Quantitative Landau model for bifurcations near a tricritical point in Couette-Taylor flow

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Using a Landau potential $F=\frac{1}{6}\psi^6+\frac{1}{4}g\psi^4+\frac{1}{2}\epsilon\psi^2$, where ψ is a suitable order parameter, one can quantitatively explain symmetry-breaking phenomena in Couette-Taylor flow near a tricritical point. Ideas from bifurcation theory have been used to show that ϵ depends quadratically and g linearly on the aspect ratio and on the angular velocity of the inner cylinder. This agrees well with earlier experimental results in quite a wide area around the tricritical point.

I. INTRODUCTION

Landau's phenomenological model for equilibrium phase transitions has also been used for a long time to quantitatively describe forward bifurcations in nonequilibrium systems.[1] Recently it has been shown to be appropriate, too, in a transition between forward and backward bifurcations, analogous to the tricritical phenomenon.[2] The structure of the coefficients is not, however, predicted in the model.

In this paper recent ideas from bifurcation theory are applied in order to find expressions for the coefficients in a Landau model describing bifurcations in Couette-Taylor flow near a tricritical point. Bifurcation theory[3] classifies multiple bifurcations into normal forms, whose coefficients have a simple structure. These normal forms can be perturbed with unfolding parameters and, as these parameters vary, the character of the bifurcations will change. The coefficients obtained this way agree quantitatively with earlier measurements[2] in a wide area around the tricritical point.

In Couette-Taylor flow between infinitely long coaxial cylinders with the inner one rotating at angular velocity ω there is a spatially ordered multivortex state if ω is greater than a critical value. In this paper the units will be fixed so that this critical value is 1. By restricting the length H of the cylinder so that the aspect ratio $L\equiv H/d$ is near 1 (d is a gap between the cylinders), one can have a symmetric flow pattern with only one vortex pair provided ω is not too small. Then if ω is increased beyond a critical value $\omega_1(L)$, the symmetry of the rolls breaks. One vortex grows at the expense of the other. Also, if ω is fixed and big enough, and one varies L (by changing H), the symmetry of the rolls can change. These bifurcations can happen continuously or abruptly, corresponding to a second-order or first-order phase transition, depending on the precise value of the parameter (ω or L) held fixed. This phenomenon has first been investigated both theoretically and experimentally by Benjamin and Mullin[4] and later by Cliffe,[5] Lücke *et al.*,[6] Schmidt,[7] and Aitta *et al.*[2]

In our work,[2] we found a tricritical point, and we noticed that our data could be fitted very accurately by a Landau model:

$$F=\tfrac{1}{6}k\psi^6+\tfrac{1}{4}g\psi^4-\tfrac{1}{2}\epsilon\psi^2-h\psi\ . \qquad (1)$$

The tricritical point occurs where g and ϵ simultaneously vanish. A small asymmetric term h was needed because the apparatus was not perfect. The order parameter, which measures the amount of symmetry breaking, was

$$\psi=\frac{\int_0^H v_z\,dz}{\int_0^H |v_z|\,dz}\ . \qquad (2)$$

The axial velocity component v_z was measured by laser Doppler velocimetry. The coefficient ϵ was defined as

$$\epsilon=\frac{\omega}{\omega_1}-1\ , \qquad (3)$$

so that it measures linear distance from the bifurcation point.

In this paper I will consider "perfect" bifurcations ($h=0$) which occur when the potential F is symmetric in ψ, and the coefficient k in (1) will be scaled out for convenience. I show that the coefficients ϵ and g may be expressed in terms of L and ω in a new way, such that the Landau model explains the bifurcation phenomena over a much wider range than with the coefficients used earlier. The expression for ϵ fits quantitatively with the experimental results of Ref. 2. ϵ depends quadratically on ω and formula (3) is a linear approximation valid for ω near ω_1. g depends linearly on ω when earlier it was assumed independent of ω. These dependences were first revealed by measuring the time evolution of the order parameter as the system relaxed to a stationary state,[8] and they motivated this work.

