Local-field rate equations for coupled optical resonators

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We present a general formalism for obtaining a coupled set of first-order rate equations for the optical field at a finite number of points in an optical resonator. We compare these local-field equations to an expansion in modes of the individual resonators and point out a shortcoming of the latter technique that has not been recognized before. We consider the special case of a set of two coupled resonators and derive the coupled rate equations for the fields in each of the two cavities with coupling coefficients. The resulting formulas are both simpler than those derived from coupled-mode theory and more accurate in the sense that they give a steady-state response which agrees with that calculated from the composite cavity modes. We show how the use of a single local-field rate equation can carry the same information as a full set of multimode rate equations in some cases, in particular, that of partial reflection back into a laser from a distant source.

I. INTRODUCTION

The springboard for the analysis of almost any laser's dynamics—modulation response, noise properties, or even a simple stability analysis—is the rate equation describing the optical field in terms of material parameters and other dynamical quantities (e.g., gain). While the ultimate arbiter of the field's behavior must be Maxwell's equations, the set of second-order partial-different wave equations that follows from Maxwell's does not lend itself to a simple solution in the face of fluctuating coefficients in the equation. Fortunately, in almost all cases, a sufficient description is provided by a set of first-order ordinary differential equations in which the field is characterized by a small number of variables, e.g., the spatial average of the field amplitude over different regions of the resonator. In fact, for many systems, even complicated sets of coupled resonators, the field dependence can be adequately described by a single variable, the amplitude of the lasing mode of the composite cavity.[1] Such a formalism has been used to analyze the dynamic response and noise properties of multielement lasers.[1,2] With small modifications, it can be extended to produce a set of multimode rate equations, expanding in terms of the modes of the composite cavity.[3]

An alternate description of a multielement optical resonator is produced by writing rate equations for several variables which somehow represent the field inside of each of the coupled elements.[4–7] This second approach conveys more of the flavor of interactions among individual lasers while the former approach takes a more global view. The resulting set of coupled rate equations is intuitively appealing—each field satisfies the rate equation for its own individual cavity, modified by the addition of a coupling term linking it to a neighboring cavity. In most treatments, the dynamical variables are chosen to be the amplitudes of the individual cavity modes (henceforth, the ICM's) which make up the lasing composite cavity mode (CCM). Such analyses have qualitatively predicted the properties of weakly coupled systems. However, they suffer from two flaws which severely limit their usefulness.

The first is that the CCM (which is the proper description for steady-state operation) is generally made up of more than one ICM from each cavity. In the case of weakly coupled systems, it consists of predominantly one ICM from each resonator, but the stronger the coupling between cavities, the stronger the need to include multiple ICM's from each cavity to accurately model the CCM. Consequently, any description of such a CCM in terms of two ICM's is going to be incomplete. There are treatments which do take into account the coupling among other longitudinal modes,[4,5] and such treatments can be expected to give more accurate results than those that do not. The second flaw, however, afflicts *all* treatments which attempt to expand in terms of individual cavity modes.

The second problem involves the following point: *The complete set of modes of the individual cavities do not constitute a complete set when those cavities are coupled together.* In fact, they *never* constitute a sufficient description of the composite cavity model. This is a subtle point that bears a bit more explaining. When we solve for the modes of an isolated resonator, we have implicitly made the stipulation that all fields outside the resonator (or within some infinitesimal distance from the mirrors) are outgoing. This is a reasonable assumption, since the only source of light in an isolated laser lies inside the resonator, and any light encountered outside the resonator is indeed outgoing. The modes thus found lie in the *discrete* spectrum of the operator corresponding to wave propagation in the resonator. They form a complete set *only* for fields which are outgoing.

When we couple two resonators, however, there arise incoming fields in each cavity. In fact, coupling cannot take place without such fields. The modes of an isolated resonator cannot adequately describe such a field. A complete description of the field inside each cavity must include the modes corresponding to the *continuous* spectrum, corresponding to fields incident upon the cavity from the outside. Such a description must include a sum over the discrete modes, plus an integral over the continuous modes. However, a representation of this sort which includes the intermodal coupling leads to a set of integrodifferential equations for the modal amplitudes which is more difficult to solve than the original problem. The conventional approach is to drop the continuous part of the spectrum and to hope that what is left is sufficiently complete to describe the field.

