Infeld-Hull factorization and the simplified solution of the Dirac-Coulomb equation

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(Received 21 January 1986; revised manuscript received 8 May 1986)

It is pointed out that the approach used by Su to obtain the simplified solution to the Dirac-Coulomb equation can be interpreted in the light of the Infeld-Hull factorization method. According to this method the first-order Dirac-Coulomb equation, after the transformation S, necessarily implies two second-order "Schrödinger-type" equations, whose solutions are the upper and lower simplified solutions of the Dirac-Coulomb equation. It is also interesting to note that the "recurrence relation" between the upper and lower solutions, first obtained by Biedenharn, is exactly the Dirac-Coulomb equation itself, after the transformation S.

Recently Su[1] has obtained the simplified solution to the Dirac-Coulomb equation where each of the upper and lower solutions contains only one term of a confluent hypergeometric function. His result agrees with Biedenharn[2] (for the continuum) and with Wong and Yeh[3] (for the bound state). The difference between these papers lies in the approach, i.e., in Su's method there is no reference to the "recurrence relation" used by Biedenharn and Wong and Yeh as a mathematical identity to reduce the second-order differential-equation solution to the first-order Dirac-equation solution.

We wish to point out that Su's approach can also be interpreted in the light of the Infeld-Hull factorization method.[4] Actually in the Infeld-Hull paper,[4] there is a discussion in Sec. 8.4 of Dirac's radial functions χ_1 and χ_2. However, they started with the untransformed Dirac equation, and thus their solution still contains the sum of two functions. The new feature in applying the Infeld-Hull method is to start with the transformed Dirac equation

$$H'\psi'=E\psi' , \tag{1}$$

where

$$H'=SHS^{-1} , \tag{2}$$

$$\psi'=S\psi . \tag{3}$$

The transformation S used by Su[1] is the inverse of the transformation used by Biedenharn[2] and Wong and Yeh.[3] As a result, the upper and lower solutions of Su become the lower and upper solutions of Wong and Yeh.

Here we use the notation of Wong and Yeh[3] and obtain, from Eq. (1),

$$\psi'=N\begin{bmatrix} i\left[E\dfrac{|\kappa|}{|\gamma|}-m\right]^{1/2} g\;\chi^{\mu}_{-\kappa} \\ \tilde{\omega}\left[E\dfrac{|\kappa|}{|\gamma|}+m\right]^{1/2} f\;\chi^{\mu}_{\kappa}\end{bmatrix} , \tag{4}$$

$$\begin{bmatrix} -\left[E\dfrac{|\kappa|}{|\gamma|}+m\right] & -i\sigma\cdot\hat{\mathbf{r}}\left[\dfrac{d}{dr}+\dfrac{1+\tilde{\omega}|\gamma|}{r}-\dfrac{EZe^2}{\tilde{\omega}|\gamma|}\right] \\ i\sigma\cdot\hat{\mathbf{r}}\left[\dfrac{d}{dr}+\dfrac{1-\tilde{\omega}|\gamma|}{r}+\dfrac{EZe^2}{\tilde{\omega}|\gamma|}\right] & E\dfrac{|\kappa|}{|\gamma|}-m\end{bmatrix}\begin{bmatrix} i\left[E\dfrac{|\kappa|}{|\gamma|}-m\right]^{1/2} g\chi^{\mu}_{-\kappa} \\ \tilde{\omega}\left[E\dfrac{|\kappa|}{|\gamma|}+m\right]^{1/2} f\chi^{\mu}_{\kappa}\end{bmatrix}=0 , \tag{5}$$

or

$$\left[\frac{d}{dr}+\frac{1+\tilde{\omega}|\gamma|}{r}-\frac{EZe^2}{\tilde{\omega}|\gamma|}\right]f=(E^2\kappa^2/\gamma^2-m^2)^{1/2}g , \tag{6}$$

$$\left[\frac{d}{dr}+\frac{1-\tilde{\omega}|\gamma|}{r}+\frac{EZe^2}{\tilde{\omega}|\gamma|}\right]g=-(E^2\kappa^2/\gamma^2-m^2)^{1/2}f , \tag{7}$$

Thus the transformed Dirac equation (6) and (7) can be regarded as the factorized Infeld-Hull first-order equations

$$O_+f=ag , \tag{8}$$

$$O_-g=bf , \tag{9}$$

with

$$O_+ = \frac{d}{dr} + \frac{1+\tilde{\omega}|\gamma|}{r} - \frac{EZe^2}{\tilde{\omega}|\gamma|} ,$$
$$O_- = \frac{d}{dr} + \frac{1-\tilde{\omega}|\gamma|}{r} + \frac{EZe^2}{\tilde{\omega}|\gamma|} , \tag{10}$$
$$a = (E^2\kappa^2/\gamma^2 - m^2)^{1/2} ,$$
$$b = -a .$$

From the Infeld-Hull method, we conclude that f and g satisfy the second-order Schrödinger-type equations

$$O_- O_+ f = abf ,$$
$$O_+ O_- g = abg . \tag{11}$$

Or

$$\left[\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} - \frac{(\gamma^2 \pm \tilde{\omega}|\gamma|)}{r^2} + \frac{2Ze^2E}{r} + E^2 - m^2 \right] \begin{bmatrix} f \\ g \end{bmatrix} = 0 . \tag{12}$$

Equation (12) is completely similar to the radial Schrödinger or Klein-Gordon equation. Thus the lower solution f and the upper solution g will each contain only one term of a confluent hypergeometric function, in complete resemblance to the Schrödinger wave function. It is also interesting to note that the "recurrence relations" of Biedenharn are exactly the same as (6) and (7).

To summarize our results, we have shown that the transformed first-order Dirac-Coulomb equation, Eqs. (6) and (7), can be put in the form of the Infeld-Hull first-order equations (8) and (9). Moreover, these two equations are just the recurrence relations of Biedenharn. The second-order equations (12), similar to the Schrödinger equation, are then a necessary consequence of the two first-order equations according to the Infeld-Hull method. Thus there is a direct relation between the Dirac-Coulomb equation and the Schrödinger-type equation.

One last remark we wish to make is that the two energetically degenerate states such as $2S_{1/2}$ and $2P_{1/2}$ are intimately related to each other, since they correspond to the upper and lower solutions. In the context of the Dirac-Coulomb equation they must necessarily have the same energy, because they must satisfy equations (6) and (7) simultaneously. Thus the Lamb shift can be seen as the most classical case of symmetry breaking.

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