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Oscillatory convection in binary mixtures

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Two-dimensional oscillatory convection in a binary fluid mixture in an infinite-plane porous layer heated from below is studied. Small-amplitude nonlinear solutions in the form of standing and traveling waves are found and their relative stability is established. Stable traveling waves are preferred near onset. The interaction of the two types of wave with steady overturning convection is also studied. As the Rayleigh number is increased the period of each type of wave approaches infinity, standing waves as $-\ln(R_c^{SW}-R)$ and traveling waves as $1/(R_c^{TW}-R)$, where R_c is the critical Rayleigh number at which the transition to finite amplitude overturning convection occurs. This transition is hysteretic. The presence of modulated traveling waves (i.e., waves with two distinct frequencies) is also predicted. These predictions are made on the basis of analyses of multiple bifurcations in the presence of O(2) symmetry. This symmetry is present in two-dimensional problems with periodic boundary conditions and a reflection symmetry in a vertical plane. The relevance of the results to recent experiments on binary fluids, both in bulk mixtures and in a porous medium, is discussed.

I. INTRODUCTION

Binary fluid mixtures offer a suitable system in which complicated dynamical phenomena near multiple bifurcations can be studied.[1,2] Two recent experiments have been remarkably successful. Rehberg and Ahlers[3] have observed the approach to a heteroclinic orbit near a codimension-two bifurcation[4,5] in their study of oscillatory convection in a He^3-He^4 mixture in a porous medium heated from below. In an experiment on a water-alcohol mixture, Walden *et al.*[6] observed a traveling roll pattern, the possibility of which was independently suggested by Knobloch *et al.*[7] Both experiments investigated oscillatory convection in a binary fluid mixture. The pattern observed in the first experiment apparently consists of stationary rolls whose velocity reverses cyclically, and may be thought of as a standing wave, i.e., as a superposition of two traveling waves of equal amplitude, one traveling to the left and the other to the right. The uniformly drifting roll pattern observed in the second experiment can similarly be thought of as a right-traveling wave; left-traveling waves can be obtained for other initial conditions.

In this paper we treat the traveling and standing-wave patterns in a unified way. Analytical progress can be made for mixtures confined in a horizontally infinite plane layer, and we use this idealization to establish a number of properties of both types of wave. We focus attention on convection in binary mixtures in a porous medium, since the results of the analysis carry over to bulk mixtures via a trivial modification. We begin by studying separately small amplitude standing and traveling waves near the critical Rayleigh number for the onset of oscillations. This analysis cannot determine the relative stability of these two solutions, i.e., the stability of one with respect to perturbations in the form of the other. The general theory that we develop answers questions of this type. The theory makes full use of the symmetries that characterize the system and clearly indicates the extent to which the multiplicity of possible solutions and their stability are determined by these symmetries. In the present problem we study two-dimensional spatially periodic solutions, with no distinction between left and right. In this case the group of symmetries is the group O(2) of rotations and reflections of a circle. The theory determines the structure of the amplitude equations near the bifurcation and the results of the earlier calculation can be used to infer the coefficients of the nonlinear terms in these equations. In this way a complete description of the behavior near bifurcation is established. The results show that in an idealized model of the system standing waves are typically unstable near the onset of convection, and that traveling waves should be observed.

We then study the interaction of both types of wave with steady overturning convection by considering the neighborhood of a particular codimension-two bifurcation in the presence of O(2) symmetry. The analysis gives a complete description of the possible nonlinear interactions between these three types of solution at small amplitude,[8] and predicts for the system under consideration a hysteretic transition from each type of wave to nonlinear overturning convection, as the Rayleigh number is increased. As the critical Rayleigh number for each transition is approached, the frequency of the wave goes to zero. However, both the rates at which this occurs and the critical Rayleigh numbers depend on the wave type. The theory also predicts a Rayleigh-number interval in which stable modulated waves exist.

This paper is organized as follows. In Sec. II the basic equations are introduced, followed by a summary of their linear-stability properties in Sec. III. In Sec. IV we study small-amplitude standing and traveling waves near the bifurcation to oscillation, and in Sec. V we develop the amplitude equation that describes the interaction of the two types of waves with steady overturning convection by considering the neighborhood of a particular codimension-two bifurcation. In Sec. VI explicit consideration of the symmetries is introduced, and general amplitude equations for steady-state, Hopf, and the codimension-two bifurcations are derived. Analysis of these equations in conjunction with the coefficients deduced from Secs. IV and V establishes a number of explicit predictions which are summarized in Sec. VII.

II. FORMULATION OF THE PROBLEM

In a binary fluid heated from below, the solute may diffuse to the cooler plate (the ordinary Soret effect) or to the warmer plate (the negative Soret effect). In the latter case a stabilizing molecular-weight gradient will be set up in response to a destabilizing temperature gradient, and will lead to a competition characteristic of all doubly diffusive systems. One consequence is the possibility of oscillatory convection.[9] In fluid mixtures the Dufour effect[1] is negligible, and the equations describing convection in a porous medium then are[10]

$$\frac{1}{\epsilon}\mathbf{u}_t + a'\mathbf{u}f(|\mathbf{u}|) = -\frac{1}{\rho}\nabla p' + (\alpha T' - \beta C')g\hat{\mathbf{z}} - \frac{\nu}{K}\mathbf{u} , \tag{2.1a}$$

$$T'_t + \mathbf{u}\cdot\nabla T' - Aw = \kappa\nabla^2 T' , \tag{2.1b}$$

$$C'_t + \mathbf{u}\cdot\nabla C' + \frac{k_T}{T}Aw = D\nabla^2 C' + \frac{k_T}{T}D\nabla^2 T' , \tag{2.1c}$$

$$\nabla\cdot\mathbf{u} = 0 . \tag{2.1d}$$

In writing Eqs. (2.1) we have employed the Boussinesq approximation, and modeled dissipation by a Darcy friction term with a coefficient ν/K, where ν is the kinematic viscosity. The quantity ϵ is the porosity. The primed variables represent departures of the temperature T, concentration C, and pressure p from the conduction profiles set up in response to an applied temperature gradient $-A\hat{\mathbf{z}}$. Thus the basic state whose stability is of interest is given by

$$\mathbf{u} = T' = C' = p' = 0 . \tag{2.2}$$

The coefficients κ and D are the thermal and solutal diffusivities, k_T is the Soret coefficient ($k_T < 0$), and $\alpha = -(\partial\rho/\partial T)_{p,C}$, $\beta = (\partial\rho/\partial C)_{p,T}$. All these quantities are assumed to be constant. The nonlinearity in the equation of motion represents a correction to the Darcy friction term and describes the increase in the friction that results from decreasing spatial scales of the motion in the interstices of the medium with increasing Reynolds number. The amplitude a' of this term is small.[10] Assuming that its functional form is differentiable at the origin,[11] we may write

$$f = \mathbf{u}\cdot\mathbf{u} + O(|\mathbf{u}|^4) . \tag{2.3}$$

Since in what follows we shall be interested in bifurcations from the conduction state (2.2), the higher-order terms in (2.3) will not be required. Thus any function $f(\mathbf{u}\cdot\mathbf{u})$ such that $f(0)=0$, $f'(0)=1$ will yield identical results.

In order to write the equations of motion in a convenient form for a study of two-dimensional convection, we introduce the stream function $\psi(x,z,t)$ such that $\mathbf{u}=(u,w)=(-\psi_z,\psi_x)$, and measure time in units of the thermal diffusion time across a layer of depth h, velocities in units of κ/h, and distances in units of the temperature difference $\Delta T = Ah$ across the layer, assumed to be fixed, and the concentration in units of the concentration difference induced by the Soret effect, $\Delta C = -(k_T/T)\Delta T$. Eliminating the pressure, we obtain

$$\nabla^2\psi_t + \sigma\nabla^2\psi = \sigma R\phi_x - a[(f\psi_x)_x + (f\psi_z)_z] , \tag{2.4a}$$

$$\theta_t - \nabla^2\theta = \psi_x - J(\psi,\theta) , \tag{2.4b}$$

$$\phi_t - \tau\nabla^2\phi = (1+S)\psi_x + [1-\tau(1+S)]\nabla^2\theta - J(\psi,\phi) , \tag{2.4c}$$

where the five dimensionless parameters are given by

$$R = \frac{\alpha g A h^2 K}{\kappa\nu}, \quad S = \frac{k_T}{T}\frac{\beta}{\alpha}, \quad \tau = \frac{D}{\kappa} ,$$
$$\sigma = \epsilon\frac{\nu}{\kappa}\left[\frac{h^2}{K}\right], \quad a = \epsilon\kappa a' \tag{2.5}$$

and

$$\phi = \theta - S\Sigma . \tag{2.6}$$

Here θ, Σ denote dimensionless T', C', and $J(g,h) \equiv g_x h_z - g_z h_x$. Note that the "separation" constant S is negative.[12] The effective Prandtl number σ is generally large in porous media, and we shall in Sec. V take the limit $\sigma \to \infty$. In addition we shall assume throughout that $0 < a \ll 1$.