What are the results of dropping modes from a coupled-mode theory? The time evolution of each mode is determined by the scattering of energy from one mode into another. If we do not include a mode in a coupled-mode theory, then while the rate equations remaining will account for the energy scattered *out* of the remaining modes (and into the "lost" modes), there is no mechanism for the scattering of energy from the "lost" modes back into the "kept" modes. In effect, then, the lost modes become "black holes," absorbing energy from the system and never returning it. Consequently, threshold gains calculated from such a theory are going to be overestimated, since they will take into account these spurious energy sinks. The amount of the overestimation depends upon the relative fraction of the CCM described by modes of the continuous spectrum. Unfortunately, for strongly coupled cavities, that amount is substantial. While this problem can be ameliorated somewhat by imposing fictitious boundary conditions upon the ICM's to force them to more closely match the CCM's (as is discussed in Ref. 4), this method requires one to *a priori* possess fairly detailed knowledge of the CCM's.

It appears, then, that coupled-mode theory cannot easily provide quantitative information about the behavior of fields in composite cavity resonators. However, one of the attributes of coupled-mode theory—a description of the field by a few variables which somehow characterize the individual cavities—is still desirable. As we will show, we can produce such a description by choosing as the dynamical variables the amplitude of one of the traveling waves in each cavity *at some fixed point* within the cavity rather than the amplitude of an individual cavity mode (hence the moniker "local-field" rate equations). This choice, plus a little dc analysis, will yield a set of rate equations and *analytic* expressions for all coupling coefficients with no long summations or involved overlap integrals. Furthermore, because it implicitly comes from the *composite* cavity modes, it encounters none of the difficulties of standard coupled-mode theory. The method is powerful and general, yet conceptually quite simple. It relies on the approximation that the optical field adiabatically follows the characteristics of the resonator, an approximation that has been widely made and recently justified.[1,3] In Sec. II we will present the derivation in abstract form. In Sec. III we apply it to the most common coupled system, a two-section longitudinally coupled-cavity laser and derive the coupling coefficients. In Sec. IV we show how a modification of the technique can incorporate multimode behavior in a single rate equation and justify the delay-differential equation composed by Lang and Kobayashi[8] to describe optical feedback. In Sec. V we summarize the analysis.

II. GENERAL THEORY

In a previous work,[1] we found that for single-mode lasers a suitable first-order rate equation is provided by first finding the steady-state eigenvalue equation for complex frequency, and then replacing the frequency $j\Omega$ by the differential operator d/dt. Elsewhere, we formally justified this approximation for both single-mode and multimode systems and calculated the lowest-order correction terms.[3] The assumption underlying this approximation can be stated in several equivalent ways. A rather formal statement of the approximation is that "linear operators commute with all variables except the electric field, which they act upon." Perhaps a more meaningful expression is that the only significant time derivatives are those of the electric field; time derivatives of other variables (for example, a fluctuating index of refraction) will be sufficiently small that they can be dropped. We shall refer to this approximation throughout the paper as the "adiabatic approximation," because the electric field inside the resonator adiabatically follows the instantaneous characteristics of the resonator. It is an assumption widely made in the literature without comment, but it is implicitly assumed in any first-order rate equation.