II. BIFURCATION PHENOMENA AND A POTENTIAL MODEL

One can explain the bifurcation phenomena near a tricritical point using a simple sixth-order potential

$$F(\psi)=\tfrac{1}{6}\psi^6+\tfrac{1}{4}g\psi^4+\tfrac{1}{2}\epsilon\psi^2\ , \qquad (4)$$

where ψ is the order parameter. The dynamics of this dis-

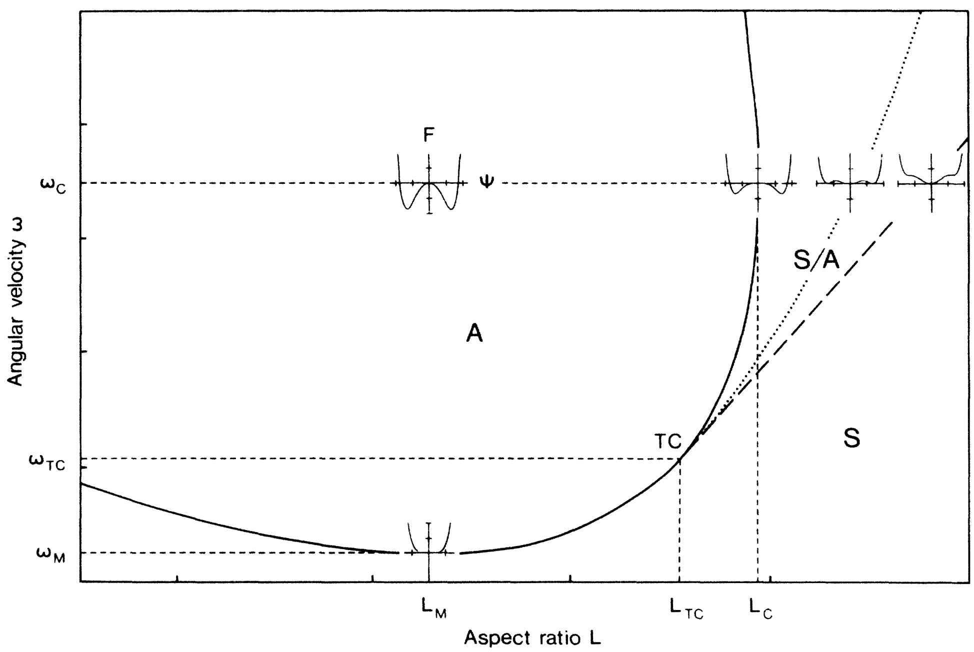

FIG. 1. Phase diagram of the system. S is the symmetric phase, A the asymmetric phase, and S/A the hysteretic phase. TC is the tricritical point. At special places on the phase diagram it is shown how the potential F depends on the order parameter ψ. F has three equal minima on the dotted line corresponding to the first-order transition line.

sipative system is assumed to be described by

$$\tau_0 \frac{d\psi}{dt} = -\frac{dF}{d\psi} , \tag{5}$$

where τ_0 is a characteristic time. In a static state

$$\frac{dF}{d\psi} = \psi^5 + g\psi^3 + \epsilon\psi = 0 . \tag{6}$$

The local minima of F represent stable states and the maxima unstable states. The shape of the potential changes qualitatively at special values of g and ϵ, and where this happens one speaks of a phase transition.

Figure 1 shows the phase diagram for a short Couette-Taylor system, whose features can be described using (4). Three phases exist: S, where the vortex pair is always symmetric; A, where it is always asymmetric; and S/A, where the symmetry/asymmetry depends on the previous history of the system. In other words S/A is a hysteretic phase. In S the potential F has one minimum at $\psi=0$; in A it has two minima, and a maximum at $\psi=0$; and in S/A there are three minima and two maxima. The solid curve is where $\epsilon=0$ and it is where the symmetry-breaking bifurcation takes place. It is referred to as the phase curve. The dashed curve, referred to as the hysteresis branch, is where

$$g^2 = 4\epsilon . \tag{7}$$

The phase curve and the hysteresis branch meet tangentially where $\epsilon = g = 0$. This is the tricritical point (TC).

