# Motion of charged particles in an axisymmetric magnetic mirror

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(Received 23 January 1985)

The basic features of the motion of charged particles in an axisymmetric mirror, when the magnetic field varies sinusoidally along lines of force, is examined and the diffusion rate for the magnetic moment near the loss cone is obtained.

## I. INTRODUCTION

In a sufficiently strong magnetic field a charged particle gyrates approximately around and along a magnetic line of force and simultaneously drifts over a surface of constant magnetic flux. For an axisymmetric magnetic mirror the drift motion around the symmetry axis is inessential and that reduces the degrees of freedom from three to two. Also for axisymmetric systems, the constancy of angular momentum prevents radial diffusion that results from nonadiabatic magnetic moment changes.

The magnetic moment $\mu$ changes during the particle motion because the ratio of the gyroradius to the characteristic length $L$ of the spatial variation of the field strength varies during the gyration period. Then the particle that is adiabatically trapped in the mirror, i.e., $\mu_0 > \mu_c$ can escape from the mirror when the adiabatic moment $\mu$ changes with time and becomes less than the critical value $\mu_c$. This nonadiabaticity can be measured by the so-called "adiabaticity parameter" $\varepsilon = 2v/[L\Omega_0(1+R)]$ where $v$ is the particle speed, $\Omega_0$ is the gyrofrequency at the midplane $Z=0$, and $R$ is the mirror ratio.

The dynamics of the particle motion can be described by a mapping.[1,2] According to this approach, the magnetic moment abruptly changes by $\Delta\mu = \xi(\mu)\sin\phi$ every time the particle crosses the midplane of the mirror on which the magnetic field has a minimum, where $\phi$ is the gyrophase of the particle on the midplane and $\xi(\mu)$ is a function of the magnetic moment.[1,2]

Here we are interested in studying diffusion of a charged particle in the course of its motion in the vicinity of the separatrix corresponding to the adiabatic loss cone ($\mu = \mu_c$). In Sec. II, we will describe a Poincaré map, which represents the change in the state of the particle (in terms of variables $\mu,\phi$) from one crossing of the midplane $(\mu,\phi)$ to the next one $(\bar{\mu},\bar{\phi})$. We will then linearize this mapping with respect to $\mu$ and obtain the standard mapping which is characterized by only one parameter, the so-called stability parameter $\sigma$. When the stability parameter[3] $\sigma$ is less than 0.971 639... there is only bounded oscillation of $\mu$ around each resonance between the gyration oscillation and the bouncing oscillation of the particle. If $\sigma$ is larger than 0.971 639... there occur resonance overlaps and chaotic changes and the diffusion can be observed in the $\sigma$ variable.

## II. POINCARÉ'S MAPPING AND THE STABILITY PARAMETER

Consider a single-parameter adiabatic orbit with the mass and charge of the particle in unity with an axisymmetric magnetic field. For simplicity we assume that, near the axis of symmetry, the magnetic field is approximated by

$$B(s) = B_0\left[\frac{(R+1)}{2} - \frac{(R-1)}{2}\cos\left[\frac{2\pi s}{L}\right]\right], \tag{2.1}$$

where $B_0$ is the value of $B$ at midplane ($s=0$), $L$ is the mirror length (a constant which is a measure of the characteristic dimension of the magnetic field inhomogeneity), $R$ is the mirror ratio, and $s$ is the particle coordinate along the field lines. In the approximation $\mu = \text{const}$, the longitudinal motion of the particle is described by the Hamiltonian

$$H(P_s,s) = \frac{P_s^2}{2} + \mu B(s), \quad P_s = V_{\parallel} = \frac{ds}{dt}. \tag{2.2}$$

The orbit of Eq. (2.2) can be expressed as

$$V_{\parallel} = \frac{ds(t)}{dt} = P_s = \frac{L}{\pi}\omega_0 k\,\mathrm{cn}(\omega_0 t,k), \tag{2.3}$$

$$s(t) = \frac{L}{\pi}\arcsin[K\,\mathrm{sn}(\omega_0 t,k)]. \tag{2.4}$$

Here sn and cn are the Jacobi elliptic functions and

$$\omega_0 = \frac{\pi}{L}v\sin\psi_0\cos\psi_c, \quad K = \cot\psi_0\tan\psi_c, \tag{2.5}$$

where $\psi_c$ is the critical pitch angle for the adiabatic loss cone given by $\sin^2\psi_c = 1/R$ and $\psi_0$ is the pitch angle at the initial moment $t=0$ at which the particle starts from the midplane $s=0$ of the magnetic mirror in the $s$ direction ($s>0$). The bounce frequency $\omega_B$ is $2\pi$/[the period of $s(t)$] and is given by

$$\omega_B = \frac{\omega_0\pi}{2K(k)}, \tag{2.6}$$

where $K(k)$ is the complete elliptic integral of the first kind. The gyrophase evolves as