We will attempt to characterize the field by a set of amplitudes $\{E_i(\vec{x}_i)\}$ of traveling waves at fixed points $\vec{x}_i$ within the composite cavity. As a practical matter, one would choose a single fixed point in each of the coupled cavities. In steady-state, it is simple to assume complex exponential behavior of the form $\exp(j\Omega t)$ and to then find a set of linear relations linking the field amplitudes $\{E_i\}$. The coefficients are typically functions of Ω. The process is well codified for several geometries;[9,10] a self-consistency requirement yields a set of linear equations of the form

$$\sum_j \Gamma_{ij}(\Omega)E_j = 0 , \tag{1}$$

where each of the Γ_{ij} depends upon the complex frequency Ω. We shall adopt a matrix notation, where each of the fields E_i is a component of the vector $\mathbf{E}$, and Γ_{ij} is a component of the matrix $\underline{\Gamma}$. The steady-state eigenvalue equation (1), written in matrix notation, becomes

$$\underline{\Gamma}(\Omega)\mathbf{E} = \mathbf{0} , \tag{2}$$

where $\mathbf{0}$ is the zero vector. Equation (2) has nontrivial solutions for $\mathbf{E}$ only if $\underline{\Gamma}$ is singular, and this requirement yields the lasing frequencies Ω.

We should like to find a set of rate equations of the form

$$\frac{d}{dt}\mathbf{E} = \underline{T}\mathbf{E} , \tag{3}$$

so our goal is to establish a correspondence between Eqs. (3) and (2). Let us begin by working backwards. If we want to solve Eq. (3), we would assume exponential variation in $\mathbf{E}$,

$$\mathbf{E} \rightarrow \mathbf{E}e^{j\Omega T} , \tag{4}$$

in which case Eq. (3) would become

$$j\Omega\mathbf{E}=\underline{T}\mathbf{E}\ . \tag{5}$$

We know from Ref. 1 that Eq. (2) holds adiabatically for small fluctuations in the steady-state solutions, provided that we replace $j\Omega$ by the differential operator. If we define $\Gamma(\Omega)\equiv\det\underline{\Gamma}(\Omega)$, then the nontrivial steady-state solutions to (2) are the eigenvectors corresponding to the roots of

$$\Gamma(\Omega)=0\ . \tag{6}$$

Let us linearize Eq. (2) about a value Ω_0 which is near a root of (6),

$$[\underline{\Gamma}(\Omega_0)+(\Omega-\Omega_0)\underline{\Gamma}_{\Omega}(\Omega_0)]\mathbf{E}=\mathbf{0}\ , \tag{7}$$

where $\underline{\Gamma}_{\Omega}$ is the derivative of $\underline{\Gamma}$ with respect to Ω. Then, we can multiply by $-j\underline{\Gamma}_{\Omega}^{-1}$ to get

$$j\Omega\mathbf{E}=j\Omega_0\mathbf{E}-j\underline{\Gamma}_{\Omega}^{-1}(\Omega_0)\underline{\Gamma}(\Omega_0)\mathbf{E}\ . \tag{8}$$

Now, if we replace $j\Omega$ by d/dt, we have

$$\frac{d}{dt}\mathbf{E}=j[\underline{I}\Omega_0-\underline{\Gamma}_{\Omega}^{-1}(\Omega_0)\underline{\Gamma}(\Omega_0)]\mathbf{E}\ , \tag{9}$$

which is in the form of Eq. (3), with

$$\underline{T}=[\underline{I}\Omega_0-\underline{\Gamma}_{\Omega}^{-1}(\Omega_0)\underline{\Gamma}(\Omega_0)]\ .$$

We can remove the "fast" (optical) oscillation by taking $\mathbf{E}=\mathbf{A}\exp(j\Omega_0 t)$, so that the components of the vector $\mathbf{A}$ are the amplitudes of a field oscillating at frequency Ω_0. This assumption is convenient because if Ω_0 is an exact root of the dispersion equation (6) and there is no time variation in $\underline{\Gamma}$, then $\mathbf{A}$ is stationary. Under this definition, the desired rate equation for the amplitudes of a field oscillating at frequency Ω_0 is

$$\frac{d}{dt}\mathbf{A}=j[-\underline{\Gamma}_{\Omega}^{-1}(\Omega_0)\underline{\Gamma}(\Omega_0)]\mathbf{A}\ . \tag{10}$$

Equation (10) not only gives us the desired rate equations and expressions for all the coupling coefficients, it also tells us just how accurate the entire rate equation (first-order differential equation) approximation really is. Here, the factor $j\Omega$ corresponds to the differential operator d/dt. For frequency ranges in which the linearization of Eq. (7) is a good match to $\underline{\Gamma}$, the approximation is good; otherwise, it is not. However, even the simplest linearization is going to be a good approximation for variations in $\Omega-\Omega_0$ over some fraction, say, $\frac{1}{10}$, of a longitudinal mode spacing. For almost all lasers, that frequency is well beyond the timescale of fluctuations of interest. Note that we do not actually have to possess an exact root of (6); we simply have to be within a domain of the root where the linearization provides a good match to $\underline{\Gamma}$.