Figures 2 and 3 are bifurcation diagrams. The solid curves correspond to stable states and dashed curves to unstable ones. Figure 2 shows qualitatively how the order parameter ψ depends on ω with L fixed. Figure 3 shows how it depends on L with ω fixed. The bifurcation parameters λ_ω and λ_L depend linearly on ω and L, respectively. In order to allow for both types of bifurcations any realistic bifurcation parameter should depend on both ω and L. This dependence can be revealed by analyzing qualitative changes in the diagrams piece by piece (as in Ref. 3, pp. 259–260).

In Figs. 2(a)–2(c) the first bifurcation turns from forward to backward. This tricritical behavior is expressed by the equation

$$\psi^5 + a\psi^3 - \lambda_\omega \psi = 0 . \tag{8}$$

It has solutions $\psi=0$ and $\psi^2 = -\frac{1}{2}a \pm (\frac{1}{4}a^2 + \lambda_\omega)^{1/2}$ which gives one pair of real roots if $a > 0$, corresponding to $L < L_{\mathrm{TC}}$, and two pairs if $a \le 0$ corresponding to $L \ge L_{\mathrm{TC}}$. In Figs. 2(c)–2(e) the first and second bifurcations meet and disappear. This is expressed by

$$-\psi^3 + \lambda_\omega^2 \psi + b\psi = 0 , \tag{9}$$

which has solutions $\psi=0$, and also $\psi = \pm(\lambda_\omega^2 + b)^{1/2}$ provided $\lambda_\omega^2 > -b$. This last condition excludes an interval of λ_ω if $b < 0$, corresponding to $L < L_C$. This interval vanishes when $b = 0$ corresponding to $L = L_C$ (the phase curve turns over). For $b > 0$, corresponding to $L > L_C$, nonzero solutions for ψ always exist.

In Figs. 3(a)–3(c), a bifurcation emerges. This is described by

$$\psi^3 + \lambda_L^2 \psi + c\psi = 0 . \tag{10}$$

The solutions are now $\psi=0$, and also $\psi = \pm(-\lambda_L^2 - c)^{1/2}$ provided $\lambda_L^2 < -c$. For $c \ge 0$ corresponding to $\omega \le \omega_M$, the only solution is $\psi=0$. Nonzero solutions for ψ exist in an interval of λ_L when $c<0$ corresponding to $\omega>\omega_M$. In Figs. 3(c)–3(e) the second bifurcation has a tricritical turning from backward to forward. The appropriate equation is

$$\psi^5+d\psi^3+\lambda_L\psi=0 \ . \tag{11}$$

Its solutions are $\psi=0$ and $\psi^2=-\frac{1}{2}d\pm(\frac{1}{4}d^2-\lambda_L)^{1/2}$ which gives one pair of real roots if $d>0$ corresponding to $\omega<\omega_{TC}$ and two pairs if $d\le 0$ corresponding to $\omega\ge\omega_{TC}$.

Since λ_ω and λ_L occur at most quadratically in all these equations, one concludes that the coefficient ϵ in Eqs. (4) and (6) should have quadratic plus linear dependence on L and ω. The phase curve ($\epsilon=0$) in Fig. 1 is then a conic.

It is convenient to define X and Y by

$$X=\frac{L-L_C}{L_C-L_M} \tag{12}$$

and

$$Y=\frac{\omega-\omega_M}{\omega_C-\omega_M} \ . \tag{13}$$

In terms of these rescaled variables the equation of a general conic is

$$pX^2+qXY+rY^2+sX+tY+u=0 \tag{14}$$

and it will be a reasonable simplification to assume that the phase curve is a parabola, which requires that

$$q^2=4pr \ . \tag{15}$$

At the points (L_M,ω_M) and (L_C,ω_C) the phase curve has horizontal (X axis) and vertical (Y axis) tangents; in (X,Y) coordinates these points are $(-1,0)$ and $(0,1)$. The parabola with these tangents is

