Universal spectra of quasirandom objects produced by off-equilibrium space divisions

Hiroshi Furukawa
Institute of Physical Sciences, Faculty of Education, Yamaguchi University, Yamaguchi 753, Japan
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A simple model of quasirandomness is presented. The model is a self-similar space division: A d-dimensional space is divided successively into smaller subspaces with randomness. Positions of subspaces are fixed. Then centers of subspaces distribute quasirandomly. Two typical cases of the model are discussed. In one case (repulsive case) the power spectrum of centers of subspaces exhibits a $k^{-\alpha}$ ($\alpha \approx d$) singularity in a broad range of a randomness parameter, where k is the wave number. This spectrum corresponds to a so-called $1/f$ noise spectrum for a one-dimensional time sequence. The k^{-d} singularity will be observed in the scattering intensity from a phase-separating system if droplets cannot move freely due to the strong correlation. This singularity does not seem to depend on details of the model, and is universal. We can show the same singularity analytically for a modified model. In the other case (attractive case) the power spectrum exhibits a $k^{-\alpha}$ ($\alpha \approx 0.4d$) singularity in a broad range of the randomness parameter. This singularity is equivalent to the typical singularity observed in the universe. The physical reason for this agreement is discussed.

I. INTRODUCTION

The particle configuration in an ideal gas is a typical example of physical randomness: There is no correlation among particles. On the other hand, the configuration of atoms in a crystal is an example of regularity. The correlation among atoms does not vanish even for infinite distance. In between these two states, there are physical states: fluid, glassy, or amorphous. In such states the correlations among atoms are, in general, of short range, however. Only near the critical pont the correlations become long range. In this sense the liquid, glassy, or amorphous state is a random state. In this paper we study a completely different random state, which we call a quasirandom state.

As an example, let us consider the breaking of a crystal. A piece of crystal is broken randomly into 2^d pieces in a single try, where d denotes the space dimension. The number of pieces is 2^{dn} after n tries. Here we set up the model. (i) The breaking is done in a self-similar way. (ii) The positions of fragments are frozen in during the whole process. The centers of gravity of the fragments are partly random due to the random breaking; but, due to the condition (ii), they retain some of the initial long-range correlation of the crystal. Therefore, the positions of the crystal pieces are half-randomly and half-regularly distributed. This is the reason why we use the term "quasirandom." The physical meaning of the freezing in of positions of fragments is that the whole breaking is completed in a relatively short time compared to that for the randomization or the equilibriation of positions of fragments. Thus the situation is intrinsically off equilibrium.

We focus our attention on the power spectrum of the model. One of the motivations of the present study is to present a prototype of the mechanism of $1/f$ noise in a physical and nonphysical systems.[1] Power spectra of the sound from a radio (music, speech, etc.) are known to exhibit a $1/f$ singularity.[2] The electric signal of a static TV screen also exhibits a similar singularity.[3] The appearance of an $f^{-\beta}$ singularity in these situations is considered a result of self-similar construction of the music or picture.[4] However, the reason why a $1/f$ spectrum results is not yet known. The $1/f$ current noise observed in electric circuit is also a mysterious phenomenon. There are many singular behaviors showing an $f^{-\beta}$ singularity of the correlation function in physical systems in off-critical regions. Our study might be more or less related to these unsolved phenomena. We may answer the question why a long-range correlation arises even in a disordered system. Apart from $1/f$ noise, we investigate two concrete problems. One is the correlation among droplets or grains in a thermodynamically unstable system. If droplets are strongly correlated to each other (in a percolating state), they coalesce into larger ones without changing their positions significantly. Therefore, the growth process of droplets may be viewed as the inverse process of the above model. Thus we speculate that a similar singularity may occur in the droplet correlation function. The other problem is the correlation among stars or galaxies in the universe. The observed correlations between stars or galaxies exhibit an $r^{-1.8}$ singularity, where r is the distance.[5] This correlation may be approximately reproduced by the present model.

The model we study in this paper is a static one. However, all physical quantities we study are snapshots of dynamical phenomena. Therefore, we consider that the present static model can represent the essence of the observed phenomena.

The mathematical definition of our model will be given in the next section. Our model is classified into two cases: the repulsive case and the attractive case. An analytic calculation of the power spectrum will be given in Sec. III.

Results of numerical simulation will be presented in Sec. IV. In Sec. V, a qualitative explanation and several applications of the model will be discussed. Section VI is devoted to a summary.

II. MODEL AND ITS MODIFICATION

For simplicity we first consider a one-dimensional case. A one-dimensional space of length L is prepared. Then this space is divided into two subspaces. The boundary between the two subspaces is set randomly within a given region of the host space. In the second step we divide each subspace further into two subspaces. The boundaries are set by the same rule as in the first step. Namely, we randomly set a boundary but in a self-similar way as in the first step. This procedure is repeated up to the nth step.

We consider two different cases of setting boundaries. We divide a subspace into three regions: one center region and two edge regions. The width of the center region is $1-2a$ for a subspace of unit length. In one case the boundary is set randomly in the center region. We call this case the repulsive one. In the other case, which we call the attractive case, the boundary is set randomly in one of the edge regions, which is randomly chosen. The above two cases are illustrated in Fig. 1. In both cases a boundary is randomly set in the shaded region. The distribution of boundaries is uniform in the repulsive case, whereas it is not in the attractive case. In the repulsive case $a=0.5$ yields a regular, evenly spaced lattice, whereas in the attractive case $a=0$ yields a lattice with only three distinct points ($x=0$, 0.5, and 1). Also, the repulsive case with $a=0$ is equivalent to the attractive case with $a=0.5$.