We solve Eqs. (2.4) in a horizontally infinite plane-parallel layer subject to idealized boundary conditions[2]

$$\psi = \psi_{zz} = \theta = \phi = 0 \quad \text{on } z = 0,1 , \tag{2.7}$$

corresponding to stress-free boundaries, and fixed temperature and concentration at the top and bottom. We shall seek spatially periodic solutions of horizontal period $d = 2\pi/k$. Owing to the structure of the advective term in the Navier-Stokes equation, the results for convection in bulk mixtures[1,7] corresponding to those derived in Secs. III–V below can be recovered by setting $a=0$, and replacing R with R/p and σ with $p\sigma$, where $p=\pi^2+k^2$. This changes the critical wavelength of the modes that first become unstable, but does not otherwise modify the calculations.

III. LINEAR THEORY

To study the stability of the conduction solution $\psi=\theta=\phi=0$ we linearize the equations and look for solutions growing as $\exp(spt)$. The conduction solution loses stability to overturning convection ($s=0$) at[10]

$$R_e = \frac{p^2}{k^2} \frac{\tau}{[\tau+(1+\tau)S]} , \quad (3.1)$$

and to oscillatory convection ($s = \pm i\omega_0$) at[10]

$$R_0 = \frac{p^2}{\sigma k^2} \frac{(1+\tau)(p+\sigma)(\tau p+\sigma)}{[p+\sigma(1+S)]} , \quad (3.2)$$

with the oscillation frequency ω_0 given by

$$\omega_0^2 = \frac{\sigma k^2}{p^2} \left[\frac{\tau+(1+\tau)S}{\sigma+(1+\tau)p} \right] (R_e - R_0) > 0 . \quad (3.3)$$

Thus the bifurcation to oscillatory convection precedes the bifurcation to steady convection. The codimension-two bifurcation that will be of interest occurs when ω_0 vanishes, i.e., when[2]

$$R_c = \frac{p^2}{\sigma k^2} [\sigma + \tau(1+\tau)(\sigma+p)] , \quad S_c = \frac{-\tau^2(\sigma+p)}{\sigma+\tau(1+\tau)(\sigma+p)} . \quad (3.4)$$

At this point the linear stability problem has *two* zero eigenvalues, and we speak of a multiple bifurcation. Note that $|S_c| < \frac{1}{2}$.

In what follows the results

$$1+\omega_0^2 = \frac{(1+\tau)(\sigma+p)[1-\tau(1+S)]}{p+\sigma(1+S)} , \quad (3.5a)$$

$$\tau^2+\omega_0^2 = \frac{-S(1+\tau)(\sigma+\tau p)}{p+\sigma(1+S)} \quad (3.5b)$$

will be useful.

IV. SMALL-AMPLITUDE THEORY

In this section we study the small-amplitude nonlinear solutions near the bifurcation points R_0, R_e. Near the point R_0 both standing- and traveling-wave solutions are readily found, and we compute them by looking for solutions of the appropriate type. This type of approach does not, however, show that no other solutions exist near R_0, nor does it address the stability of one type of solution relative to infinitesimal perturbations of the other type. These issues are addressed by the theory developed in Sec. VI. Steady overturning convection is found near the point R_e.

A. Standing waves

We seek nonlinear oscillatory solutions of Eqs. (2.4) and (2.7) with frequency ω using a perturbation expansion of the form[13]

$$R = R_0 + \epsilon R_1 + \epsilon^2 R_2 + \cdots , \quad (4.1a)$$

$$\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots , \quad (4.1b)$$

$$\psi = \epsilon\psi_1 + \epsilon^2\psi_2 + \epsilon^3\psi_3 + \cdots , \quad (4.1c)$$

and so on. The quantities R_j, ω_j ($j \ge 1$) are determined by the requirement that the solutions we find are periodic in time. When this is done, the expansion (4.1a) determines the amplitude of such solutions as a function of $R - R_0$. Substituting (4.1) into Eqs. (2.4) and equating powers of ϵ yields a sequence of problems. At $O(\epsilon)$ we recover the linear problem with R_0 and ω_0 given by (3.2) and (3.3), and

$$\psi_1 = \mathrm{Re}\, e^{i\omega pt} \sin(kx)\sin(\pi z) \quad (4.2a)$$

$$\theta_1 = \mathrm{Re} \frac{k}{p(1+i\omega_0)} e^{i\omega pt} \cos(kx)\sin(\pi z) \quad (4.2b)$$

$$\phi_1 = \mathrm{Re} \frac{p(\sigma+i\omega_0 p)}{k\sigma R_0} e^{i\omega pt} \cos(kx)\sin(\pi z) . \quad (4.2c)$$

At $O(\epsilon^2)$ the condition that time-periodic solutions exist is always satisfied, and we therefore take

$$R_1 = \omega_1 = 0 . \quad (4.3)$$

This is, of course, because the bifurcation is a Hopf bifurcation. The quantity ψ_2 is found to be proportional to ψ_1, so that by choosing ϵ to be the amplitude of ψ_1 we may take $\psi_2 = 0$. The nonlinear terms required for θ_2, ϕ_2 are most conveniently calculated using complex notation and replacing $\mathrm{Re}\,a\,\mathrm{Re}\,b$ by $\mathrm{Re}[\frac{1}{2}a(b+b^*)]$ or $\mathrm{Re}[\frac{1}{2}(a+a^*)b]$, where required.[13] We obtain

$$\theta_2 = -\mathrm{Re} \frac{k^2\pi}{4p^2} \left[\frac{1}{1+i\omega_0} \right] \left[\frac{1}{\tilde{\omega}} + \frac{e^{2i\omega pt}}{2i\omega_0+\tilde{\omega}} \right] \sin(2\pi z) , \quad (4.4a)$$

$$\phi_2 = -\mathrm{Re} \frac{\pi}{4\sigma R_0} (\sigma + pi\omega_0) \left[\frac{1}{\tilde{\omega}\tau} + \frac{e^{2i\omega pt}}{2i\omega_0+\tilde{\omega}\tau} \right] \sin(2\pi z) + \mathrm{Re}[1-\tau(1+S)] \frac{k^2\pi}{4p^2} \left[\frac{1}{1+i\omega_0} \right] \times \left[\frac{1}{\tilde{\omega}\tau} + \frac{\tilde{\omega} e^{2i\omega pt}}{(2i\omega_0+\tilde{\omega})(2i\omega_0+\tilde{\omega}\tau)} \right] \sin(2\pi z) , \quad (4.4b)$$

where $\tilde{\omega} \equiv 4\pi^2/p$, $0 < \tilde{\omega} < 4$. At $O(\epsilon^3)$, we have the problem

$$\begin{bmatrix} \nabla^2\partial_t + \sigma\nabla^2 & 0 & -\sigma R_0 \partial_x \\ -\partial_x & \partial_t - \nabla^2 & 0 \\ -(1+S)\partial_x & -[1-\tau(1+S)]\nabla^2 & \partial_t - \tau\nabla^2 \end{bmatrix} \begin{bmatrix} \psi_3 \\ \theta_3 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} p^2 i\omega_2\psi_1 + \sigma R_2 \phi_{1x} \\ -pi\omega_2\theta_1 - J(\psi_1,\theta_2) \\ -pi\omega_2\phi_1 - J(\psi_1,\phi_2) \end{bmatrix} + \begin{bmatrix} O(a) \\ 0 \\ 0 \end{bmatrix} . \quad (4.5)$$

On the right side there will now be terms which lie in the kernel of the operator on the left. The condition that such resonant terms are absent determines R_2, ω_2. We restrict (4.5) to the resonant terms, and multiply from the left by the row vector

$$M^T=\{-p^2(\tau+i\omega_0)(1+i\omega_0),-pk\sigma R_0[1-\tau(1+S)],pk\sigma R_0(1+i\omega_0)\}\ . \tag{4.6}$$

The left side then vanishes, and hence the required condition is that M^T (the right-hand side) equals zero. This condition can be written in the form

$$p\left[2\omega_0\omega_2 p+\frac{k^2}{16}\right](\sigma+p+\tau p+pi\omega_0)+\sigma k^2R_2[\tau+(1+\tau)S+(1+S)i\omega_0]$$
$$=\frac{\pi^2k^2}{8}\frac{1}{2i\omega_0+\tilde{\omega}\tau}\left[(\sigma+pi\omega_0)(1+i\omega_0)-\frac{(\sigma+\tau p)(1-i\omega_0)[\tilde{\omega}(1+\tau)+(2+\tilde{\omega})i\omega_0]}{2i\omega_0+\tilde{\omega}}\right]+O(a)\ . \tag{4.7}$$

This complex equation determines both R_2,ω_2, and reduces to the results of Ref. 14 in the limit $\sigma\to\infty$. To derive it is helpful to work in terms of ω_0 rather than R_0,S. For this the expressions (3.2) and (3.5) are useful.