 

$$\phi(t)=\phi_0-\int_0^t dt' B(s(t'))$$
$$=\phi_0-B_0\left[Rt-\frac{(R-1)}{\omega_0}E(\Lambda(t),k)\right], \quad (2.7)$$

where $\Lambda(t)=\arcsin[\mathrm{sn}(\omega_0 t,k)]$ and $E(\Lambda,t)$ is the incomplete elliptic integral of the second kind. From Eq. (2.7), the gyrofrequency averaged over a bounce period becomes

$$\langle\Omega\rangle=RB_0\left[1-\frac{(R-1)}{R}\frac{E(k)}{K(k)}\right], \quad (2.8)$$

where $E(k)$ is the complete elliptic integral of the second kind.

The ratio of $\langle\Omega\rangle$ to the bounce frequency then becomes

$$\frac{\langle\Omega\rangle}{\omega_B}=\frac{\Omega_0}{\omega_0}\frac{2R}{\pi}\left[K(k)-\frac{(R-1)}{R}E(k)\right], \quad (2.9)$$

where for a particle of unit mass, charge and $c=1$, we have $B_0=\Omega_0$.

The rate of change of the magnetic moment is[4,5]

$$\frac{d\mu}{dt}=\frac{v_\perp}{BR_c}\left[v^2-\frac{v_\perp^2}{2}\right]\sin\Phi-\frac{v_\parallel v_\perp^2}{2B^2}\frac{\partial B}{\partial s}\sin 2\Phi, \quad (2.10)$$

where $R_c$ is the radius of a magnetic line of force and $\Phi$ is the perturbation phase, which is related to the Larmor phase $\phi$ by the approximate relation[2]

$$r\sin\Phi=r_g\sin\phi. \quad (2.11)$$

Here $r_g$ is the distance from the Larmor center to the symmetry axis of the field. Equation (2.10) is exact and assumes no change in particle position. It holds, in particular, for trajectories which enclose the symmetry axis of the field ($r_g<\rho_\perp$, $\rho_\perp$ is the Larmor radius).

To find $\Delta\mu$, the change in the magnetic moment, we integrate (2.10) over a half period of the bounce frequency. By integrating (2.10), we obtain the following expression for the change in the magnetic moment:

$$\Delta\mu=\xi(\mu)\sin\phi_0 \quad (2.12)$$

with

$$\xi(\mu)=-\frac{3\pi}{8}r_c(2\mu\Omega_0)^{1/2}\exp\left[-\frac{\nu}{\varepsilon}\right] \quad (2.13)$$

and

$$\nu=\frac{1}{\pi\sin\psi_0}\frac{1}{\sqrt{R-1}}\left[\frac{2}{(R+1)}F(\psi_0,k')+\frac{2(R-1)}{(R+1)}E(\psi_0,k')-\sin\psi_0\left[\frac{2R(R-1)}{(R+1)^2}\right]^{1/2}\right]. \quad (2.14)$$

According to Eq. (2.12), the change in the magnetic moment over half a bounce period $\Delta\mu$ depends on the Larmor phase $\phi$, at the time at which the median plane is intersected. For a complete description of the motion we must also find the change in the phase between one intersection of the median plane and the next. This change can be written approximately as

$$\Delta\phi_0\equiv d(\mu)\cong\frac{\pi\langle\Omega\rangle}{\omega_B}=\frac{2\Omega_0 R}{\omega_0}\left[K(k)-\frac{(R-1)}{R}E(k)\right]. \quad (2.15)$$

If we are interested in the motion of a particle in a close vicinity of the adiabatic loss cone, then

$$(k')^2=1-k^2\simeq\left[\frac{\mu_m}{\mu_m-\mu_c}\right]\left[\frac{\mu-\mu_c}{\mu_c}\right]\ll 1,$$

where $\mu_m$ is the value of $\mu$ as $s=0$. For $k'\ll 1$ we can approximate elliptic functions as

$$K(k')\simeq\ln\left[\frac{4}{k'}\right],\quad E(k')\simeq 1$$

and we can rewrite Eq. (2.15) as

$$d(\mu)=\frac{L\Omega_0}{\sqrt{\mu}}\frac{R}{\sqrt{R-1}}\frac{1}{\sqrt{2\pi}}\times\left[\ln\left[\frac{16}{\left[\frac{\mu_m}{\mu_m-\mu_c}\right]\left[\frac{\mu-\mu_c}{\mu}\right]}\right]-\frac{2(R-1)}{R}\right]. \quad (2.16)$$