$$p(X^2+2XY+Y^2+2X-2Y+1)=0 \tag{16}$$

and this is the final equation of the phase curve. The solutions of (16) are

$$Y_{1,2}=(1\pm\sqrt{-X}\,)^2 \tag{17}$$

or alternatively

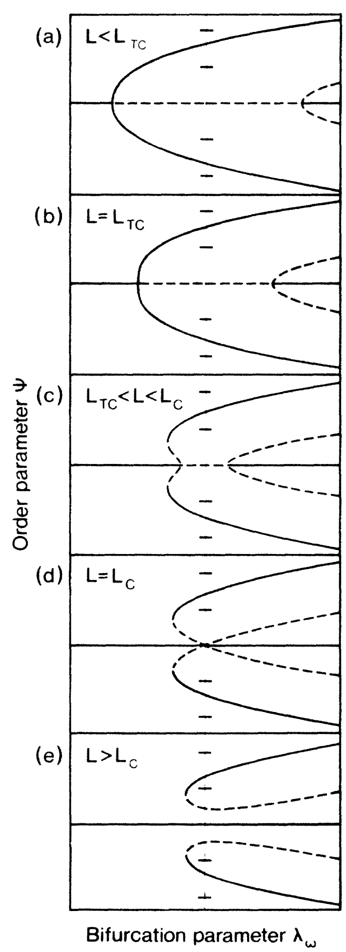

FIG. 2. Order parameter ψ as a function of bifurcation parameter λ_ω. Solid curves mean stable and dashed curves unstable states. The figure can be understood as vertical cuts through the phase diagram at various values of L. L_{TC} and L_C are as in Fig. 1. These bifurcation diagrams were revealed by Benjamin and Mullin (Ref. 4) and later verified by Cliffe (Ref. 5).

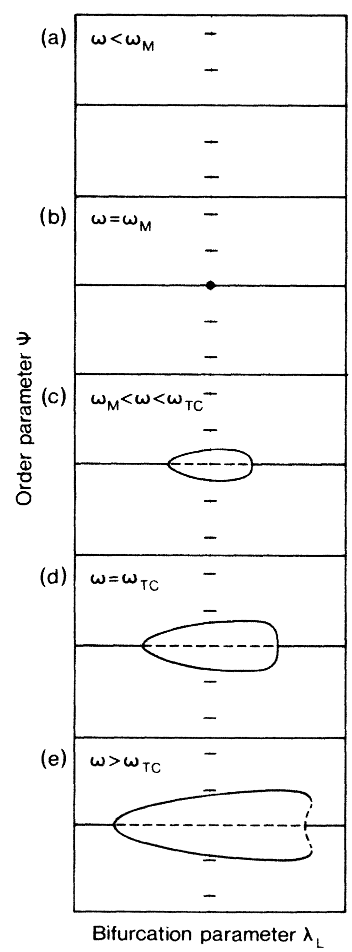

FIG. 3. Order parameter ψ as a function of bifurcation parameter λ_L. The figures can be understood as horizontal cuts through the phase diagram at various values of ω. These bifurcation diagrams have not been explicitly reported before.

$$X_{1,2} = -(1 \pm \sqrt{Y})^2 \ . \tag{18}$$

The expression for ϵ,

$$\epsilon = p(X^2 + Y^2 + 2XY + 2X - 2Y + 1) \tag{19}$$

can be written, using (17), as

$$\epsilon = p(Y - Y_1)(Y - Y_2) \tag{20}$$

and using (13) too, as

$$\epsilon = \frac{p}{(\omega_C - \omega_M)^2}(\omega - \omega_1)(\omega - \omega_2) \ . \tag{21}$$

The angular velocity corresponding to the first bifurcation is ω_1 and to the second one ω_2. Although ϵ has quadratic dependence on ω the earlier formula (3) for ϵ is approximately right for ω close to ω_1.