We now generalize the above model to the d-dimensional case. We consider a division of a d-dimensional cubic space of volume L^d into 2^d subspaces in a single step. Each subspace has $2d$ (hyper) rectangular sides. Boundaries in a single dimension are set by the same rule as in the case of the above one-dimensional model. After n steps we have $N=2^{nd}$ subspaces. In Fig. 2 the two-dimensional model is illustrated for the repulsive case.

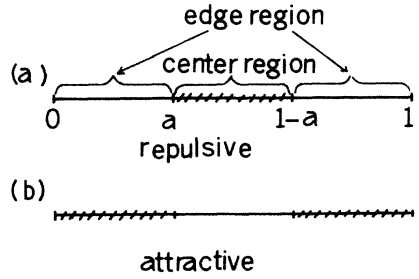

FIG. 1. Rule for setting boundaries in a one-dimensional space. (a) The repulsive case. A boundary is randomly set in the center region. (b) The attractive case. A boundary is randomly set in one of the edge regions chosen at random. In both cases center region and edge regions are self-similarly set in subspaces. This procedure is repeated successively.

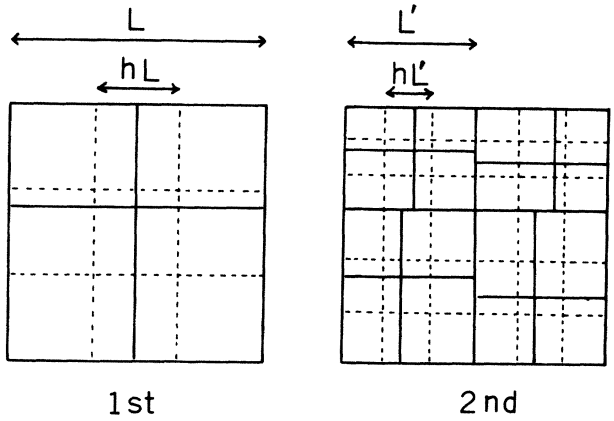

FIG. 2. Two-dimensional space division. Dashed lines indicate boundaries between edge regions and a center region, and $h=1-2a$. A subspace is divided into four subspaces by two straight lines which are drawn between two parallel dashed lines (repulsive case). In the attractive case straight lines are drawn outside the region between two parallel broken lines. This procedure is repeated.

It is not easy to treat this model analytically. Therefore, we shall modify the model so that we can obtain an analytic form of the power spectrum. This modification may make the model somehow unrealistic. But we can show a mathematical structure of the "universal" spectrum. In the modified model the space is divided into 2^d subspaces with the same size in a single step: The space division of the modified model is that of $a=0.5$. However, the mass of a particle in a subspace in the modified model is determined to give the same center of gravity as in the original space-division model. We consider the one-dimensional case. In the above original model a one-dimensional space of length L is divided into two subspaces of lengths xL and $(1-x)L$ ($a\le x\le 1-a$ in the repulsive case and $1\le x\le a$ and $1-a\le x\le 1$ in the attractive case), respectively. At the center of each subspace a "particle" with unit mass is placed. In the modified model particles with mass m and $2-m$ are placed at centers of two subspaces with the same size, respectively (see Fig. 3). The mass m is determined by the lever rule:

$$\tfrac{1}{2}x+\tfrac{1}{2}(1+x)=\tfrac{1}{4}m+\tfrac{3}{4}(2-m)\,, \tag{2.1}$$

which gives $m=2(1-x)$. Since $a\le x\le 1-a$ in the repulsive case and $0\le x\le a$ and $1-a\le x\le 1$ in the attractive case, we find that

$$2a\le m\le 2(1-a) \tag{2.2a}$$

in the repulsive case, and

$$0\le m\le 2a,\quad 2(1-a)\le m\le 2 \tag{2.2b}$$

in the attractive case. That is, the range of variation of $m/2$ is the same as that of x: the position of the boundary between the two subspaces in the original model. For $a>0.05$ in the repulsive case the two models may

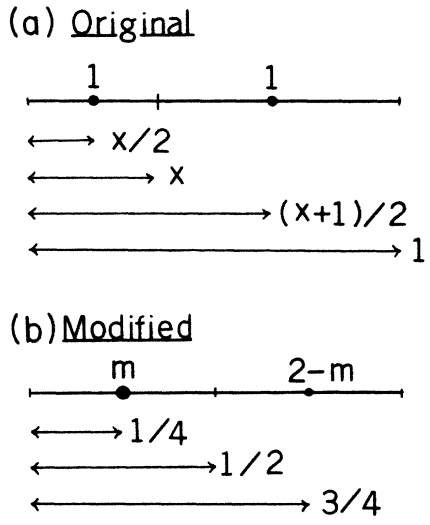

FIG. 3. Modification of model. (a) The original model. Closed circles indicate particles with unit mass. The two subspaces have length x and $1-x$, respectively. (b) The modified model. The two subspaces have the same length $\frac{1}{2}$. Particle masses m and $2-m$ in the two subspaces are given so that the center of the gravity of the system may be the same as in the original model.

well approximate to each other. Then the property of the original model for $a>0.05$ in the repulsive case may be suitably examined by the analytic treatment of the modified model.