The computation of R_2 determines when the bifurcation to standing waves is subcritical ($R_2<0$) and when it is supercritical ($R_2>0$). In the former case the solutions are unstable with respect to infinitesimal standing-wave perturbations, while in the latter case they are stable. Both possibilities occur as parameters are varied, though near the codimension-two bifurcation the branch is always supercritical.[14] The stability with respect to perturbations in the form of traveling waves is deferred to Sec. VI.

B. Traveling waves

We seek small-amplitude nonlinear waves traveling to the right of the form $\psi(x,z,t)=\psi(\xi,z)$, etc., where $\xi=x-pct$, and c is the phase velocity or drift speed of the wave. Since the wave is nonlinear, c depends on the amplitude of the wave. In terms of the variable ξ Eqs. (2.4) become

$$pc\nabla^2\psi_\xi-\sigma\nabla^2\psi+\sigma R\phi_\xi=a[\partial_\xi(u^2\psi_\xi)+\partial_z(u^2\psi_z)]\ , \tag{4.8a}$$

$$pc\theta_\xi+\nabla^2\theta+\psi_\xi=J(\psi,\theta)\ , \tag{4.8b}$$

$$pc\phi_\xi+\tau\nabla^2\phi+(1+S)\psi_\xi+[1-\tau(1+S)]\nabla^2\theta=J(\psi,\phi)\ , \tag{4.8c}$$

subject to the boundary conditions (2.7). We solve these equations by a perturbation expansion of the form (4.1), with the phase velocity given by

$$c=\omega/k\ . \tag{4.9}$$

Substituting these expressions into (4.8) and equating coefficients of $O(\epsilon)$, we once again recover the linear-eigenvalue problem with $R_0,\omega_0\equiv kc_0$ given by (3.2) and (3.3), and

$$\psi_1=\mathrm{Re}\,e^{ik\xi}\sin(\pi z)\ , \tag{4.10a}$$

$$\theta_1=\mathrm{Re}\frac{-p}{ik\sigma R_0}(\sigma-pikc_0)e^{ik\xi}\sin(\pi z)\ , \tag{4.10b}$$

$$\phi_1=\mathrm{Re}\frac{ik}{p}\frac{1}{1-ikc_0}e^{ik\xi}\sin(\pi z)\ . \tag{4.10c}$$

At $O(\epsilon^2)$, we again find that the requirement that solutions be periodic in ξ imposes no constraint on R_1,c_1, and hence take

$$R_1=c_1=0\ . \tag{4.11}$$

By choosing ϵ to be the amplitude of ψ_1, we can take $\psi_2=0$, and then

$$\theta_2=\mathrm{Re}A\sin(2\pi z),\quad \phi_2=\mathrm{Re}B\sin(2\pi z)\ , \tag{4.12}$$

where

$$A=-\frac{k^2}{8\pi p}\left[\frac{1}{1-ikc_0}\right],$$
$$B=-\frac{p}{8\pi\sigma R_0\tau}(\sigma-ikc_0)+\frac{1-\tau(1+S)}{1-ikc_0}\frac{k^2}{8\pi p\tau}\ . \tag{4.13}$$

Finally, at $O(\epsilon^3)$, we seek spatially periodic solutions of the problem

$$\begin{bmatrix}-pc_0\nabla^2\partial_\xi+\sigma\nabla^2 & 0 & -\sigma R_0\partial_\xi\\ -\partial_\xi & -pc_0\partial_\xi-\nabla^2 & 0\\ -(1+S)\partial_\xi & -[1-\tau(1+S)]\nabla^2 & -pc_0\partial_\xi-\tau\nabla^2\end{bmatrix}\begin{bmatrix}\psi_3\\ \theta_3\\ \phi_3\end{bmatrix}=\begin{bmatrix}pc_2\nabla^2\psi_{1\xi}+\sigma R_2\phi_{1\xi}-a[\partial_\xi(u_1^2\psi_{1\xi})+\partial_z(u_1^2\psi_{1z})]\\ pc_2\theta_{1\xi}-J(\psi_1,\theta_2)\\ pc_2\phi_{1\xi}-J(\psi_1,\phi_2)\end{bmatrix}. \tag{4.14}$$

Restricting (4.14) to the resonant terms yields the problem

$$\begin{bmatrix}p^2ikc_0-p\sigma & 0 & -ik\sigma R_0\\ -ik & p(1-ikc_0) & 0\\ -(1+S)ik & p[1-\tau(1+S)] & p(\tau-ikc_0)\end{bmatrix}\begin{bmatrix}\hat{\psi}_3\\ \hat{\theta}_3\\ \hat{\phi}_3\end{bmatrix}=\begin{bmatrix}-p^2ikc_2\hat{\psi}_1+ik\sigma R_2\phi_1+aQ\hat{\psi}_1\\ pikc_2\hat{\theta}_1+\frac{\pi}{2}ik(A+\bar{A})\hat{\psi}_1\\ pikc_2\hat{\phi}_1+\frac{\pi}{2}ik(B+\bar{B})\hat{\psi}_1\end{bmatrix}, \tag{4.15}$$

where

$$Q=(9k^4+2\pi^2k^2+9\pi^4)/16 , \tag{4.16}$$

and we have written

$$(\psi_3,\theta_3,\phi_3)=\mathrm{Re}(\hat{\psi}_3,\hat{\theta}_3,\hat{\phi}_3)e^{ik\xi}\sin(\pi z) .$$

If we multiply this equation from the left by the row vector

$$M^T=\{p^2(1-ikc_0)(\tau-ikc_0) ,\quad -p[1-\tau(1+S)]ik\sigma R_0, ik\sigma R_0 p(1-ikc_0)\} \tag{4.17}$$

the left side again vanishes. In order that the right side also vanish, R_2 and c_2 must satisfy the condition

$$p(2c_0c_2p+\tfrac{1}{8})(\sigma+p+\tau p-ikc_0p) +\sigma R_2[\tau+(1+\tau)S-(1+S)ikc_0] =\frac{ap}{k^2}(\tau-ikc_0)(1-ikc_0)Q , \tag{4.18}$$

which has the solution

$$R_2=\frac{ap}{\sigma}\frac{Q}{k^2}\left[\frac{(1+\tau)(\sigma+p)+p(\tau^2+k^2c_0^2)}{p+\sigma+\sigma S}\right]+O(a^2) ,$$
$$c_2=-\frac{1}{16pc_0}+O(a) . \tag{4.19}$$

Observe that the phase velocity of the wave decreases with amplitude ($c_2<0$), and that $R_2>0$. Thus the branch of traveling waves bifurcates supercritically, and we conclude that near R_0 the traveling waves are stable with respect to infinitesimal perturbations that are themselves periodic waves traveling in the same direction. On the other hand, no conclusion can be drawn about the stability of these waves with respect to waves traveling in the opposite direction. This problem is answered in Sec. IV on the basis of more general considerations.

Note that if $a=0$, i.e., there is no nonlinearity in the equation of motion, then the bifurcation to traveling waves is degenerate, and no conclusion about the bifurcating solutions can be drawn at this order. This is a familiar feature from studies of traveling waves in thermosolutal convection and Soret-driven convection in bulk mixtures.[7,15,16] As a result the above expansion is formally valid only for $\epsilon \ll (a/\sigma)^{1/2}$; when $a=0$ a higher-order calculation becomes necessary.[7]

C. Steady overturning convection

For completeness we give here the corresponding result for steady overturning convection near R_e. Setting the time derivatives equal to zero, and expanding as in Eqs. (4.1) we obtain

$$R_2=\frac{p}{8}\frac{1+(1+\tau^{-1}+\tau^{-2}+\tau^{-3})S}{[1+(1+\tau^{-1})S]^2}+O(a/\sigma) . \tag{4.20}$$

Since $S<S_c$ for oscillations, we conclude that when oscillations are present, $R_2<0$, and the bifurcation to steady convection is always subcritical. These solutions are unstable near R_e.

V. THE CODIMENSION-TWO BIFURCATION

In this section we study the interaction of standing and traveling waves with steady overturning convection. The analysis is carried out for large Prandtl numbers σ, for which the equations of motion reduce to[17]

$$0=R\phi_x-\nabla^2\psi , \tag{5.1a}$$

$$\theta_t=\psi_x+\nabla^2\theta-J(\psi,\theta) , \tag{5.1b}$$

$$\phi_t=(1+S)\psi_x+\tau\nabla^2\phi+[1-\tau(1+S)]\nabla^2\theta-J(\theta,\phi) , \tag{5.1c}$$

subject to the boundary conditions (2.7). We focus attention on the codimension-two bifurcation which occurs at[2]

$$R_c=\frac{p^2}{k^2}(1+\tau+\tau^2),\quad S_c=\frac{-\tau^2}{1+\tau+\tau^2} , \tag{5.2}$$

and study the solutions of (5.1) as a function of the small parameters $R-R_c, S-S_c$ denoting the departure from the multiple-bifurcation point.