It can be shown that $\underline{\Gamma}$ can always be chosen such that $\Gamma(\Omega)$ possesses no finite singularities. Therefore, a Taylor-series expansion of Γ has an infinite radius of convergence in the Ω plane. Consequently, we can take a Taylor series at *any* point Ω_i and by replacing each factor of $j\Omega$ by the operator d/dt, we can construct a higher-order rate equation of arbitrarily high accuracy over an arbitrarily large frequency domain, all within the adiabatic approximation. As a practical matter, the equation is simplest and most useful if we linearize about one of the roots of Γ that corresponds to a low-threshold CCM, since the first term of the series is all that is needed. However, for certain situations, e.g., distant optical feedback, a slight variation on this technique can give a close fit to Γ over a number of longitudinal mode spacings, as we will see in Sec. IV. For now, however, we will take an explicit example of two longitudinally coupled resonators and produce the rate equations for the optical field.

III. TWO-SECTION COUPLED CAVITY

As an example, to illustrate the simplicity of this technique, we shall derive local-field rate equations for a two-section longitudinally coupled cavity laser, illustrated in Fig. 1. It consists of two sections of lengths L_1 and L_2 terminated on the outside by mirrors with amplitude reflectivities r_0 and r_3, respectively. They are separated by a gap, which is characterized by its transmission and reflection coefficients t_{12}, t_{21}, r_{11}, and r_{22}, as illustrated. We shall derive rate equations for the amplitudes of the traveling wave fields in each cavity that are traveling toward the gap, as measured at the gap (E_1 and E_2 in the figure). We denote the waves propagating away from the gap by E_1' and E_2'.

For an optical field at frequency Ω, we can write the relations linking the fields by inspection.[9] They are

$$E_1=r_0e^{\varphi_1}E_1'\ , \tag{11a}$$

$$E_1'=r_{11}E_1+t_{21}E_2\ , \tag{11b}$$

$$E_2=r_3e^{\varphi_2}E_2'\ , \tag{11c}$$

$$E_2'=r_{22}E_2+t_{12}E_1\ , \tag{11d}$$

where $\varphi_1\equiv-2j\Omega\mu_1L_1/c$, $\varphi_2\equiv-2j\Omega\mu_2L_2/c$, and μ_1 and μ_2 are the (complex) indices of refraction in each cavity. Elimination of E_1' and E_2' yields

$$E_1=\frac{e^{-\varphi_1}}{r_0r_{11}}E_1+\frac{t_{21}}{r_{11}}E_2\ , \tag{12a}$$

$$E_2=\frac{e^{-\varphi_2}}{r_3r_{22}}E_2+\frac{t_{12}}{r_{22}}E_1\ . \tag{12b}$$

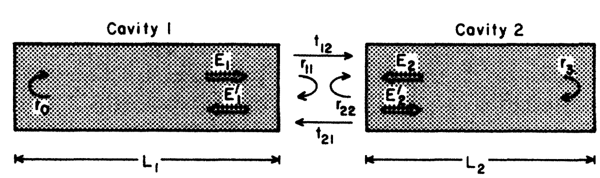

FIG. 1. Schematic of a two-section coupled-cavity laser. The gap is characterized by reflectivities r_{11} and r_{22} and transmissivities t_{21} and t_{22}. The reflectivities at the ends are r_0 and r_3. The lengths are L_1 and L_2, and the (complex) indices of refraction (incorporating gain or loss) are μ_1 and μ_2. All fields are measured at the gap in each laser cavity. Primed fields are the amplitudes of the waves traveling away from the gap; unprimed fields are the amplitudes of the waves traveling toward the gap.