In d dimensions the mass M in a subspace of the modified model is of the form

$$M=m_1 m_2 \cdots m_d \ , \tag{2.3}$$

where m's are independent variables with unit mean $\langle m_j\rangle=1$ $(j=1,2,\ldots,d)$. Equation (2.3) is due to the fact that space divisions in the original model are done independently in each dimension.

III. POWER SPECTRUM

The power spectrum S_k is given by

$$S_k=\frac{1}{N}\left|\sum_{j=1}^{N} e^{i\mathbf{k}\cdot\mathbf{r}_j}\right|^2$$

$$=\frac{1}{V}\int\int G(\mathbf{r},\mathbf{r}')e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}d\mathbf{r}\,d\mathbf{r}' \ , \tag{3.1}$$

where $\mathbf{r}_j$ is the position of the center of the jth subspace, which we represent by the jth particle with unit mass; G is the correlation function; and V is the total volume of the system. In this section we calculate G in an analytic way.

We now consider the correlation function for the modified model. The step number is assumed to be n, so that the number of the subspaces is 2^{dn}. The mass of the particle in a subspace of the modified model is of the form

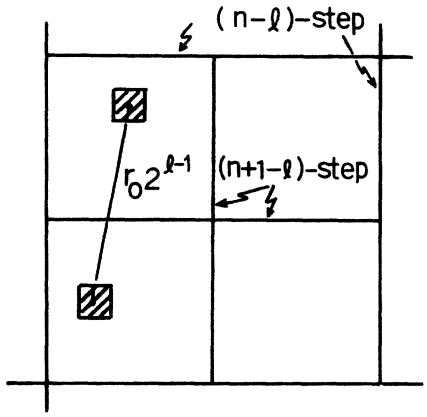

FIG. 4. Two different subspaces in the modified model. Shaded squares indicate two different subspaces, which belong to the same outer subspace up to the $(n-l)$-step division, but belong different respective outer subspaces after the $(n+1-l)$-step division. $r_0 2^{l-1}$ indicates an average distance between the two subspaces.

$$M_1 M_2 \cdots M_n \ ,$$

where M's are independent of each other, and $\langle M_j\rangle=1$ $(j=1,\ldots,n)$. Let us consider another subspace. As illustrated in Fig. 4, let the two subspaces belong to the same outer subspaces up to the $n-l$ step division, and then the two subspaces belong to different outer subspaces after the $n+1-l$ step division. The mass of the particle in the latter subspace is of the form

$$M_1 M_2 \cdots M_{n-l} M'_{n+1-l} M'_{n+2-l} \cdots M'_n \ .$$

Therefore, the correlation between two subspaces is the following:

$$G_l=\langle M_1^2 M_2^2 \cdots M_{n-l}^2 M_{n+1-l} M'_{n+2-l} \times M'_{n+2-l} \cdots M_n M'_n\rangle$$
$$=\langle M^2\rangle^{n-l}\langle M_{n+1-l} M'_{n+1-l}\rangle \ . \tag{3.2}$$

Here we have used

$$\langle M_{n+2-l} M'_{n+2-l} \cdots M_n M'_n\rangle = \langle M_{n+2-l}\rangle\langle M'_{n+2-l}\rangle \cdots \langle M_n\rangle\langle M'_n\rangle = 1 \ .$$

Although M_{n+1-l} and M'_{n+1-l} are not independent of each other, the correlation $\langle M_{n+1-l} M'_{n+1-l}\rangle$ is independent of n and l. We have

$$\langle M_{n+1-l} M'_{n+1-l}\rangle=\langle mm'\rangle^d=(2-\langle m^2\rangle)^d \ .$$

The average distance r between the two subspaces is given by

$$r=r_0 2^{l-1} \ , \tag{3.3}$$

where $r_0(=L2^{-n})$ is a constant. Then the correlation function as a function of r is given as

$$G(r)\equiv G_{l(r)}=\langle MM'\rangle\langle M^2\rangle^{n-l(r)}=C\left[\frac{r}{r_0}\right]^{-c}, \tag{3.4}$$

where $C=\langle MM'\rangle\langle M^2\rangle^{n-1}$ and

$$c=\log_2\langle M^2\rangle . \tag{3.5}$$

Notice that the r denotes an average distance between two subspaces. Therefore, the correlation function as a function of the bare distance may not be such as (3.4), but may be accompanied by a small correction. Since the model is constructed in a self-similar way and therefore we can expect that the power spectrum or the correlation function obeys a power law, such a correction is not important. This is because the dependence of r as shown by the relation (3.3) is the only one that leads to the power-law correlation (3.4). From (2.3) we have $\langle M^2\rangle=\langle m^2\rangle^d$. Thus $c=d\log_2\langle m^2\rangle$. Therefore, we have the power spectrum

$$S_k=\int G(r)e^{i\mathbf{k}\cdot\mathbf{r}}d\mathbf{r}\propto k^{-\beta d}, \tag{3.6}$$

where

$$\beta=1-\log_2\langle m^2\rangle . \tag{3.7}$$

For the repulsive case we have

$$\langle m^2\rangle=\frac{1}{2(1-2a)}\int_{2a}^{2-2a}m^2dm=\tfrac{4}{3}(1-a+a^2), \tag{3.8a}$$

and for the attractive case we have

$$\langle m^2\rangle=\frac{1}{4a}\left[\int_0^{2a}m^2dm+\int_{2(1-a)}^{2}m^2dm\right]$$
$$=2-2a+\tfrac{4}{3}a^2 . \tag{3.8b}$$