The method we use is an extension of the iterative procedure employed in the first study of this kind.[4,18] In the present problem it leads directly to the normal form of the amplitude equation. We introduce a small parameter ϵ such that $R-R_c$, $S-S_c$ are both $O(\epsilon^2)$, and write $(R-R_c)/R_c=\epsilon^2\delta_R$, $(S-S_c)/S_c=\epsilon^2\delta_S$. Since the frequency of oscillation is then of $O(\epsilon)$ we define a slow time $t'=\epsilon pt$, and write

$$\psi=\mathrm{Re}[\epsilon v_1(t',\epsilon)e^{ikx}\sin(\pi z) +\epsilon^2 v_2(t',\epsilon)\sin(2\pi z)+O(\epsilon^3)] , \tag{5.3a}$$

$$\theta=\mathrm{Re}[\epsilon w_1(t',\epsilon)e^{ikx}\sin(\pi z) +\epsilon^2 w_2(t',\epsilon)\sin(2\pi z)+O(\epsilon^3)] , \tag{5.3b}$$

$$\phi=\mathrm{Re}[\epsilon z_1(t',\epsilon)e^{ikx}\sin(\pi z) +\epsilon^2 z_2(t',\epsilon)\sin(2\pi z)+O(\epsilon^3)] . \tag{5.3c}$$

Here $v_i(t',\epsilon), w_i(t',\epsilon), z_i(t',\epsilon)$, $i=1,2,\ldots$, are complex amplitudes depending on both t' and ϵ. The spatial part of the expansion follows directly from the analysis below, but is written explicitly in (5.3) to save space. In the calculation that follows we retain the complex notation (cf. Sec. IV A). We substitute into Eqs. (5.1) and equate terms of like spatial structure:

$$0=R_c ikz_1+pv_1+\epsilon^2\delta_R ikR_c z_1 , \tag{5.4a}$$

$$0=4\pi^2 v_2 , \tag{5.4b}$$

$$\epsilon p w_1'=ikv_1-pw_1+\frac{ik\pi}{2}\epsilon^2(v_1w_2^*+v_1w_2-w_1v_2^* -w_1v_2)+O(\epsilon^4) , \tag{5.4c}$$

$$\epsilon p w_2'=-4\pi^2 w_2+\frac{ik\pi}{4}(v_1^*w_1-v_1w_1^*)+O(\epsilon^2) , \tag{5.4d}$$

$$\epsilon p z_1'=(1+S_c)ikv_1-\tau p z_1-p[1-\tau(1+S_c)]w_1 +\epsilon^2\delta_S S_c ikv_1+\epsilon^2 p\tau\delta_S S_c w_1 +\frac{ik\pi}{2}\epsilon^2(v_1z_2+v_1z_2^*)+O(\epsilon^4) , \tag{5.4e}$$

$$\epsilon p z_2' = -4\pi^2 \epsilon z_2 - 4\pi^2[1-\tau(1+S_c)]w_2 - \frac{\pi}{2R_c}|v_1|^2 + O(\epsilon^2) , \quad (5.4f)$$

where the prime denotes derivatives with respect to the slow time t'. Using the fact that ϵ is small, we observe that Eqs. (5.4a), (5.4c), (5.4d), and (5.4f) imply

$$w_1 = \frac{ik}{p} v_1 + \epsilon g(t', \epsilon) , \quad (5.5a)$$

$$w_2 = \frac{-k^2}{8\pi p} |v_1|^2 + \epsilon h(t', \epsilon) , \quad (5.5b)$$

$$z_1 = \frac{-p}{ikR_c} v_1 + \epsilon k(t', \epsilon) , \quad (5.5c)$$

$$z_2 = \epsilon l(t', \epsilon) , \quad (5.5d)$$

where the functions g, h, k, and l are $O(1)$ at leading order. These functions can be calculated in powers of ϵ by a simple iterative procedure. We illustrate the method by computing h. Substituting (5.5a) into (5.4d), we find

$$w_2' = -\tilde{\omega} h + \frac{ik\pi}{4p}(v_1^* g - v_1 g^*) + O(\epsilon) , \quad (5.6)$$

where $\tilde{\omega} \equiv 4\pi^2/p$. Similarly, from (5.4c), we obtain at leading order that

$$w_1' = -g + O(\epsilon) . \quad (5.7)$$

Substituting for w_1' and w_2' from (5.5a) and (5.5b) and eliminating g we find

$$h = \frac{\pi k^2}{4p^2 \tilde{\omega}} \left[1 + \frac{2}{\tilde{\omega}} \right] \frac{d}{dt'} |v_1|^2 + O(\epsilon) . \quad (5.8)$$

Similarly,

$$l = -\frac{[1-\tau(1+S_c)]}{\tau} h + O(\epsilon) . \quad (5.9)$$

The function g has to be calculated to higher order and makes use of (5.8) and (5.9). The result is

$$g = -\frac{ik}{p} v_1' + \frac{\epsilon ik}{p} v_1'' - \frac{\epsilon i k^3}{8p^2} v_1 |v_1|^2 + \frac{\epsilon^2 i k^3}{16 p^2} \left[1 + \frac{2}{\tilde{\omega}} \right] v_1 \frac{d}{dt'} |v_1|^2 + \frac{\epsilon^2 i k^3}{8p^2} \frac{d}{dt'} v_1 |v_1|^2 - \frac{\epsilon^2 ik}{p} v_1''' + O(\epsilon^3) . \quad (5.10)$$

With the above results we know z_1, w_1 and z_2 in terms of v_1 to sufficiently high order in ϵ. Substituting the resulting expressions into Eqs. (5.4e) we find that the $O(\epsilon)$ and $O(\epsilon^2)$ terms cancel. This is because with the chosen R_c, S_c the linear problem has two zero eigenvalues. The remaining terms can be written, after canceling ϵ^2 throughout, in the form

$$v_1'' = [\tau\delta_R - (1+\tau)\tau^2\delta_S] v_1 + \epsilon(\delta_R - \tau^3\delta_S) v_1' + \frac{k^2}{8p}|v_1|^2 v_1 - \frac{\epsilon k^2}{16p}\left[\frac{1+\tau}{\tau}\right]\left[1+\frac{2}{\tilde{\omega}}\right] v_1 \frac{d}{dt'}|v_1|^2 - \frac{\epsilon k^2}{8p}\frac{d}{dt'} v_1 |v_1|^2 + \epsilon v_1''' + O(\epsilon^2) . \quad (5.11)$$

Finally, we observe that this equation implies

$$v_1'' = [\tau\delta_R - (1+\tau)\tau^2\delta_S] v_1 + \frac{k^2}{8p}|v_1|^2 v_1 + O(\epsilon) , \quad (5.12)$$

and so we can eliminate v''' from (5.11). We then obtain the amplitude equation in the normal form

$$v'' = \mu v + \epsilon \nu v' + [A|v|^2 + \epsilon C(v v'^* + v' v^*)] v + O(\epsilon^2) \quad (5.13)$$

with

$$A = \frac{k^2}{8p}, \quad C = -\frac{k^2}{16p}\left[\frac{1+\tau}{\tau}\right]\left[1+\frac{2}{\tilde{\omega}}\right] , \quad (5.14)$$

and

$$\mu = \tau\delta_R - (1+\tau)\tau^2\delta_S, \quad \nu = (1+\tau)\delta_R - \tau^2\delta_S . \quad (5.15)$$

In Sec. VI we obtain the structure of the generic normal form describing this bifurcation on the basis of symmetry arguments, and describe the nature of the solutions, and the appropriate bifurcation diagram corresponding to varying δ_R (i.e., the Rayleigh number) for fixed δ_S (i.e., the separation constant). Here we content ourselves to note that when v is real the analysis we have done describes the codimension-two bifurcation with $Z(2)$ reflectional symmetry that describes the interaction of standing waves and stationary convection. In this case Eqs. (5.13)–(5.15) reduce to those obtained in Ref. 2.

VI. AMPLITUDE EQUATIONS AND NORMAL FORMS

In this section we study the two bifurcations considered in Secs. IV and V, the codimension-one Hopf bifurcation and the codimension-two Takens-Bogdanov bifurcation, from a general point of view. For problems with periodic boundary conditions on a line the translations introduce an additional symmetry that is absent in the standard treatment of these bifurcations. It is this new symmetry that is responsible for the wealth of new phenomena observed near these bifurcations.