It will be convenient next to rotate and shift axes by defining

$$\mu = X + Y \tag{22}$$

and

$$\nu = Y - X - \tfrac{1}{2} \ . \tag{23}$$

Then

$$\epsilon = p(\mu^2 - 2\nu) \tag{24}$$

and the phase curve becomes the standard parabola[9]

$$\nu = \tfrac{1}{2}\mu^2 \ . \tag{25}$$

In order to get an expression for g one needs to use the normal form for this bifurcation problem. A normal form is a Taylor series with the minimum number of essential terms to describe a multiple bifurcation. The higher-order terms are not simply truncated but transformed away by a smooth change of coordinates. The nearby simpler bifurcations can be captured by adding unfolding terms and the universal unfolding has the minimum number of such terms. A detailed classification of normal forms with their universal unfoldings is given in Ref. 3. However, there the bifurcations are considered in one dimension; here the bifurcations occur in the (L,ω) plane. The relevant normal form for our tricritical bifurcation is

$$H_0 = \psi^5 + 2m\lambda\psi^3 + \lambda^2\psi \tag{26}$$

and its universal unfolding is

$$H = \psi^5 + 2m\lambda\psi^3 + \lambda^2\psi + \alpha\psi + \beta\psi^3 \ . \tag{27}$$

m must be negative because the bifurcations are basically forward. α and β are the unfolding parameters and the bifurcations are assumed to be perfect. $H=0$ is equivalent to equation (6) if one identifies

$$g = 2m\lambda + \beta \tag{28}$$

and

$$\epsilon = \lambda^2 + \alpha \ . \tag{29}$$

Notice that in this parametrization ϵ has no term linear in λ.

In principle, one may be interested in bifurcations along any line in the (μ,ν) plane although experimentally it is easiest to work along lines where L is constant or ω is constant. On the line

$$\nu = l\mu + c \tag{30}$$

one has

$$\epsilon = p(\mu^2 - 2l\mu - 2c) \ , \tag{31}$$

which can be expressed in the form (29) by choosing

$$\lambda = \sqrt{p}(\mu - l) \tag{32}$$

and

$$\alpha = -p(l^2 + 2c) \ . \tag{33}$$

By using formula (32) for λ in (28) one obtains

$$g = 2m\sqrt{p}(\mu - l) + \beta \ . \tag{34}$$

This means that g has linear dependence on μ. Generally this dependence can be presented as

$$g = R\mu + S \ , \tag{35}$$

where R and S are fixed constants independent of the line (30). Then

$$m = \frac{R}{2\sqrt{p}} \tag{36}$$

and

$$\beta = Rl + S \ . \tag{37}$$

The constants R, S, and p are as yet undetermined. The unfolding parameters α and β have a clear significance here. They specify a line in the (μ,ν) plane along which one can study the bifurcation structure. In particular, β is proportional to the slope l of the line and for given β, α determines the intercept c on the ν axis. When α is 0 the line is tangent to the phase curve, and when β is 0 too, this tangent passes through the tricritical point.

The tricritical point is where $\epsilon = g = 0$, which gives $\mu_{\mathrm{TC}} = -(S/R)$, $\nu_{\mathrm{TC}} = \frac{1}{2}(S^2/R^2)$. The tricritical point has a symmetric position on the parabola if $S=0$, and if $S>0$, it is shifted up and if $S<0$, down, since R is negative.

Now one can determine the nature of the hysteresis branch. Its equation $g^2 = 4\epsilon$ becomes in terms of μ and ν

$$(R^2 - 4p)\mu^2 + 2RS\mu + S^2 + 8p\nu = 0 \ . \tag{38}$$

This is a straight line if $R = -2\sqrt{p}$, that is, if $m = -1$. m describes the curvature of the hysteresis branch. If $m < -1$ the branch curves down and if $-1 < m < 0$, up.