Consequently, we can write the matrix $\underline{\Gamma}(\Omega)$ as

$$\underline{\Gamma}(\Omega)=\begin{bmatrix} \dfrac{e^{-\varphi_1}}{r_0 r_{11}}-1 & \dfrac{t_{21}}{r_{11}} \\ \dfrac{t_{12}}{r_{22}} & \dfrac{e^{-\varphi_2}}{r_3 r_{22}}-1 \end{bmatrix} . \tag{13}$$

For most cases of interest, the gap is short enough that the transmission and reflection coefficients of the gap are independent of frequency over the gain linewidth of the lasing medium. Consequently, we can drop their derivatives. The derivative of $\underline{\Gamma}$ becomes

$$\underline{\Gamma}_{\Omega}=\begin{bmatrix} \dfrac{2j\mu_1 L_1}{c}\dfrac{e^{-\varphi_1}}{r_0 r_{11}} & 0 \\ 0 & \dfrac{2j\mu_2 L_2}{c}\dfrac{e^{-\varphi_2}}{r_3 r_{22}} \end{bmatrix} . \tag{14}$$

Since $\underline{\Gamma}_{\Omega}$ is diagonal, inverting it is trivial. The matrix that enters the first-order rate equation (10), $-\underline{\Gamma}_{\Omega}^{-1}\underline{\Gamma}$, is given by

$$\begin{bmatrix} \dfrac{c}{2j\mu_1 L_1}(r_0 r_{11}e^{\varphi_1}-1) & \dfrac{-c}{2j\mu_1 L_1}\dfrac{t_{21}}{r_{11}}r_0 r_{11}e^{\varphi_1} \\ \dfrac{-c}{2j\mu_2 L_2}\dfrac{t_{12}}{r_{22}}r_3 r_{22}e^{\varphi_2} & \dfrac{c}{2j\mu_2 L_2}(r_3 r_{21}e^{\varphi_2}-1) \end{bmatrix} . \tag{15}$$

If we write the rate-equation system for amplitudes of a field oscillating at frequency Ω_0 [i.e., the analog of Eq. (10)] as

$$\frac{dE_1}{dt}=j\omega_1 E_1+j\kappa_{21}E_2 , \tag{16a}$$

$$\frac{dE_2}{dt}=j\omega_2 E_2+j\kappa_{12}E_1 , \tag{16b}$$

then we have

$$\omega_1=\frac{c}{2j\mu_1 L_1}(r_0 r_{11}e^{\varphi_1}-1) , \tag{17a}$$

$$\omega_2=\frac{c}{2j\mu_2 L_2}(r_3 r_{22}e^{\varphi_2}-1) , \tag{17b}$$

$$\kappa_{21}=\frac{-c}{2j\mu_1 L_1}t_{21}(r_0 e^{\varphi_1}) , \tag{17c}$$

$$\kappa_{12}=\frac{-c}{2j\mu_2 L_2}t_{12}(r_3 e^{\varphi_2}) , \tag{17d}$$

and the eigenvalue equation defining the lasing frequency, $\Gamma(\Omega)=0$, is

$$\left[\frac{e^{-\varphi_1}}{r_0 r_{11}}-1\right]\left[\frac{e^{-\varphi_2}}{r_3 r_{22}}-1\right]=\frac{t_{21}t_{12}}{r_{11}r_{22}} . \tag{18}$$

The last term on the right-hand side is significant; we will give it a special designation,

$$K\equiv\frac{t_{21}t_{12}}{r_{11}r_{22}} . \tag{19}$$

It is a well-known result of microwave theory that for a lossless two-port network characterized by transmission and reflection coefficients t_{21}, t_{12}, r_{11}, and r_{22}, K is always negative and real.