A. Symmetry considerations

Consider a continuous translationally invariant system on a line, with no distinction between left and right, and seek spatially periodic solutions with period $d = 2\pi/k$ near a parameter value at which a mode of wave number k loses stability with one real eigenvalue crossing into the right half plane. Near this bifurcation the problem is described by a single complex amplitude $v(t)$ of the linearized vertical velocity $w(x,t)$, defined by

$$w(x,z,t)=(ve^{ikx}+v^*e^{-ikx})\sin(\pi z)\ . \tag{6.1}$$

Then invariance with respect to horizontal translations by an amount l: $x \to x+l$ (mod$d$) induces the action

$$\mathrm{SO}(2)\colon\ v \to e^{ikl}v\ , \tag{6.2}$$

i.e., a rotation of the amplitude v, while invariance with respect to reflections in $x=0$ induces the action

$$Z(2)\colon\ v\to v^*\ . \tag{6.3}$$

Hence the dynamical equation for the amplitude v must be equivariant with respect to the representation (6.2) and (6.3) of the group $\mathrm{SO}(2)\times Z(2) \simeq \mathrm{O}(2)$, i.e., the group of rotations and reflections of a circle.

B. Steady-state bifurcation with O(2) symmetry

Since the only quantity that transforms like v is v itself, it follows that any smooth equivariant vector field has to be of the form

$$\dot v = g(\sigma,\lambda)v\ , \tag{6.4}$$

where g is a C^∞ invariant function, i.e., a function of the only invariant $\sigma \equiv |v|^2$ of the representation (6.2) of SO(2), as well as of the bifurcation parameter $\lambda \propto R-R_e$. In order for (6.4) to be equivariant under (6.3), the function g must be real. Equation (6.4) can be written in terms of real variables r,φ defined by $v=re^{i\varphi}$, with the result

$$\dot r = g(r^2,\lambda)r,\quad \dot\varphi=0\ . \tag{6.5}$$

Hence the bifurcation is a pitchfork. Note that at $\lambda=0$, there are in fact two zero eigenvalues, and that the finite-amplitude solutions break both the $Z(2)$ and SO(2) symmetries.

C. Hopf bifurcation with O(2) symmetry

At the point R_0 the general solution of the linear problem is a linear superposition of left- and right-traveling waves. We shall write it in the form

$$w(x,z,t)=[(v+w)e^{ikx}+(v^*+w^*)e^{-ikx}]\sin(\pi z)\ , \tag{6.6}$$

where

$$\begin{bmatrix}\dot v\\ \dot w\end{bmatrix}=\begin{bmatrix} i\omega_0 & 0\\ 0 & -i\omega_0\end{bmatrix}\begin{bmatrix} v\\ w\end{bmatrix}, \tag{6.7}$$

and v,w are the complex amplitudes of the two waves. Because the amplitudes are complex, each eigenvalue occurs twice, and there are therefore four eigenvalues on the imaginary axis simultaneously; the center manifold is therefore four dimensional.

The translation and reflection symmetries induce the following action of O(2) on the amplitudes v,w:

$$\mathrm{SO}(2)\colon\ (v,w)\to e^{ikl}(v,w)\ , \tag{6.8a}$$

$$Z(2)\colon\ (v,w)\to(w^*,v^*)\ . \tag{6.8b}$$

As before the aim of the theory is to construct amplitude equations which are equivariant under the representation (6.8) of O(2). We begin by observing that there are three independent invariants of (6.8):

$$\sigma_1=|v|^2+|w|^2,\quad \sigma_2=vw^*,\quad \sigma_3=v^*w\ . \tag{6.9}$$

The most general smooth equivariant vector field is then of the form

$$\begin{bmatrix}\dot v\\ \dot w\end{bmatrix}=\begin{bmatrix} g_1 & g_2\\ g_2^* & g_1^*\end{bmatrix}\begin{bmatrix} v\\ w\end{bmatrix}, \tag{6.10}$$

where $g_j=g_j(\sigma_1,\sigma_2,\sigma_3)$, $j=1,2$, are C^∞ complex functions. This is therefore the form of the system after reduction to the four-dimensional center manifold at $R=R_0$. The equations can be further simplified by smooth near-identity nonlinear equivariant coordinate changes of the form

$$\begin{bmatrix}\tilde v\\ \tilde w\end{bmatrix}=\begin{bmatrix} v\\ w\end{bmatrix}+\begin{bmatrix} h_1 & h_2\\ h_2^* & h_1^*\end{bmatrix}\begin{bmatrix} v\\ w\end{bmatrix}, \tag{6.11}$$

where $h_j=h_j(\sigma_1,\sigma_2,\sigma_3)$, $j=1,2$, are C^∞ complex functions that vanish at the origin.

We indicate explicitly the necessary coordinate changes for the cubic terms in (6.10). To this order we have

$$\begin{aligned} \dot v &= i\omega_0 v + A|v|^2v + B|w|^2v + Cv^2w^* + D|v|^2w\\ &\quad + E|w|^2w + Fw^2v^* + O(5)\ , \end{aligned} \tag{6.12}$$

and carry out a coordinate change of the form

$$\begin{bmatrix} v\\ w\end{bmatrix}=\begin{bmatrix}\tilde v\\ \tilde w\end{bmatrix}+\begin{bmatrix} \alpha|v|^2v+\beta|w|^2v+\gamma v^2w^*+\delta|v|^2w+\epsilon|w|^2w+\eta w^2v^*\\ \alpha^*|w|^2w+\beta^*|v|^2w+\gamma^*w^2v^*+\delta^*|w|^2v+\epsilon^*|v|^2v+\eta^*v^2w^*\end{bmatrix}+O(5)\ , \tag{6.13}$$

where the complex coefficients $\alpha,\dots,\eta$ are to be chosen to achieve maximum simplification of (6.12). In terms of $\tilde v,\tilde w$ Eq. (6.12) becomes

$$\begin{aligned}\dot{\tilde v} &= i\omega_0(\tilde v - 2\gamma\tilde v^2\tilde w^* + 2\delta|\tilde v|^2\tilde w + 2\epsilon|\tilde w|^2\tilde w + 4\eta\tilde w^2\tilde v^*)\\ &\quad + A|\tilde v|^2\tilde v + B|\tilde w|^2\tilde v + C\tilde v^2\tilde w^* + D|\tilde v|^2\tilde w + E|\tilde w|^2\tilde w + F\tilde w^2\tilde v^* + O(5)\ ,\end{aligned}$$

showing that the last four terms in (6.12) can be eliminated by a suitable coordinate change. Dropping the tildes, we see that (6.10) can be written in the form

$$\begin{bmatrix}\dot v\\ \dot w\end{bmatrix}=\begin{bmatrix} i\omega_0+A|v|^2+B|w|^2 & 0\\ 0 & -i\omega_0+B^*|v|^2+A^*|w|^2\end{bmatrix}\begin{bmatrix} v\\ w\end{bmatrix}+O(5)\ , \tag{6.14}$$

with an unfolding which we write in the form

$$\dot{v}=v[\lambda+i\omega+a\,|w|^2+b(|v|^2+|w|^2)] +O(5)+\lambda O(3) \tag{6.15a}$$

$$\dot{w}=w[\lambda-i\omega+a^*\,|v|^2+b^*(|v|^2+|w|^2)] +O(5)+\lambda O(3)\ , \tag{6.15b}$$

where $\lambda \propto R-R_0$, $\omega-\omega_0=O(\lambda)$, and a and b are complex constants to be determined from the equations of motion. These are the amplitude equations in normal form to third order in the amplitude.

The above procedure generalizes to all orders in perturbation theory, with the result that (6.10) can be written in the Poincaré-Birkhoff normal form

$$\begin{bmatrix}\dot{v}\\ \dot{w}\end{bmatrix}=\begin{bmatrix}g(\sigma_1,|w|^2) & 0\\ 0 & g(\sigma_1,|v|^2)\end{bmatrix}\begin{bmatrix}v\\ w\end{bmatrix} \tag{6.16}$$

by a suitable coordinate change of the form (6.11). Observe that in this form the equations are independent of the phases, i.e., they are invariant under phases changes $t\to t+\theta/\omega_0$. This symmetry induces the action

$$(v,w)\to(e^{i\theta}v,e^{-i\theta}w)\ , \tag{6.17}$$

which we shall call $S(1)$ to distinguish it from the SO(2) translational symmetry. If the presence of this symmetry is assumed at the outset, then by appropriately coupling spatial translations with phase shifts, we obtain the new symmetries[16]

$$(v,w)\to(e^{2i\theta}v,w),\quad (v,w)\to(v,e^{-2i\theta}w)\ . \tag{6.18}$$

For example, the first symmetry states that displacing a right traveling wave a distance l to the left and then allowing it to evolve in time for the time kl/ω_0 leaves it unchanged. Since the invariants of (6.18) are the functions $|v|^2$, $|w|^2$, we see that an assumed invariance with respect to phase would lead to the Poincaré-Birkhoff normal form (6.16). However, the phase-shift symmetry is only a symmetry of the normal form, and is not an exact symmetry of the physical system. As a result there will be terms in the tail of the Taylor series that break this symmetry. Under certain conditions these terms can become important.