The exponent (3.7) with (3.8) must be very accurate for the modified model, but is an approximation for the original model. We consider, however, that the dependence of (3.6) on dimension d must be valid also for the original model. This will be ascertained numerically as shown in the next section.

IV. NUMERICAL SIMULATION

We show in Fig. 5 two-dimensional patterns of centers of subspaces for several values of a for the original space-division model. For the numerical simulation the power spectrum (3.1) is computed under the periodic boundary condition, and therefore wave-number components $\{k\}$ satisfy

$$k_\mu L/(2\pi)=1,2,3,\ldots\quad(\mu=1,2,\ldots,d) . \tag{4.1}$$

If arbitrary wave numbers are used, then the power spectra exhibit k^{-2} dependence at small wave numbers. This can be found as follows. We consider a regular lattice. The density fluctuation ρ_k of the regular sequence is given by

$$\rho_k=N^{-1/2}\sum_{m=1}^{N}\exp(iksm)$$
$$=N^{-1/2}[1-\exp(ikL)]/[1-\exp(iks)] ,$$

where s is the nearest-neighbor distance, $L=Ns$, and N is the total number of lattice points. Therefore, if arbitrary wave numbers are taken, then the power spectrum is given by

$$S_k=N^{-1}[1-\cos(ks)]^{-1}\approx N^{-1}k^{-2}s^{-2} .$$

However, if the periodic boundary condition (3.1) is taken, then such a k^{-2} dependence does not appear. The effect of k^{-2} dependence to the power spectrum disappears in the limit of large system size. Thus the k^{-2} dependence at small wave number for a finite system size should be removed. This can be done by the use of periodic boundary conditions.

In the previous short communication,[6] the periodic boundary condition was not used, and therefore the k^{-2} dependence of the power spectrum was not removed. Therefore, the previous estimate of the exponent β for large value of a in the repulsive case contains an ambiguity. In Fig. 6 the power spectrum in the one-dimensional repulsive case with $a=0.35$ are shown for several values of the step number of the space division n. In Fig. 7 one-dimensional spectra in the repulsive case for $n=13$ are shown for several values of a. In Fig. 8 one-dimensional power spectra in the attractive case are shown for several values of a. All power spectra are for the original space-division model. They are averaged over 25 different runs. Also wave numbers are coarse grained. in Fig. 9 the exponent β estimated from these spectra is plotted. In this figure the dashed lines indicate the theoretical value of β for the modified model, i.e., (3.7) with (3.8). We have also done the numerical simulation for the modified model with the step number $n=13$. The estimation of β for the modified model is more accurate than that for the original model due to the large number of data for the modified model. This is because we can use the fast Fourier transformation for the numerical computation for the modified model. We found that the numerical simulation and the theoretical calculation (3.7) in the modified model agrees well with each other. We can find that the exponent β in the repulsive case for the original model agrees well with that for the modified model for $a>0.05$. But they do not agree with each other in the attractive case. This is because the particle configuration in the original model is not uniform in the attractive case.

We now present results for the two-dimensional original model. In two dimensions, we have made simulations only for $n=6$. The number of divided subspaces is $N=4^6=4096$. In Fig. 10 we show S_k's for the original model corresponding to patterns in Fig. 5 as functions of the scalar wave number $k=(k_1^2+k_2^2)^{1/2}$. Here we have computed S_k only for positive values of the one-dimensional wave numbers k_1 and k_2. The S_k's are averaged over 25 runs, but no coarse graining is taken over the wave number. The dashed lines have the slope -2β (i.e., $S_k\propto k^{-2\beta}$), where β is the exponent in the one-