Let us compare Eqs. (17) for the coupling coefficients of the local fields with the coupling coefficients derived from coupled-mode theory (not including the continuous spectrum):[4]

$$\kappa_{\nu\sigma}=-\sum_{\mu}\left[(\Omega_{1\nu}-\omega_\mu)\int_{V_1}\mu_1^2 e_{1\nu}e_{c\mu}dV \times\int_{V_2}\mu_2^2 e_{2\sigma}e_{c\mu}dV\right] , \tag{20}$$

where $\Omega_{1\nu}$ is the resonant frequency of the νth ICM, ω_μ is the resonant frequency of the μth CCM, μ_1 is the index of refraction used to define the ICM in cavity 1, μ_c is the actual index of refraction seen by the CCM, $e_{1\nu}$ is the field pattern of the νth ICM, and $e_{c\mu}$ is the field pattern of the μth CCM. The advantage of local-field equations begins to appear. In coupled-mode theory, one must solve for the complete field patterns of both the ICM's and CCM's, as well as the lasing eigenfrequencies ω_μ, perform many overlap integrals between the different modes, and finally sum over all of the composite cavity modes. As many as 400 terms in the summation[4] may be necessary before the expression (20) converges. On the other hand, one need only solve for a root of the eigenvalue equation (and in fact, only get *close* to a root) to use local-field equations.

By ignoring the presence of self-coupling coefficients, Marcuse[5] has heuristically calculated cross-coupling coefficients based on considerations of power flow. The cross-coupling coefficients for a given system can vary, depending upon how one chooses to normalize the fields (the effect of a change in field normalization is to multiply one and divide the other cross-coupling coefficient by the same constant); nevertheless, there are irreconcilable differences between Eqs. (17c) and (17d) and the heuristic formulas. The latter are lacking the final parenthetical expression in each of (17c) and (17d).

As a comparison of the local-field rate-equation-derived coefficients, the heuristic formulas, and numerical results based on coupled-mode theory, we consider the particular case of two identical cavities. We take $\mu_1 L_1=\mu_2 L_2\equiv\mu L$, $\varphi_1=\varphi_2\equiv\varphi$, and $r_0=r_3=1$. The reference planes around the gap can be chosen so that $r_{11}=-r_{22}\equiv r$, and r is positive real. In this case, the secular equation can be solved analytically,

$$(e^{-\varphi}-r)(e^{-\varphi}+r)=t_{21}t_{12}\rightarrow e^{-\varphi}(r^2+t_{21}t_{12})^{1/2} . \tag{21}$$

The intercavity coupling coefficients become

$$\begin{aligned}\kappa_{21}&=\frac{-c}{2j\mu L}t_{21}\frac{1}{(r^2+t_{21}t_{12})^{1/2}} ,\\ \kappa_{12}&=\frac{-c}{2j\mu L}t_{12}\frac{1}{(r^2+t_{21}t_{12})^{1/2}} .\end{aligned} \tag{22}$$

Allowing for differences in field normalizations, these are the same as Marcuse's heuristic formulas, with the addition of a correction factor $1/(r^2+t_{21}t_{12})^{1/2}$. As we mentioned, for a lossless gap, $K\equiv t_{12}t_{21}/r_{11}r_{22}$ must be negative real. Thus, our definitions of r_{11} and r_{22} require that the product $t_{12}t_{21}$ be positive real; since $|r|^2+|t|^2=1$ for a lossless gap, the correction factor is simply 1, and the heuristic formulas are correct.

However, for lossy gaps, numerical calculations show that the heuristic formulas fall short by a factor "close to one-half."[5] For a lossy gap, $|r|^2+|t|^2<1$; consequently, the correction factor is going to increase somewhat. How large will it get? As the gap losses increase, the transmission will become negligibly small compared to the reflectivity, which will approach the dielectric reflectivity of one surface of the gap. In this limit, then, the correction term will become

$$\frac{1}{(r^2+t_{21}t_{12})^{1/2}} \to \frac{1}{r} \tag{23}$$

and r simply approaches the dielectric reflectivity of the first surface of the gap. For a GaAs air interface, that reflectivity is $r=0.55$, which exactly accounts for the discrepancy between heuristic and numerical results.