The general theory described above enables us to draw important conclusions. We begin by studying the solutions of Eqs. (6.15) truncated at third order, which we write as the four real equations

$$\dot{x}_1=x_1(\lambda+a_r x_2^2+b_r A^2)\ , \tag{6.19a}$$

$$\dot{x}_2=x_2(\lambda+a_r x_1^2+b_r A^2)\ , \tag{6.19b}$$

$$\dot{\varphi}_1=\omega+a_i x_2^2+b_i A^2\ , \tag{6.19c}$$

$$\dot{\varphi}_2=-\omega-a_i x_1^2-b_i A^2\ , \tag{6.19d}$$

where $(v,w)=x_j\exp(i\varphi_j)$, $j=1,2$, and subscripts r,i denote real and imaginary parts, respectively. The total amplitude A^2 is defined by

$$A^2=x_1^2+x_2^2\ , \tag{6.20}$$

and is proportional to $N-1$, where N is the Nusselt number. As explained above the phases decouple from the amplitude equations (6.19a) and (6.19b). These equations possess four stationary solutions of the form (x_1,x_2): the conduction solution (0,0), the left- and right-traveling waves $(x,0)$, $(0,x)$ and the standing wave (x,x). There are no limit cycles in (x_1,x_2). The amplitude A and stability properties of these solutions as a function of the bifurcation parameter λ are summarized in Table I, and exhibited in the form of bifurcation diagrams in the (a_r,b_r) plane in Fig. 1.[20] Observe that two solution branches bifurcate from $\lambda=0$; this is made possible by the doubling of the eigenvalues due to the O(2) symmetry. Stable solutions are found near the bifurcation only when both branches bifurcate supercritically, and the stable solution is the one with the higher Nusselt number. This conclusion is independent of the nature of the boundary conditions, provided these preserve the symmetry properties.

The primary solution branches can be classified by their symmetries.[20] Since the normal form has the symmetry $SO(2)\times Z(2)\times S(1)$, it is possible to have spatiotemporal symmetries. The standing waves break the SO(2) translational symmetry, but respect the reflection and phase-shift symmetries. The traveling waves break the $Z(2)$ reflectional symmetry, but are invariant under translations followed by appropriate phase shifts [i.e., under the twisted subgroup of $SO(2)\times S(1)$].[21]

The results obtained in Sec. IV are of course special cases of the general theory described above. However, we can use these results to obtain the coefficients a_r,b_r for convection in binary mixtures in a porous medium, and then use the theory to deduce the relative stability of standing and traveling waves near the bifurcation. If we take $\lambda=R-R_0$, then from Table I we conclude that

$$R^{\rm SW}=R_0-(a_r+2b_r)x^2\ ,\qquad R^{\rm TW}=R_0-b_r x^2\ , \tag{6.21}$$

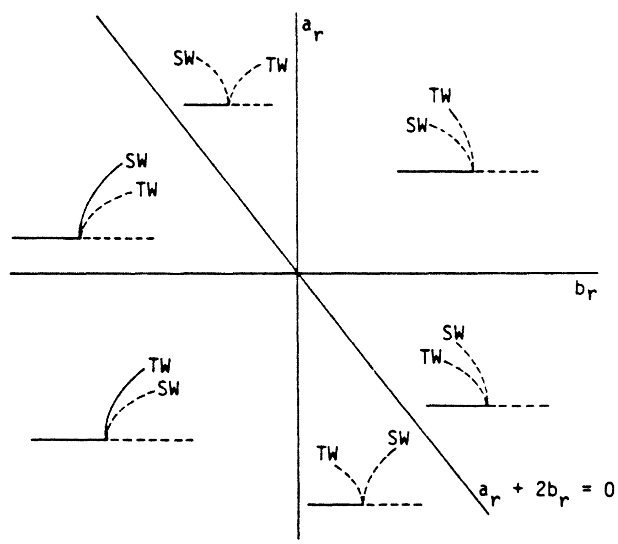

FIG. 1. Bifurcation diagrams in the (a_r,b_r) plane for the Hopf bifurcation with O(2) symmetry, showing the amplitude A of standing (SW) and traveling (TW) waves, as a function of the Rayleigh number. Continuous lines indicate stable solutions, dashed lines unstable ones.

TABLE I. Solution branches as a function of the bifurcation parameter λ.

Solution	Equation	Eigenvalues	Name
(0,0)	$A^2=0$	λ,λ	ϕ
$(x,0)$	$\lambda+b_r A^2=0$	$-2\lambda, -\lambda a_r/b_r$	TW
(x,x)	$\lambda+(\frac{1}{2}a_r+b_r)A^2=0$	$-2\lambda, \lambda a_r/(\frac{1}{2}a_r+b_r)$	SW

where x is the amplitude of the left-traveling wave component of $w(x,z,t)$. On comparing with the results of Sec. IV, we conclude that, depending on the parameters, the coefficient a_r can take either sign, and that b_r is small and negative. Thus when $a_r \gtrsim 0$ there are no stable solutions near bifurcation, but when $a_r < 0$ stable traveling waves should be observed near bifurcation. This is the case for $S \lesssim S_c$. Since b_r is small in magnitude the standing waves are stable only in a very narrow parameter regime.

The amplitude equations (6.15) can also be derived directly from the governing equations using a procedure similar to that used in Sec. V.

D. The Takens-Bogdanov bifurcation with O(2) symmetry

The interaction of standing and traveling waves with steady states in the nonlinear regime can be studied by unfolding the codimension-two bifurcation which occurs at $R=R_c$, $S=S_c$. Since the Hopf bifurcation with O(2) symmetry will occur in the unfolding we must introduce two complex amplitudes (v,w) as in (6.6), in terms of which the linearized problem at $R=R_c$, $S=S_c$ takes the form

$$\begin{bmatrix}\dot v\\ \dot w\end{bmatrix}=\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}v\\ w\end{bmatrix}. \tag{6.22}$$

However, because the eigenvalues are real (note that there are *four* zero eigenvalues), the appropriate representation of the O(2) group is

$$(v,w)\rightarrow e^{ikl}(v,w)\,, \tag{6.23a}$$

$$(v,w)\rightarrow(v^*,w^*)\,, \tag{6.23b}$$

as in our discussion of the steady-state bifurcation. To construct amplitude equations for (v,w) reflecting the invariance of the problem under spatial translations and reflections, we first write down the three basic invariants under (6.23):

$$\sigma_1=|v|^2,\quad \sigma_2=|w|^2,\quad \sigma_3=vw^*+v^*w\,. \tag{6.24}$$

The most general smooth equivalent vector field is then given by

$$\begin{bmatrix}\dot v\\ \dot w\end{bmatrix}=\begin{bmatrix}g_1&g_2\\g_3&g_4\end{bmatrix}\begin{bmatrix}v\\ w\end{bmatrix}, \tag{6.25}$$

where by (6.23b) the functions $g_j=g_j(\sigma_1,\sigma_2,\sigma_3)$, $j=1,2,3,4$, are smooth real functions of the three invariants.

If we expand the functions $g_j(\sigma_1,\sigma_2,\sigma_3)$, $j=1,2,3,4$, in a Taylor series about the origin, and demand that the linear problem be of the form (6.22), then to third order we obtain the equations

$$\dot v=w+(a_1|v|^2+b_1|w|^2)v+c_1v^2w^* +(a_2|v|^2+b_2|w|^2)w+c_2v^*w^2+O(5)\,, \tag{6.26a}$$

$$\dot w=(a_3|v|^2+b_3|w|^2)v+c_3v^2w^* +(a_4|v|^2+b_4|w|^2)w+c_4v^*w^2+O(5)\,, \tag{6.26b}$$

where a_j,b_j,c_j, $j=1,2,3,4$, are real coefficients. This is the form of the amplitude equations on the four-dimensional (real) center manifold. As we did for the Hopf bifurcation, we can simplify these equations by a smooth O(2)-equivariant coordinate change of the form

$$\begin{bmatrix}\tilde v\\ \tilde w\end{bmatrix}=\begin{bmatrix}v\\ w\end{bmatrix}+\begin{bmatrix}h_1&h_2\\h_3&h_4\end{bmatrix}\begin{bmatrix}v\\ w\end{bmatrix}, \tag{6.27}$$

where h_j, $j=1,2,3,4$, are smooth real functions vanishing at the origin. To simplify the third-order terms in (6.26), we expand (6.27) in the form

$$v=\tilde v+(\alpha_1|v|^2+\beta_1|w|^2)v+\gamma_1v^2w^* +(\alpha_2|v|^2+\beta_2|w|^2)w+\gamma_2v^*w^2+O(5)\,, \tag{6.28a}$$