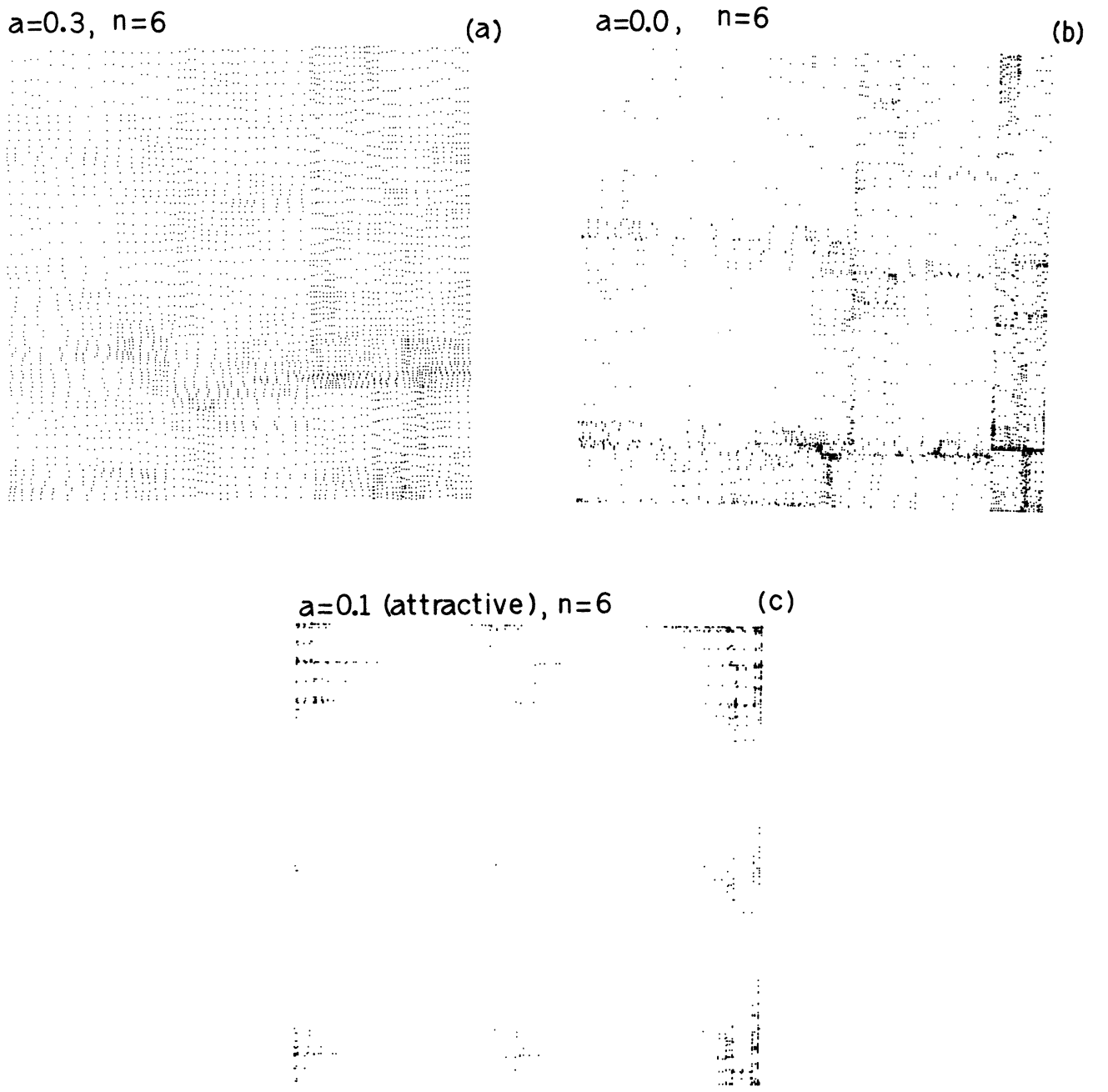

FIG. 5. Set of centers of subspaces in two-dimensional original model with $n=6$ ($N=4^6$). (a) Repulsive case with $a=0.3$; (b) repulsive case with $a=0$, equivalent to attractive case with $a=0.5$; (c) attractive case with $a=0.1$.

dimensional model for the same value of a. One can find that (3.6) hold for the original space-division model, though there is a lot of scatter in the data due to the small range of wave number.

V. DISCUSSION

A. Conservation law with quasirandomness

In the limit of the weak randomness ($a \to 0.5$ in the repulsive case) we observe k^{-d} spectrum. There is only a weak continuous change in the spectrum near $a=0.5$. There are several reasons for the appearance of this spectrum. One is the continuous nature of the model. A regular configuration changes continuously into a random configuration by changing a parameter a from 0.5 to smaller values. This gives continuous change in the value of $\langle m^2 \rangle$ in (3.7). This continuity also guarantees a finite density of state near the k^{-d} spectrum. In reality this continuity comes from the conservation of mass, volume, or other physical quantities such as momentum or energy.

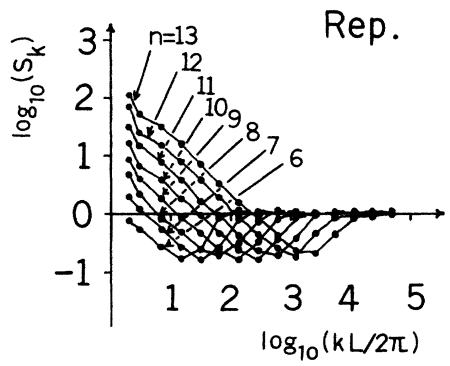

FIG. 6. Log-log plot of one-dimensional power spectra S_k's in the repulsive case of the original model, for $a=0.35$ and for several values of step number n. The data are averaged over 25 different runs, and wave numbers are coarse grained.

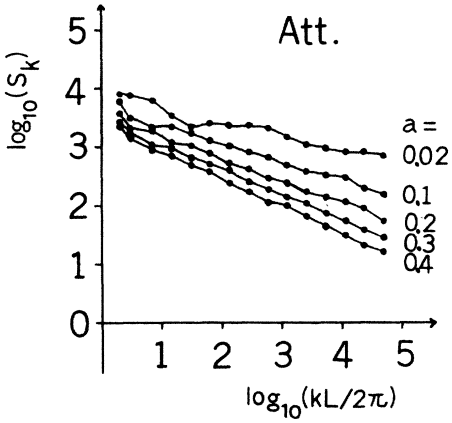

FIG. 8. Log-log plot of one-dimensional power spectra S_k's in the attractive case of the original model, for $n=13$ and for several values of a. The data are averaged over 25 different runs, and wave numbers are coarse grained.