Thus the motion of a particle in close vicinity of the separatrix (i.e., the adiabatic loss cone for the case of the magnetic mirror) can be described approximately by

$$\bar{\mu}=\mu+\xi(\mu)\sin\phi,$$
$$\bar{\phi}=\phi+d(\mu), \quad (2.17)$$

where $d(\mu)$ is given by (2.16), $\xi$ by (2.13), and we omit the subscript 0 from the gyrophase $\phi_0$. Equation (2.17) is the Poincaré mapping, which describes the change in the state of the particle [in terms of the variables $(\bar{\mu},\bar{\phi})$ from one crossing of the midplane $(\mu,\phi)$ to the next one $(\mu,\phi)$].

If we expand $d(\mu)$ around some value $\mu_1$ as

$$d(\mu)=d(\mu_1)+(\mu-\mu_1)d'(\mu_1)+\frac{(\mu-\mu_1)^2}{2!}d''(\mu_1),$$

and define the new action variable

$$I=d(\mu_1)+(\mu-\mu_1)d'(\mu_1),$$

we obtain the so-called "standard mapping"

$$\bar{I}=I+\sigma\sin\phi,$$
$$\bar{\phi}=\phi+\bar{I}, \quad (2.18)$$

where we assume that

$$|(\mu-\mu_1)^2 d''(\mu_1)/2I|\ll 1.$$

The standard mapping has only one parameter $\sigma$, the so-called "stability parameter"[1,2] given by

$$\sigma=\xi d'(\mu_1)=-\frac{2\pi A\xi}{(\mu_1-\mu_c)}. \quad (2.19)$$

Here

$$A=\frac{RL(\Omega_0)^{1/2}}{2\sqrt{2}\pi^2\sqrt{R-1}}\frac{1}{\mu_c}=\frac{1}{\varepsilon}\left[\frac{2}{(R+1)}\frac{R}{\sqrt{R-1}}\frac{1}{\sqrt{2}\pi^2}\frac{1}{\sin\psi_c}\right] \quad (2.20)$$

and

$$\mu-\mu_c=\frac{v^2}{2\Omega_0}(\sin^2\psi-\sin^2\psi_c)=\frac{v^2}{\Omega_0}\sin\psi_c\cos\psi_c\Delta\psi\ . \quad (2.21)$$

Substituting $\xi(\mu)$ from Eq. (2.13) into Eq. (2.19) and taking $\mu_1=\mu_0$ (initial value), we get, by using (2.20) and (2.21),

$$\sigma=\frac{\alpha}{\varepsilon^2}\exp\left[-\frac{\nu}{\varepsilon}\right], \quad (2.22)$$

where ε is an adiabaticity parameter, and

$$\alpha=\frac{3}{2}\frac{R^2}{(R+1)^2}\frac{1}{(R-1)}\frac{r_c}{L}\frac{1}{\Delta\psi_0}, \quad (2.23)$$

where $\Delta\psi_0=\psi_0-\psi_c$. Using the approximate relation

$$\frac{16}{\left[\frac{\mu_m}{\mu_m-\mu_c}\right]\left[\frac{\mu-\mu_c}{\mu_c}\right]}\simeq\frac{8\sin\psi_c\cos\psi_c}{\Delta\psi}, \quad (2.24)$$

where $\Delta\psi=\psi-\psi_c$, we can write the rotation number by Eq. (2.16), viz.,

$$r(\mu)=\frac{d(\mu)}{2\pi}=A\left[\ln\left[\frac{8\sin\psi_c\cos\psi_c}{\Delta\psi}\right]-\frac{2(R-1)}{R}\right]. \quad (2.25)$$

The local diffusion rate is given by

$$D_\mu=\frac{\langle(\Delta\mu)^2\rangle}{2t}=D_0\chi\ , \quad (2.26)$$

where $t$ is the number of iterations, $D_0=W_B\xi^2/4\pi$ is the quasilinear diffusion rate, and $\chi$ is the correction factor. Equation (2.26) is valid only if $\sigma\gtrsim 4$. Near the chaos border[1,2]

$$\chi(\sigma)=\left[1-\frac{1}{\sigma}\right]^2. \quad (2.27)$$

Integral surfaces in an axially symmetric field of a magnetic dipole were studied in detail by Störmer[6] in connection with the analysis of cosmic-ray motion in the Earth's magnetic field. Similar calculations for a magnetic mirror are described by Artsimovich.[7] The conduction of absolute confinement may be written in the form