$$w=\tilde w+(\alpha_3|v|^2+\beta_3|w|^2)v+\gamma_3v^2w^* +(\alpha_4|v|^2+\beta_4|w|^2)w+\gamma_4v^*w^2+O(5)\,, \tag{6.28b}$$

and write (6.26) in terms of the new variables:

$$\begin{aligned}\dot{\tilde v}&=\tilde w+(a_1+\alpha_3)|\tilde v|^2\tilde v+(a_2-2\alpha_1+\alpha_4)|\tilde v|^2\tilde w+(b_1-\alpha_2+\beta_3-2\gamma_1)|\tilde w|^2\tilde v+(b_2-\beta_1+\beta_4-\gamma_2)|\tilde w|^2\tilde w\\&\quad+(c_1-\alpha_1+\gamma_3)\tilde v^2\tilde w^*+(c_2-\alpha_2+\gamma_4)\tilde w^2\tilde v^*+O(5)\,,\\ \dot{\tilde w}&=a_3|\tilde v|^2\tilde v+(a_4-2\alpha_3)|\tilde v|^2\tilde w+(b_3-\alpha_4-2\gamma_3)|\tilde w|^2\tilde v+(b_4-\beta_3-\gamma_4)|\tilde w|^2\tilde w+(c_3-\alpha_3)\tilde v^2\tilde w^*\\&\quad+(c_4-\alpha_4)\tilde w^2\tilde v^*+O(5)\,.\end{aligned} \tag{6.29}$$

It is convenient to choose the real coefficients α_j, β_j, γ_j, $j=1,2,3,4$, to eliminate all the cubic terms in (6.29a). We can also eliminate the fourth and sixth terms in (6.29b). This leaves four terms in (6.29b) which cannot be removed by a transformation of the form (6.28). Hence to third order, we are left with the dynamical system[8]

$$\dot{v}=w \; , \tag{6.30a}$$

$$\dot{w}=[A\,|v|^2+B\,|w|^2+C(vw^*+v^*w)]v+D\,|v|^2w \; , \tag{6.30b}$$

where

$$A=a_3, \quad B=b_3-a_2+2c_1-2c_4 \; , \qquad C=c_3+a_1, \quad D=a_4+a_1-c_3 \; , \tag{6.31}$$

and the tildes on (v,w) have been dropped.

In order to study the dynamics near this codimension-two bifurcation, we need to unfold the normal form (6.30). Without loss of generality, we add small linear equivariant terms to (6.30b):

$$\dot{v}=w \; , \tag{6.32a}$$

$$\dot{w}=\mu v+\nu w+[A\,|v|^2+B\,|w|^2+C(vw^*+v^*w)]v +D\,|v|^2w \; . \tag{6.32b}$$

Here μ,ν are the two unfolding parameters, and are linearly related to $R-R_c$, $S-S_c$. Equations (6.32) are the desired amplitude equations near this codimension-two bifurcation. As in Sec. V, we can measure the proximity to the bifurcation in terms of a small parameter ϵ, and introduce a slow time $t'=\epsilon t$, and scale v and the unfolding parameters according to

$$v\rightarrow\epsilon v, \quad \mu\rightarrow\epsilon^2\mu, \quad \nu\rightarrow\epsilon^2\nu \; . \tag{6.33}$$

With this scaling the amplitude equation (6.32) becomes[8]

$$v''=(\mu+A\,|v|^2)v+\epsilon[\nu v'+C(vv'^*+v'v^*)v +D\,|v|^2v']+O(\epsilon^2) \; , \tag{6.34}$$

the prime denoting differentiation with respect to the slow time t'. Note that the coefficient B does not enter at leading order in ϵ. To this order the solutions therefore depend on the choice of the three coefficients A,C,D and the unfolding parameters μ,ν. This is in contrast to the $Z(2)$-equivariant problem, in which the solutions depend on two coefficients only.

A complete discussion of the dynamics described by the normal form (6.34) is given in Ref. 8.[22] Here we use these results to make specific predictions for convection in binary fluid mixtures in a porous medium. The results of Sec. V give the coefficients A,C in the normal form; in addition, from Sec. IV we deduce that $D<0$ and is of order a/σ. In this case the normal form (6.34) is nondegenerate, and the analysis of Ref. 8 applies. We show in Fig. 2 the (D,M) plane for $A>0$, where $M=2C+D\simeq 2C$. The plane divides into eight regions in which different bifurcation diagrams are found. Since $D,M<0$, and $0<D/M\ll 1$, the convection problem we are investigating lies in region II$^-$. The corresponding (μ,ν) plane (Fig. 3) is divided into a number of regions in which different types of solutions are found. The plane is traversed by the line

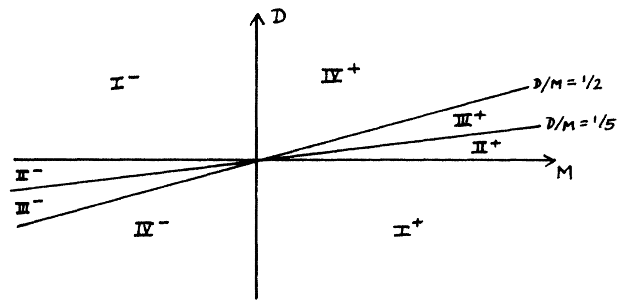

FIG. 2. The (D,M) plane for $A>0$ splits into 8 regions in which the amplitude equation (6.34) exhibits qualitatively different dynamics. The equation is degenerate on the boundaries separating the different regions. The convection system studied falls into region II$^-$.

$$\tau\nu-(1+\tau)\mu=\tau^2(1+\tau+\tau^2)\delta_S \; , \tag{6.35}$$

corresponding to increasing the Rayleigh number R for fixed $\delta_S>0$. By following this line from left to right a succession of transitions is seen to occur, and is represented in the bifurcation diagram shown in Fig. 4. Both traveling (TW) and standing (SW) waves bifurcate supercritically, with the larger-amplitude TW branch initially stable. As R is increased a secondary Hopf bifurcation takes place producing a stable branch of modulated (MW) waves (i.e., waves with two independent frequencies). The MW branch terminates on the TW branch in a global bifurcation at $R_c^{\rm MW}$, at which the new frequency vanishes. The unstable TW branch terminates at $R_c^{\rm TW}$ on the branch of steady overturning convection (SS) which bifurcates subcritically from R_e; as R approaches $R_c^{\rm TW}$ the frequency of the traveling waves approaches zero as $(R_c^{\rm TW}-R)$.[8] This is in contrast to the (unstable) SW branch which also terminates on the SS branch, but

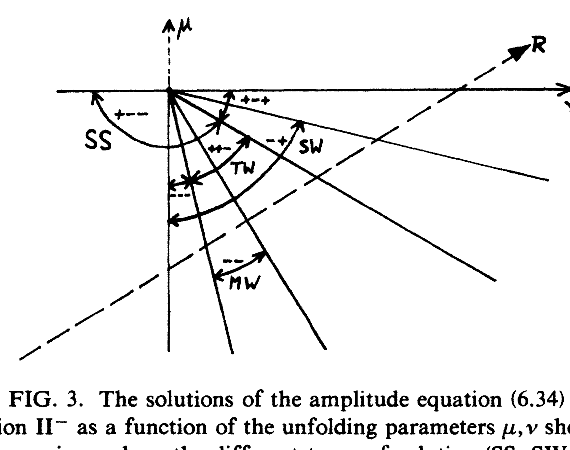

FIG. 3. The solutions of the amplitude equation (6.34) in region II$^-$ as a function of the unfolding parameters μ,ν showing the regions where the different types of solution (SS, SW, TW, MW) are located, and their stability. Codimension-one bifurcations occur along the lines separating the different regions. The broken line indicates the succession of transitions as the Rayleigh number R is increased, for fixed $S<S_c$ [Eq. (6.35)].

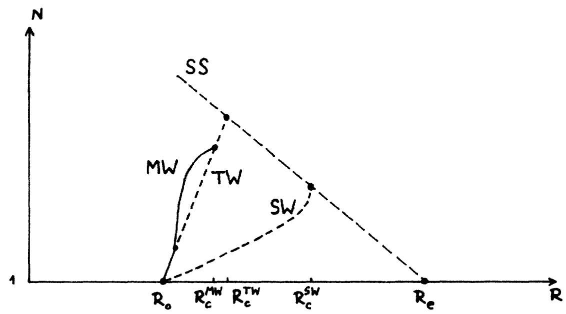

FIG. 4. The bifurcation diagram showing the Nusselt number as a function of the Rayleigh number R, for fixed $S < S_c$.

with frequency vanishing as $-1/\ln(R_c^{SW}-R)$.[4] Note that R_c^{TW} and R_c^{SW} differ in general. Thus for the present idealized system the initial Hopf bifurcation leads to stable traveling waves, followed by a secondary bifurcation which produces modulated waves. These waves disappear at R_c^{MW}, where an abrupt (hysteretic) transition to large amplitude overturning convection takes place. This new solution is outside the domain of validity of the bifurcation analysis.