It is important to note that the correction term is real only for the cases of equal optical path length, equal gain, and equal reflectors on both lasers, and the gap must be either lossless or an integral number of quarter-wavelengths long. These are rather specialized circumstances, and are unlikely to occur in a practical device. In general, one must use the exact formulas (17) to be assured of the correct coupling coefficients.

The need for rate equations above and beyond the steady-state dispersion equation (18) arises in considerations of the dynamics of the device. Both the self- and cross-coupling coefficients depend upon the carrier density; knowledge of their dependence thereon is necessary for a small-signal analysis. Because of their accuracy and simplicity, the formulas in (17) are more suitable to such treatments than either heuristic or numerical formulas.

IV. DISTANT FEEDBACK AND THE DELAY-DIFFERENTIAL EQUATION

Let us now consider the case of a distant feedback mirror, corresponding, for example, to a small reflection off of a distant optical element. In this case, the longitudinal mode spacing of the composite system may only be a few megahertz, while the bandwidth of the laser may be a few gigahertz. In this case, we must approximate Γ over many cycles of its periodicity (over a range of Ω that encompasses many roots) to get a valid rate equation. However, we can accomplish exactly that.

We derived our rate equation by assuming exponential time variation. This assumption converted time differentials into factors $j\Omega$, and the adiabatic approximation justified in Ref. 1 allows us to convert them back again. However, we can generalize this process by viewing $\underline{\Gamma}(\Omega)$ as the Fourier transform of a linear operator. Thus, if we can approximate $\underline{\Gamma}$ by a sum of functions of Ω that are transforms of linear operators, then inverting the transform gives us a set of linear rate equations in terms of those linear operators.

To illustrate this technique, we shall calculate the rate equation for a short cavity coupled to a lossy, much longer cavity (cavity 2 is taken to be the longer cavity). Let us eliminate E_2 from the equation system so as to consider only a single field. Elimination of E_2 from Eq. (12) yields

$$\left[\left[\frac{e^{-\varphi_1}}{r_0 r_{11}}-1\right]-K\left[\frac{e^{-\varphi_2}}{r_3 r_{22}}-1\right]^{-1}\right]E_1=0 . \tag{24}$$

By characterizing the field by a single dynamical variable, we have reduced the matrix equation to a scalar equation $\Gamma E_1=0$. The assumption of low return reflectivity means that

$$r_3 r_{22} e^{\varphi_2} \ll 1 ,$$

so we can drop the second "1" in Eq. (24), which becomes

$$\left[\left[\frac{e^{-\varphi_1}}{r_0 r_{11}}-1\right]-K\left[\frac{r_3 r_{22}}{e^{-\varphi_2}}\right]\right]E_1=0 . \tag{25}$$

Now, the right parenthetical term in Eq. (25) is much smaller than 1; the left parenthetical term, which can be written as

$$e^{-\varphi_1-\ln(r_0 r_{11})}-1 , \tag{26}$$

is therefore also much smaller than 1 when Eq. (25) is satisfied. Therefore, since the exponential is equal to 1 plus something much less than 1, it can be approximated by the first two terms of its Taylor series:

$$e^{-\varphi_1-\ln(r_0 r_{11})} \approx 1-(\varphi_1+2jN\pi)-\ln(r_0 r_{11}) , \tag{27}$$

where $-2jN\pi$ is the nearest integral multiple of $2j\pi$ to the lasing frequency. Using these two approximations, we find that Γ becomes

$$\Gamma(\Omega)=-\varphi_1-2jN\pi-\ln(r_0 r_{11})-Kr_3 r_{22} e^{\varphi_2} . \tag{28}$$

Now we recall the definition of $\varphi_1=-2j\Omega\mu_1 L_1/c$, and multiply by $c/2\mu_1 L_1$ to get

$$\left[j\Omega-j\left[\frac{cN\pi}{2\mu_1 L_1}-j\frac{c}{2\mu_1 L_1}\ln(r_0 r_{11})\right] -\frac{c}{2\mu_1 L_1}Kr_{22}r_3 e^{-2j\Omega\mu_2 L_2/c}\right]E_1=0 . \tag{29}$$

