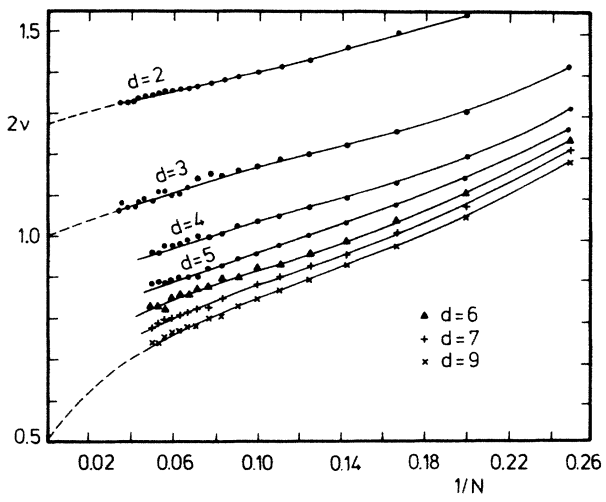

FIG. 3. Exponent 2ν calculated by successive slopes vs $1/N$ for $d=2$ to 9. The dashed lines give extrapolations to the very accurately or exactly known values at $d=2$, 3, and 9.

gyration determined from the successive slopes

$$2\nu=\ln(\langle R_N^2\rangle/\langle R_{N-1}^2\rangle)/\ln[N/(N-1)]$$

taken from values of $\langle R_N^2\rangle$ given in Table I. The dashed lines show extrapolations to the very accurate value $2\nu=1.2812$ (Ref. 6) and the exact values 1 and $\frac{1}{2}$ in $d=2$, 3, and 9, respectively. We have deliberately avoided the case $d=8$ because of the anticipated logarithmic correction difficulty. We see that the corrections to scaling increases with the dimension at least for the cases $d=2$, 3, and 9 where exact or accurate exponents are known. This is in agreement with analysis of exact series in two and three dimensions.[12,14] Our result for ν favors $d_c=8$ to $d_c=6$ since the latter case would require larger corrections to scaling for $d=6$ and $d=7$ than for $d=9$ in order that they all extrapolate to the mean-field value $\nu=\frac{1}{4}$. It is seen from Fig. 3 that our exponent ν is also consistent with the prediction of the Flory formula (3) for all $d\leq d_c$, if corrections to scaling similar to that for $d=9$ are taken into account.

IV. CONCLUSION

We have shown that our incomplete-enumeration method generates unbiased random lattice animals in arbitrary dimensions. We obtained with moderate computational time values of a_N and $\langle R_N^2\rangle$ for size N of the order of 20 and dimensions up to 7 to about 2% and 0.2% accuracy, respectively. With sufficient computation time, we show that these values can be obtained for size up to $N=20$ with 0.25% and 0.04% accuracy, respectively, even for dimensions as high as 9. Furthermore our method has the advantage that it can be applied with more or less equal ease to any type of lattice. Our result for the exponent ν is consistent with upper critical dimension $d_c=8$ but it also shows that corrections to scaling are rather large at high dimensions. It is also consistent with the predictions of the Flory formula if these corrections to scaling are taken into account. We can conclude that except for some slight discrepancy with the two-dimensional phenomenological renormalization result, the Flory formula is very accurate for all dimensions up to $d_c=8$.

ACKNOWLEDGMENTS

The author thanks Dietrich Stauffer for useful advice, Deepak Dhar for discussions, and the Sonderforschungbereich 125 for financial support.

*On leave from Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing, China.

[1]D. Stauffer, *Introduction to Percolation Theory* (Taylor and Francis, London, 1985).
[2]T. C. Lubensky and J. Isaacson, Phys. Rev. Lett. **41**, 829 (1978); **42**, 410 (1979); Phys. Rev. A **20**, 2130 (1979).
[3]G. Parisi and N. Sourlas, Phys. Rev. Lett. **46**, 871 (1981).
[4]B. H. Zimm and W. H. Stockmayer, J. Chem. Phys. **17**, 1301 (1949).
[5]J. Isaacson and T. C. Lubensky, J. Phys. (Paris) Lett. **41**, L469 (1980).
[6]B. Derrida and D. Stauffer, J. Phys. (Paris) **46**, 1623 (1985); J. Kertesz, J. Phys. A **19**, 599 (1986).
[7]D. S. Gaunt, M. F. Sykes, and H. Ruskin, J. Phys. A **9**, 1899 (1976).
[8]D. Dhar and P. M. Lam (unpublished).
[9]J. L. Martin, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York,

1974); S. Redner, J. Stat. Phys. **29**, 309 (1982); E. S. Demme and K. Diemer, J. Undergrad. Res. Phys. **3**, 25 (1984).

[10]S. K. Ma, J. Stat. Phys. **26**, 221 (1981).

[11]For a review and extensive references see *Monte Carlo Methods,* edited by K. Binder (Springer-Verlag, Berlin, 1979).

[12]P. M. Lam (unpublished).

[13]H. P. Peters, D. Stauffer, H. P. Hoelter, and K. Loewenich, Z. Phys. B **34**, 399 (1979).

[14]A. Margolina, F. Family, and V. Privman, Z. Phys. B **54**, 321 (1984).