VII. DISCUSSION

In this paper we have studied oscillatory convection in a binary mixture in a porous medium heated from below to illustrate both the techniques and the richness of behavior that can be exhibited by such systems. The theory described here emphasizes the role played by the symmetries of the system, the O(2) symmetry induced by horizontal translations mod$(2\pi/k)$ and reflections in a vertical plane, and shows clearly how the structure of the amplitude equations and hence the dynamics of the system, is determined by the symmetry. Thus the structure of the amplitude equations will not change when the boundary conditions are changed, provided the symmetries remain unaffected.[23] A number of conclusions that are independent of the boundary conditions can therefore be drawn. For example, if both SW and TW branches bifurcate supercritically, the one with the larger Nusselt number will be stable. We have seen that the symmetry produces multiple branches in cases where a single branch would be present in the absence of symmetry, and allows one to classify the bifurcating solutions in terms of their symmetries. We have shown now the amplitude equations can be put into normal form by means of near-identity coordinate changes, and how the normal form can be extracted from the partial differential equations. The perturbation methods described are the simplest and most direct methods for deriving the amplitude equations. The results obtained are justified by the general theory. The general theory also makes apparent any degeneracies in the amplitude equations, since it describes their generic structure.

The theory enabled us to make specific predictions for the system under consideration. However, the imposition of realistic no-slip no-solute-flux boundary conditions at top and bottom is expected to affect the values of the coefficients in the normal form of the amplitude equations. Thus conclusions based on the relative sizes of the coefficients computed using idealized boundary conditions are not a reliable guide to the experiments. However, it is possible to extract from the present analysis those features that might be expected to be robust and observable in the experiments.[3,6] These are summarized below.

(i) When standing waves are present the Nusselt number oscillates with frequency 2ω, where ω is the frequency of the basic oscillation. When traveling waves are present, the Nusselt number is constant except for small-amplitude modulation, also at the frequency 2ω, due to the presence of the sidewalls. This constant Nusselt number is unrelated to the Nusselt number for stable steady overturning convection at the same Rayleigh number. In the experiment,[6] this modulation is observed at the 1% level. In small-aspect-ratio experiments in which the pattern is not visualized some care should therefore be taken to distinguish standing waves from (spatially) modulated traveling waves. This can be done as in (ii) below.

(ii) The branch of standing waves terminates in a heteroclinic orbit joining two unstable steady states at R_c^{SW}. As R_c^{SW} is approached, the frequency of the oscillation vanishes as $-1/\ln(R_c^{SW}-R)$. This qualitative behavior is seen in Ref. 3.

(iii) The branch of traveling waves terminates in a steady-state bifurcation and the frequency of the waves vanishes as $(R_c^{TW}-R)$. Evidence for this qualitative behavior is given in Ref. 6.

(iv) The transition to steady overturning convection is hysteretic.

(v) Bifurcation from traveling to modulated (i.e., two frequency) waves can occur.[8]

(vi) Chaos via the Šil'nikov mechanism can occur on the SW branch, and is associated with a heteroclinic orbit of saddle-focus type.[24]

(vii) Chaos on the TW branch near R_0 is most likely to occur near parameter values for which the Hopf bifurcation at R_0 is doubly degenerate, and could appear after two successive Hopf bifurcations produced triply periodic waves.[25]

These observations are robust for two-dimensional translationally invariant systems in which R_0, R_e, and R are all of the same order of magnitude. We expect, on the basis of experimental[6] and theoretical[23] considerations, that much of the phenomena associated with the traveling waves will persist in the presence of sidewalls, and we have listed these above. An exception is provided by the modulated waves in which the translation invariance prohibits frequency locking.[26] In the presence of sidewalls we expect such waves to exhibit locking. In addition, if substantially three-dimensional motions take place, different dynamical phenomena should be observed. Further experiments will undoubtedly test these suggestions in the near future.

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[1]E. Knobloch, Phys. Fluids **23**, 1918 (1980).

[2]H. Brand, P. C. Hohenberg, and V. Steinberg, Phys. Rev. A **30**, 2548 (1984).

[3]I. Rehberg and G. Ahlers, Phys. Rev. Lett. **55**, 500 (1985).

[4]E. Knobloch and M. R. E. Proctor, J. Fluid Mech. **108**, 291 (1981).

[5]J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer-Verlag, New York, 1983).

[6]R. Walden, P. Kolodner, A. Passner, and C. M. Surko, Phys. Rev. Lett. **55**, 496 (1985). Note, however, that in this experiment the traveling waves do not bifurcate from the conduction solution.

[7]E. Knobloch, in *Double Diffusive Motions,* Proceedings of the Joint ASCE-ASME Mechanics Conference, Albuquerque, New Mexico, 1985, edited by N. E. Bixler and E. A. Spiegel (ASME, New York, 1985), FED **24**, 17 (1985); E. Knobloch, A. Deane, J. Toomre, and D. R. Moore, in *Multiparameter Bifurcation Theory,* Vol. 56 of *Contemporary Mathematics,* edited by M. Golubitsky and J. Guckenheimer (American Mathematical Society, Providence, Rhode Island, in press).

[8]G. Dangelmayr and E. Knobloch, Philos. Trans. R. Soc. London (to be published).

[9]D. T. J. Hurle and E. Jakeman, J. Fluid Mech. **47**, 667 (1971); J. K. Platten and G. Chavepeyer, *ibid.* **60**, 305 (1973).

[10]H. Brand and V. Steinberg, Physica **119A**, 327 (1983).

[11]The result of Sec. IV B that the traveling waves bifurcate supercritically when the nonlinearity increases the friction above its Darcy value is expected to be independent of this assumption.

[12]In Refs. 1 and 9, S is defined with the opposite sign.

[13]The most direct way of doing this calculation is described by E. Knobloch, N. O. Weiss, and L. N. Da Costa, J. Fluid Mech. **113**, 153 (1981).

[14]H. Brand and V. Steinberg, Phys. Lett. **93A**, 333 (1983).

[15]C. Bretherton and E. A. Spiegel, Phys. Lett. **96A**, 152 (1983).

[16]J. W. Swift, *Bifurcation and Symmetry in Convection,* Ph.D. thesis, University of California, Berkeley, 1984.

[17]In the limit $\sigma \rightarrow \infty$ the modes that first become unstable in an infinite layer at R_0 and R_e have the same wave number. This is not otherwise the case.

[18]The relation of this method to center-manifold reduction and normal-form theory is discussed by J. Guckenheimer and E. Knobloch, Geophys. Astrophys. Fluid Dyn. **23**, 247 (1983).

[19]E. Knobloch, in *Multiparameter Bifurcation Theory,* Vol. 56 of *Contemporary Mathematics,* edited by M. Golubitsky and J. Guckenheimer (American Mathematical Society, Providence, Rhode Island, in press).

[20]These results were obtained by M. Golubitsky and I. Stewart, Arch. Rat. Mech. Anal. **87**, 107 (1985) using a different method; see also D. Ruelle, *ibid.* **51**, 136 (1973); S. Schecter, in *The Hopf Bifurcation and Its Applications,* edited by J. E. Marsden and M. McCracken (Springer-Verlag, New York, 1976); A. K. Bajaj, SIAM J. Appl. Math. **42**, 1078 (1982); P. Peplowsky, Physica **6D**, 364 (1983); T. Erneux and B. J. Matkowsky, SIAM J. Appl. Math. **44**, 536 (1984); see also Refs. 7 and 16.

[21]D. Rand, Arch. Rat. Mech. Anal. **79**, 1 (1982).

[22]A partial discussion of the standing waves is given by J. Guckenheimer, in Ref. 19, using a different method.

[23]The presence of sidewalls breaks the translation symmetry and leads to spatial modulation (see Ref. 6). This effect can be studied within the context of the theory developed here by analyzing the breakdown of O(2) to $Z(2)$ and will be described elsewhere.

[24]D. R. Moore, J. Toomre, E. Knobloch, and N. O. Weiss, Nature **303**, 663 (1983); J. Fluid Mech. (to be published). These papers study thermosolutal convection, which is very closely related to the system studied here (see Ref. 1). I. Rehberg and G. Ahlers, in Ref. 19, have reported observations of period doubling backwards into chaos that may be related to the Šil'nikov mechanism; see P. Glendinning and C. Sparrow, J. Stat. Phys. **35**, 645 (1984) for a discussion of this mechanism.

[25]Because of the O(2) symmetry this three-torus is not subject to the type of breakdown described by S. Newhouse, D. Ruelle, and F. Takens, Commun. Math. Phys. **64**, 35 (1978). Instead chaos is associated with a global bifurcation of the torus, and transversal intersections caused by the breakdown of the phase-shift symmetry $S(1)$. See Ref. 19.

[26]M. Golubitsky and M. Roberts (private communication).