The potential can now be written in the form

$$F = \tfrac{1}{6}\psi^6 + \tfrac{1}{4}(R\mu + S)\psi^4 + \tfrac{1}{2}p(\mu^2 - 2\nu)\psi^2 \ , \tag{39}$$

so the order parameter in a static state is either $\psi = 0$ or

$$\psi^2 = -\tfrac{1}{2}(R\mu + S) \pm [(\tfrac{1}{4}R^2 - p)\mu^2 + \tfrac{1}{2}RS\mu + \tfrac{1}{4}S^2 + 2p\nu]^{1/2} \ . \tag{40}$$

Two special situations are worth mentioning. One is the bifurcation structure in Ref. 2 which has been measured in the Y direction. Then $l=1$ and

$$\lambda=\sqrt{p}\,(X+Y-1)\,, \tag{41}$$

$$\alpha=4pX\,, \tag{42}$$

and

$$\beta=R+S\,. \tag{43}$$

The other is the ideal situation, where the hysteresis branch is straight ($m=-1$) and the tricritical point lies on the symmetry axis of the phase curve. Then $X_{TC}=-\frac{1}{4}$, $Y_{TC}=\frac{1}{4}$, or alternatively $\mu_{TC}=\nu_{TC}=0$, which gives $S=0$. Then the hysteresis branch is $\nu=0$. The equation for the order parameter also simplifies, and it is either $\psi=0$ or

$$\psi^2=\sqrt{p}\,[\mu\pm(2\nu)^{1/2}]\,. \tag{44}$$

One can calculate the equation for the first-order transition line and in this ideal situation it is $\nu=\frac{1}{8}\mu^2$ with $\mu\geq 0$.

III. EXPERIMENTAL EVIDENCE

This work was motivated by the preliminary analysis of some measurements,[8] which show how the order parameter varies with time when the system relaxes towards its stationary state. This analysis revealed that ϵ depends quadratically [Fig. 4(a)] and g linearly [Fig. 4(b)], on ω, when L has a fixed value. Qualitatively the agreement of the model and experiment is obvious, but quantitative conclusions concerning the values of parameters R, S, and p must be deferred until a proper analysis has been done.

The model can be compared in other ways to earlier experimental results. The idealized model with $m=-1$ gives just those bifurcations for which one has evidence directly (Fig. 2) or indirectly (Fig. 3). This means that for our experimental setup,[2] m must be very near -1, because a strong curvature of the hysteresis branch will give new branches in the bifurcation diagrams (compare Fig. VI. 7.4 in Ref. 3). Also the measurements[7,8] show that the hysteresis branch is almost straight, at least within the experimental accuracy, and it is not obvious whether m is greater or less than -1.

Our measurements[2] were done only in the neighborhood of the tricritical point, and we do not have systematic measurements of the order parameter at the second bifurcation. This gives some inaccuracy to the determination of the scaling parameters L_M, ω_M, L_C, and ω_C. The fitting procedure we used gave for each L a critical angular velocity ω_1 corresponding to the perfect bifurcation. These (L,ω_1) values have now been converted to the (μ,ν) plane by first making a reasonable guess for the scaling parameters. These converted data points have been fitted (using a least-squares method) to the parabola