We define the following quantities:

$$\tau \equiv \frac{2\mu_2 L_2}{c} ,$$

$$\kappa \equiv \frac{c}{2\mu_1 L_1}Kr_2 r_3 , \tag{30}$$

$$\omega_0 \equiv \frac{cN\pi}{2\mu_1 L_1}-j\frac{c}{2\mu_1 L_1}\ln(r_0 r_{11}) .$$

Then Eq. (29) can be written as

$$(j\Omega-j\omega_0-\kappa e^{-j\Omega\tau})E_1=0 . \tag{31}$$

Now, using the Fourier-transform relations, we still find that the factor $j\Omega$ converts to the differential operator d/dt; however, we can now interpret the exponential $\exp(-j\Omega\tau)$ as a time delay of τ. The relevant rate equation is

$$\frac{dE_1(t)}{dt}=j\omega_0 E_1(t)+\kappa E_1(t-\tau) \ . \tag{32}$$

This equation is, of course, the delay-differential equation of Lang and Kobayashi.[8] Explicitly evaluating the coupling coefficient, we get

$$\kappa=\frac{c}{2\mu_1 L_1}t_{21}t_{12}\frac{r_3}{r_{11}} \ . \tag{33}$$

For a single dielectric interface (the case considered in Ref. 8), this expression is the same as was given in the reference. Although the equation was heuristically derived in Ref. 8 by adding a delayed-feedback term to a standard rate equation, we have formally justified it and have extended it to the case of a lossy coupling. Consequently, the delay-differential equation is equivalent to the full set of multimode rate equations for the modes of the composite cavity.

V. CONCLUSIONS

While it may desirable to analyze coupled systems in terms of the fields in the individual cavities, grave problems are encountered if one attempts to use the modes of the individual cavities and achieve quantitatively correct equations. Either one is forced to use an integral representation of the field in each cavity or use an incomplete set of basis functions. Even in the latter case, the number of longitudinal modes required for a given accuracy may be undesirably large; such is almost certainly the case in strongly coupled lasers.

An alternate treatment, which does not run into the problems of coupled-mode theory, is to derive local-field rate equations. Such a treatment can yield quite simple expressions for the self- and cross-coupling coefficients. The formalism relies on the development of a set of frequency-dependent relations between the different field amplitudes and the subsequent conversion of those relations into a set of first-order differential equations for the fields. Since the fields from which the basic relations are derived are based on the composite cavity modes, a local-field theory possesses none of the difficulties of a theory based upon individual cavity modes.

We treated the case of two longitudinally coupled laser cavities and calculated the coupling coefficients between the fields in the two cavities. We calculated the corrections of Marcuse's heuristic formulas and showed that they agreed with numerical calculations, while providing simple analytic formulas for the self- and cross-coupling coefficients.

We then showed how the formalism can be extended to generally derive linear rate equations from approximations of $\underline{\Gamma}(\Omega)$. As an example, we derived and justified the previously heuristically derived delayed-differential equation of Lang and Kobayashi and calculated the coupling coefficient of the delayed term resulting from a general coupling element.

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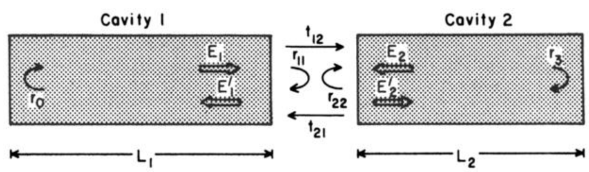

FIG. 1. Schematic of a two-section coupled-cavity laser. The gap is characterized by reflectivities r_{11} and r_{22} and transmissivities t_{21} and t_{22}. The reflectivities at the ends are r_0 and r_3. The lengths are L_1 and L_2, and the (complex) indices of refraction (incorporating gain or loss) are μ_1 and μ_2. All fields are measured at the gap in each laser cavity. Primed fields are the amplitudes of the waves traveling away from the gap; unprimed fields are the amplitudes of the waves traveling toward the gap.