Therefore in a system with a conservative quantity, we can expect a k^{-d} spectrum with much generality. Also, the k^{-d} spectrum is not an equilibrium spectrum. It should disappear in equilibrium due to the randomization of positions of subspaces or due to other dissipative mechanisms.

Another reason is the fractal nature of the model, which we here call quasirandomness. To the present model the fractal is not necessarily a suitable concept, because the fractal dimension D does not give the value of the exponent β Owing to the conservation of mass, or the volume, the fractal dimension D of the present model seems to be always equal to the spatial dimension d. This situation is the same as in the case of the devil's staircase.[4]

Here we derive k^{-d} spectrum in a heuristic way for the original model in the limit of large system size. The d-dimensional power spectrum S_k satisfies the following three conditions. (i) $S_k \approx N$ for $k \lesssim k_0 = 2\pi/L$, (ii) $S_k \approx 1$ for $k \gtrsim k_1 = N^{1/d} k_0$, and (iii) $S_k \propto k^{-\beta}$ for $k_0 \lesssim k \lesssim k_1$. Here condition (i) occurs because the largest distance between centers of subspaces is L. Therefore, for $k \lesssim k_0$, $\exp(i\mathbf{k}\cdot\mathbf{r}_j)$ in (3.1) is replaced by 1. This gives condition (i). Condition (ii) occurs because the smallest distance between centers of subspaces is approximately $N^{-1/d}L$. For $k > k_1$, S_k reduces to the correlation function of a noninteracting gas. This gives condition (ii). Condition (iii) occurs because the space division is self-similar. By setting $S_k = Ck^{-\alpha}$, we have $N = Ck_0^{-\alpha}$ from (i), and

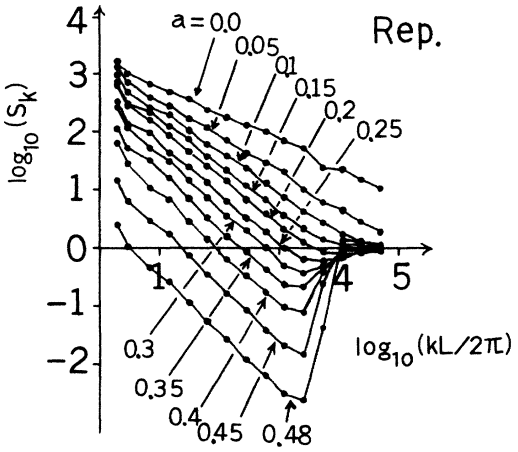

FIG. 7. Log-log plot of one-dimensional power spectra S_k's in the repulsive case of the original model for $n=13$ and for several values of a. The data are averaged over 25 different runs, and wave numbers are coarse grained.

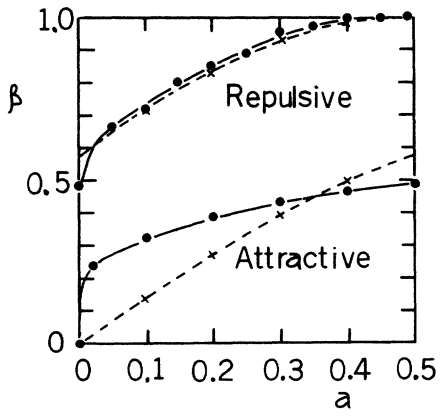

FIG. 9. β vs a. Closed circles represent β for the original model. The solid curve is drawn to guide the eye only. The dashed curves indicate the analytic calculation of β, i.e., (3.7) with (3.8), which is accurate for the modified model. Crosses represent numerical results for the modified model.

$1=Ck_1^{-\alpha}$ from (ii). Thus we have $N=(k_1/k_0)^{\alpha}$ $=(N^{1/d})^{\alpha}$ and therefore we have $\alpha=d$. As is found from this explanation of the k^{-d} spectrum this power spectrum does not depend on details of the model. A factor significant to the k^{-d} spectrum is the quasirandomness of the distribution of centers of subspaces: The correlation reduces to that of a random object for the smallest distance between subspaces (due to the random divisions of subspaces), but it is not for larger distances (due to the freeze in of subspaces). The inverse Fourier transform of k^{-d} is $\ln r$. Therefore,the correlations between centers of any distant subspaces do not vanish. This long-range correlation can be understood as in Sec. I. Namely, the positions of subspaces are fixed, and therefore the subspaces retain the memory of the long-range correlation of the host space. But the memory of the long-range correlations is partly smeared by the random divisions of subspaces. This is the reason why such a weak but long-range correlation as $\ln r$ comes out.