$$\nu=A+B\mu+C\mu^2\,, \tag{45}$$

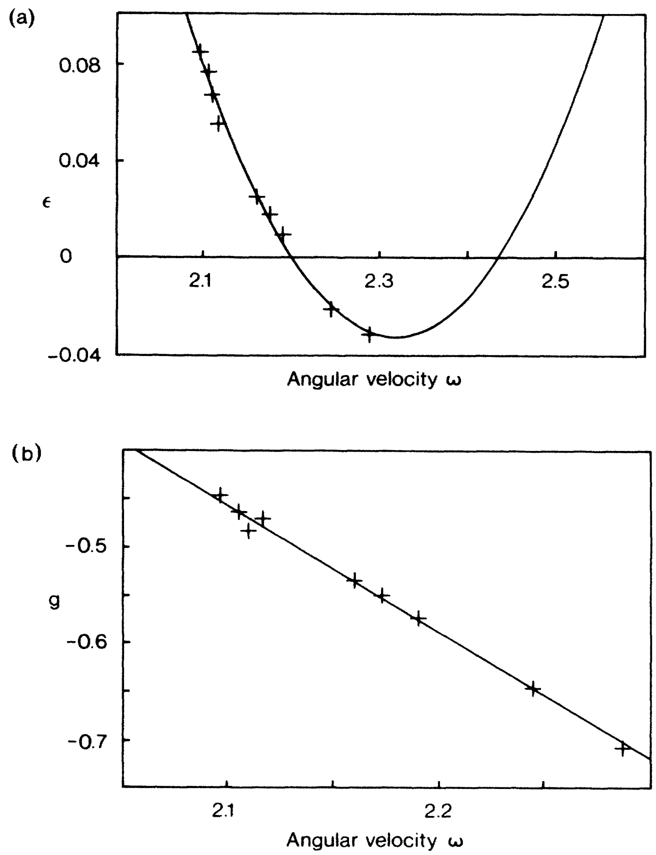

FIG. 4. The coefficients ϵ and g obtained from preliminary analysis of how the system relaxes toward equilibrium at various values of ω and a fixed L, here at the hysteresis area.

where, in the model, $A=0$, $B=0$, and $C=\frac{1}{2}$. Then the scaling parameters have been readjusted until the least-squares fit values for A, B, and C are nearest the model with the smallest standard deviation. The result of this final fit is shown in Fig. 5 as the solid curve and the data points are marked by + symbols. It gives $A=0.00170$, $B=-0.00173$, $C=0.50043$, and $L_M=1.1275$, $\omega_M=1.8499$, $L_C=1.2956$, $\omega_C=2.4979$. Note that the scaling parameters are not really known with this accuracy, because small changes of the parameters do not affect the fit much.

The hysteresis branch in the ideal situation is the μ axis. An approximation to the experimental hysteresis branch has been obtained in the following way. The data analysis in Ref. 2 gave values for g and k as functions of L alone, with ϵ depending on ω as in Eq. (3). k was not scaled out. The equation for the hysteresis branch would therefore be $g^2=-4k\epsilon$, and its solution is $\omega=\omega_1\{1-g^2(L)/[4k(L)]\}$. Using this formula, together with the data, one finds the set of points on the hysteresis branch marked by circles in Fig. 5. The same scaling parameters have been used as earlier. The points have been fitted to the line $\nu=D$ giving $D=0.00077$ which is close to zero.

One should note that the experimental points and the solid curves do not come from the same model, and the

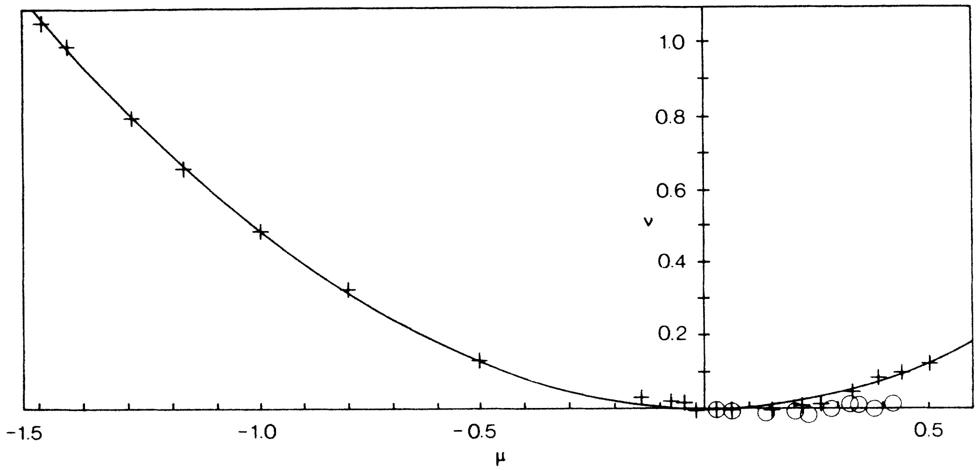

FIG. 5. Quadratic fit to the phase curve data (Ref. 2). Circles represent the hysteresis branch.

old model cannot be very accurate when $\omega \neq \omega_1$ especially in the hysteresis area. However, one can see that the new model for the potential agrees well with the experimental results over the whole range, although the agreement is not perfect near the bottom of the parabola, where the tricritical point is.