$$r_{g0}cR_{\rm St},\ \ R_{\rm St}^2\equiv\frac{V^2}{\Omega_0}\left[\sin^2\psi_0-\frac{1}{R}\right], \quad (2.28)$$

$$\varepsilon^{-1}<\varepsilon_{\rm St}^{-1},\ \ \epsilon_{\rm St}^{-2}\equiv\frac{\left[\sin^2\psi_0-\frac{1}{R}\right]}{\left[\frac{2}{(R+1)}\frac{r_{g0}}{L}\right]}, \quad (2.29)$$

where $R_{\rm St}$ is the Störmer radius, $\varepsilon_{\rm St}$ is the adiabaticity parameter at the Störmer cutoff, and $r_{g0}$ is the displacement of the guiding center from the mirror axis at the initial moment $t=0$ at which the particle starts from the midplane. The Störmer cutoff comes from a constraint imposed by canonical momentum conservation for axis encircling particles. The domain of phase space in which a particle is confined due to exact integrals of motion is usually called the Störmer zone.

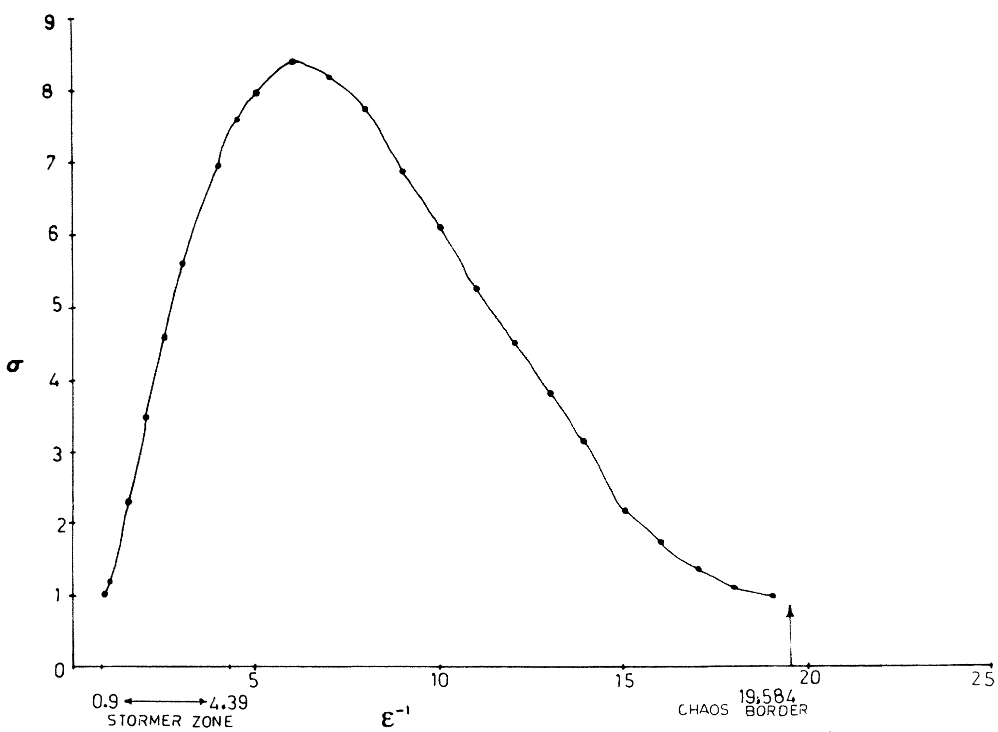


FIG. 1. Plot of stability parameter σ vs adiabacity parameter $\varepsilon^{-1}$.

Near the Störmer cutoff[1,2]

$$\chi(\mu)\simeq\frac{r_g^2(\mu)}{r_{g0}^2}=\frac{\mu-\mu_{St}}{\mu_0-\mu_{St}}\ , \tag{2.30}$$

where

$$\mu_{St}=\frac{\Omega_0}{2}\left[\rho_{\perp 0}^2-r_{g0}^2\right] \tag{2.31}$$

is the value of $\mu$ at the Störmer cutoff. Since both correction factors approach zero as $\sigma\rightarrow 1$ or $\mu\rightarrow\mu_{St}$, respectively, the diffusion rate changes considerably even if $\mu_0-\mu_c \ll \mu_0$. In the latter case a better value for particle diffusion for $\mu=\mu_0$ to $\mu=\mu_c$ would be the average diffusion rate. Here we are approximating $\xi$=const over the interval $\mu_0<\mu<\mu_c$. By averaging over $\mu$ space, we get[8]

$$\frac{1}{\langle D\rangle}=\frac{1}{\Delta\mu}\int\frac{d\mu}{D\mu}\ . \tag{2.32}$$

Substituting (2.26) into the equation (2.32) and using the relation (2.19) we get

$$\frac{1}{\langle D\rangle}=\frac{\sigma}{D_0}(\sigma-1)^{-1} \tag{2.33}$$

for $\sigma \gg 1$, $\langle D\rangle$ approaches the quasilinear limit $D_0$.