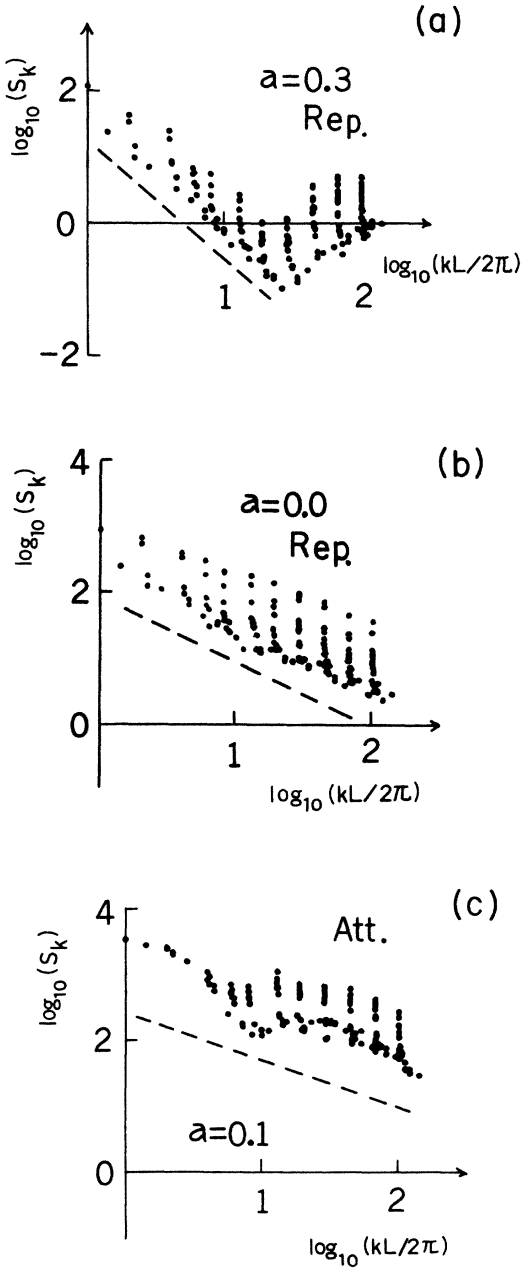

FIG. 10. Log-log plot of the two-dimensional power spectra S_k's for the original model with $n=6$. (a) Repulsive case with $a=0.3$; (b) repulsive case with $a=0$, equivalent to attractive case with $a=0.5$; (c) attractive case with $a=0.1$. No coarse graining over wave numbers is taken. Data are averaged over 25 different runs. Dashed lines indicate slopes -2β, where β is the corresponding one-dimensional exponent.

We may also explain the value $\beta\approx 0.5$ at $a=0$ in the repulsive case (equivalent to attractive $a=0.5$) in one dimension as follows. The upper limit of the wave number for which (3.6) is valid is determined by the smallest of the two segments obtained in each space divison. The average of this value may be evaluated as 4^{-n} times the total length L after n steps. The evaluation is as follows. For $a=0$ the nearest-neighbor distance becomes smaller on the average by a factor $\frac{1}{4}$ in each step of the space division, since the boundary is set randomly on a half-side of a subspace. The maximum wave number is $4^n k_0$ for $a=0$, whereas $2^n k_0$ for $a\approx 0.5$. $S_k\approx 1$ at each maximum wave number and $S_{k_0}\approx 2^n$ $(=N)$. This means that β for $a=0$ is half that for $a\approx 0.5$, i.e., $\beta\approx 0.5$ for $a=0$.

B. $1/f$ noises

The present model may give one of prototypes of $1/f$ noise. Let us consider a $1/f$ current noise. We first prepare a regular fluctuationless current. This current is disturbed by the interaction with the surroundings. The disturbance may occur in the same way as the self-similar space division, where centers of the subspaces are replaced by centers of divided current pieces. When the disturbance is not too strong, the current power spectrum exhibits a $1/f$ singularity.

Previously we have proposed a current bifurcation model of $1/f$ noise.[7] In that model the current bifurcates successively into two pieces: One is frictionless and the other is frictional. As a result, the current $V(t)$ obeys the equation

$$\left[\gamma\frac{d}{dt}\right]^{1/2} V(t)=F(t)\,, \tag{5.1}$$

where γ is the damping coefficient associated with frictional motion and $F(t)$ is the random force. Then the power spectrum $\langle |V_\omega|^2\rangle$ has a $1/\omega$ singularity. This model resembles the present space-division model, though the space-division model applies to configurational space, whereas the bifurcation model applies to momentum space. At this moment we have found no clear relationship between the two models.

A $1/f$ noise which seems to have the same origin as the above is observed in a depinned charge density wave (CDW).[8] When the CDW is pinned, it behaves a a rigid body. But it decomposes into many segments when it is depinned, exhibiting a $1/f$ current or voltage noise.