The order parameter data in Ref. 2 can, in principle, be refitted by using this new model, g should be expressed as

$$g = P\omega + Q \;, \tag{46}$$

where

$$P = \frac{R}{\omega_C - \omega_M} \tag{47}$$

and

$$Q = R\frac{L - L_C}{L_C - L_M} - R\frac{\omega_M}{\omega_C - \omega_M} + S \;. \tag{48}$$

Equation (21) lets one write

$$\epsilon = T\omega^2 + U\omega + V \;, \tag{49}$$

where

$$T = \frac{p}{(\omega_C - \omega_M)^2} \;, \tag{50}$$

$$U = -T(\omega_1 + \omega_2) \;, \tag{51}$$

$$V = T\omega_1\omega_2 \;. \tag{52}$$

To make the fit, the lowest-order asymmetric term h should be included to allow for small imperfections in the experimental apparatus. Then fitting to

$$\psi^5 + (P\omega + Q)\psi^3 + (T\omega^2 + U\omega + V)\psi + h = 0 \tag{53}$$

will give the constants P, Q, T, U, V, and h. P and T should not depend on L if the model and the order parameter are appropriate. The curvature of the hysteresis branch is given by

$$m = \frac{P}{2\sqrt{T}} \;. \tag{54}$$

ω_1 and ω_2 could be calculated from the coefficients T, U, and V. Knowing both of them would let one construct the phase curve in a much wider area than earlier. It would then be possible to find the scaling parameters L_M, ω_M, L_C, and ω_C, more accurately, using the least-squares fit (45) discussed earlier. With the scaling parameters known, one could finally obtain the parameters R, S, and p.

With this model one could also study quantitatively the dynamics (5) near the tricritical point. By measuring the time evolution of the order parameter, when the system is allowed to relax to different final states, one could find the parameters ω_1, ω_2, and m, and, if one knows the scaling parameters, also R, S, and p. This would give an independent test of the model.

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[1]See, for instance, J. P. Gollub and H. Freilich, Phys. Fluids **19**, 618 (1976); or J. Westreid, Y. Pomeau, M. Duboid, C. Normand, and P. Berge, J. Phys. (Paris) **39**, 725 (1978).

[2]A. Aitta, G. Ahlers, and D. S. Cannell, Phys. Rev. Lett. **54**, 673 (1985).

[3]M. Golubitsky, and D. G. Schaeffer, *Singularities and Groups in Bifurcation Theory, Vol. 1* (Springer-Verlag, New York, 1985), Figs. VI. 3.1—3.3 and VI. 7.4, and their analysis have

been especially useful.

[4]T. B. Benjamin and T. Mullin, Proc. R. Soc. London, Ser. A **377**, 221 (1981).

[5]K. A. Cliffe, J. Fluid Mech. **135**, 219 (1983), and private communications.

[6]M. Lücke, M. Mihelcic, K. Wingerath, and G. Pfister, J. Fluid Mech. **140**, 343 (1984).

[7]H. Schmidt, Diplomarbeit, University of Kiel, West Germany, 1983 (unpublished).

[8]A. Aitta, G. Ahlers, and D. S. Cannell (unpublished).

[9]This can be compared to the results concerning tricritical phase transitions in ^3He-^4He mixtures. E. Riedel presents in Phys. Rev. Lett. **28**, 675 (1972) the second-order transition line as $\mu_1 = r_c \mu_2^{1/\phi_t}$ where r_c is a constant and μ_1 and μ_2 "scaling fields" which are defined to be normal and tangential to the critical line at the tricritical point. It is conjectured that the appropriate scaling index is $\phi_t \approx \frac{1}{2}$.