The averager diffusion near the Störmer cutoff is given by

$$\frac{1}{\langle D\rangle}=\frac{1}{D_0}\frac{1}{(\mu_0-\mu_c)}\frac{\Omega_0 r_{g0}^2}{2}\ln\left|\frac{(\mu_c-\mu_{St})}{(\mu_0-\mu_{St})}\right|$$
$$=\frac{1}{D_0}\frac{\varepsilon_{St}^2}{\varepsilon^2}\left|\ln\left[1-\frac{\varepsilon^2}{\varepsilon_{St}^2}\right]\right|\ . \tag{2.34}$$

## III. DISCUSSION AND CONCLUSION

We choose values of the parameters[9] $R=1.5$, $r_{g0}/L=0.05$, $\gamma=0.3315$, and

$$\cos\psi_0=0.550(\langle\cos\psi_c\rangle\cong 0.5773)\ .$$

Substituting these values into (2.23) and (2.33), we get

$$\alpha=1.6718 \text{ and } \varepsilon_{St}^{-1}\simeq 4.39\ .$$

In Table I we give the values of the diffusion coefficient obtained by using the above formulas for values of the adiabaticity parameter near the Störmer radius, namely $\varepsilon^{-1}=4.4$, 4.6, and 5.0 and for values of the adiabaticity parameter ranging from $\varepsilon^{-1}=12$ to 19.

In Fig. 1 we plot a graph of $\sigma$ versus $\varepsilon^{-1}$ for those values of $\varepsilon^{-1}$ corresponding to $\sigma>1$. There is a region of absolute confinement or perfect trapping (Störmer zone) for $\varepsilon^{-1}<4.39$ in which particles cannot reach the loss cone and escape in spite of diffusion inside the trap (if $\sigma>\sigma_{cr}$). For $\varepsilon^{-1}\simeq 19.584\ldots$ corresponding to $\sigma=\sigma_{cr}=0.971\,635\ldots$ there is no diffusion at all for the initial $\mu=\mu_0$.

TABLE I. The border of particle-motion stability in an axisymmetric magnetic mirror.

| $\varepsilon^{-1}$ | $\sigma$ | $\ln D_0$ | $\ln D_\mu$ |
|---|---|---|---|
| 4.5 | 7.62 | −3.10 | −3.60 |
| 4.6 | 7.78 | −3.10 | −3.52 |
| 5.0 | 7.94 | −3.20 | −3.48 |
| 12.00 | 4.50 | −4.70 | −4.80 |
| 13.00 | 3.79 | −5.10 | −5.23 |
| 14.00 | 3.15 | −5.25 | −5.41 |
| 14.50 | 2.88 | −5.45 | −5.63 |
| 15.00 | 2.14 | −5.55 | −5.82 |
| 16.00 | 1.74 | −5.90 | −6.27 |
| 17.00 | 1.41 | −6.15 | −6.68 |
| 18.00 | 1.09 | −6.40 | −7.48 |
| 19.00 | 0.9906 | | |
| 19.5845 | 0.9716... | | |

In this paper we have studied the nonadiabatic escape of charged particles from an axisymmetric magnetic trap, when the magnetic field varies sinusoidally along lines of force, by using Chirikov's diffusion model. For the motion of a particle in a magnetic mirror the adiabatic loss cone is a separatrix and a domain of chaotic motion always exists near the adiabatic loss cone. In Table I we have given the values of averaged diffusion coefficient for values of adiabaticity parameter $\varepsilon^{-1}=4.5$, 4.6, and 5 near the Störmer radius (which corresponds to $\varepsilon^{-1}=\varepsilon_{St}^{-1}\simeq 4.39$) and for values of adiabaticity parameter ranging from $12\leq\varepsilon^{-1}\leq 19$. There is a region of absolute confinement (Störmer zone) in which the particle is trapped in a magnetic mirror ($\varepsilon^{-1}\leq 4.39$).

## ACKNOWLEDGMENTS

I would like to acknowledge Dr. B. V. Chirikov for continuous encouragement and useful discussions during my visit to the Institute of Nuclear Physics, Novosibirsk, under the exchange program between the U.S.S.R. Academy of Sciences and Indian National Science Academy. I am also thankful to Dr. G. Rowlands for useful comments.

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