C. k^{-d} correlation in phase separation

We consider a different application of the k^{-d} spectrum in d dimensions, namely the pattern formation that

occurs during phase separation. In a phase-separating system droplets or clusters coalesce successively into larger ones until a completely phase-separated state is attained.[9] When the volume fraction of the minority phase is large, droplets are percolated. As a result, a droplet cannot move freely. Under this condition droplets coalesce successively into larger ones without changing their positions. To evaluate the correlation among droplets, we consider the equivalent correlation among smaller droplets in a preceding stage of the phase separation. For this purpose we consider the inverse process of successive coalescence of droplets. Since droplets hardly move, the inverse process is represented by successive divisions of droplets into many smaller ones. Therefore, the present space-division model can be applied. Thus the power spectrum of the centers of gravity of droplets may exhibit a k^{-d} singularity if droplets cannot move freely. Combining this correlation function with the mass continuity law, one may expect that the structure function or the scattering function has asymptotic form k^d at small k.[10] Such a k^d dependence of the scattering function at small k has already been reported.[11]

D. Correlation among stars in the universe

A plausible example of the attractive case is the distribution of stars or galaxies in the universe. It is well known that the correlation function of stars or galaxies in the universe is found to be $\propto r^{-1.8.5}$ The Fourier transform of this correlation is $k^{-1.2}$. The exponent 1.2 corresponds to $\beta=0.4$ (in one dimension) of the present model. The value $\beta=0.4$ is a typical value of the attractive case of the present model. We now present a plausible model for the formation of the structure in the universe. We start with a homogeneous cold dust interacting through the gravitational force. This system is unstable, and undergoes the Jeans instability.[12] In the first stage of the instability a large scale mode becomes unstable. Then some parts of the universe have high densities. It is known that the characteristic time of the instability depends on the mass density. This is due to the long-range nature of the gravitational force. As the density becomes larger, the instability proceeds faster. Therefore, a denser part undergoes a faster instability. This causes successive instability toward smaller scales. As scales become smaller the density becomes larger and the characteristic time becomes shorter. As a result, in the formation of small structures, any larger structures can be regarded as frozen in. Now let us consider our model of the space division. All subspaces are simultaneously divided. Therefore, all subspaces at the same step contain the same number of inner subspaces. In other words, all subspaces of the same step contain the same number of points (centers of subspaces) which correspond to stars. Therefore, the successive space division is equivalent to successive clusterization of N points. This clusterization can be applied to the above model of the clusterization of stars, since the larger scale structures are regarded as frozen in in the universe.

The present discussion is not based on a dynamical model. Therefore, we cannot determine the suitable value of a. However, if we choose a typical value $\beta=0.4$ of the attractive case (i.e., $a\approx 0.25$) for the original model, then we recover a $k^{-1.2}$ correlation in three dimensions, corresponding to the $r^{-1.8}$ correlation observed in the universe.

VI. SUMMARY

We have investigated the power spectra of quasirandom objects produced by a self-similar space division. Two ways of performing the space division are considered. In one way, space divisions are made so that the boundaries of subspaces remain as far apart from each other as possible (the repulsive case). In the other way, divisions are made so that boundaries of subspaces remain close to each other (attractive case). In both cases the divisions are accompanied by randomness. Since the positions of the created subspaces are frozen in, the inner subspaces keep memories of the positions of outer subspaces. Therefore, the destruction of the initial order is imperfect. This is the reason why in the repulsive case long-range correlations result corresponding to a k^{-d} spectrum in d dimensions. The finite measure of the approximate k^{-d} spectrum (as a function of the randomness parameter a) is due to the continuous nature of the model, and this must be closely related to the conservation law of volume, mass, or other physical quantities. This spectrum can be regarded as one of the prototypes of the $1/f$ noise spectrum. The possibility of detecting the k^{-d} spectrum of the k^d asymptotic shape at small k of the scattering function of a phase-separating system is also discussed. We have applied the attractive case of the present model to the distribution of stars or galaxies in the universe: the correlation $r^{-\alpha}$ with $\alpha\approx 1.8$, which is typically observed in the universe, has been discussed.

The present study provides us with an interesting aspect of the decay process of an ordered object, as well as the ordering processes of a disordered object. It is interesting to find that such process can obey the power law k^{-d} which is not an equilibrium spectrum but is an off-equilibrium spectrum. It would be interesting to investigate the connection between the present study and other fields of physics, such as patterns of precipitation, dislocations in a crystal, discommensurations, and so on.

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[1]W. H. Press, Comments Astrophys. **4**, 103 (1978).
[2]R. F. Voss, and J. Clarke, Nature **258**, 317 (1975).
[3]T. Musha, *Nature of Fluctuations, Blue Backs B-442* (Kodansha, Tokyo, 1980), in Japanese.
[4]B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, New York, 1983).
[5]S. M. Fall, Rev. Mod. Phys. **51**, 21 (1979).
[6]H. Furukawa, Prog. Theor. Phys. **75**, 195 (1986). In this paper

however, the abscissas in Figs. 2 and 3 should be shifted to the right by the amount $\log_{10}[10/2\pi] \approx 0.20$.

[7]H. Furukawa, Prog. Theor. Phys. **73**, 1141 (1985); Phys. Lett. **110A**, 316 (1985).

[8]A. Maeda, M. Naito, and S. Tanaka, J. Phys. Soc. Jpn. **54**, 1912 (1985).

[9]K. Binder, and D. Stauffer, Phys. Rev. Lett. **33**, 1006 (1974); H. Furukawa, *ibid.* **43**, 136 (1979).

[10]H. Furukawa, Phys. Rev. B **33**, 638 (1986).

[11]P. Fratzl, J. L. Lebowitz, J. Marro, and M. H. Kalos, Acta Metall. **31**, 1849 (1983).

[12]J. H. Jeans, *Astronomy and Cosmogony* (Cambridge University Press, Cambridge, 1928).