Simulation of large Eden clusters

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Simulations of Eden clusters with more than 10^9 sites by the use of a Cray Research Cray-2 supercomputer show that the thickness of the surface layer of round two-dimensional clusters, averaged over all directions, increases as the radius for sizes above 50 000 000 sites. In simulations of flat surfaces, on the other hand, the thickness increases as the square root of the height for strip widths above 500. The initial growth of such strips is not described by one single power law and asymptotically has a "dynamic" exponent close to $\frac{3}{2}$, i.e., it increases as predicted theoretically with (height)$^{1/3}$. In the range covered, our three-dimensional flat-surface thickness is not consistent with a logarithmic law, and also in four dimensions the thickness does not approach a constant. Round clusters with more than 10^9 sites on the square lattice confirm the expected behavior due to anisotropy. Higher dimensions up to $d=12$ were also studied.

I. INTRODUCTION

Growth models of various kinds have been investigated actively during the last years.[1] The simplest of them seems to be the Eden model.[2] Here we start from one occupied site at the origin, and at every step occupy one other site. This site is selected randomly from the set of sites which have at least one nearest neighbor on the lattice occupied in one of the previous time steps. In this way one forms one single cluster and can study its properties as a function of s, the "mass" or number of occupied sites in that cluster. We call the unoccupied neighbors of occupied cluster sites the perimeter sites or surface of this cluster.

The simplicity of this process allows every unoccupied neighbor again and again the chance to become occupied. Thus every fixed site of the lattice will finally become occupied, if we only wait long enough. Therefore in the interior of the cluster there are no holes left, all sites are occupied. On the other hand, there are no small isolated clusters existing separately from the main cluster. Thus apart from a narrow surface layer, we have a fully compact structure, with the mass s increasing as $s \propto (\text{radius})^d$ and the perimeter P as $P \propto (\text{radius})^{d-1}$ in d dimensions.[3,4]

More interesting is the structure of the surface layer, in particular its thickness W defined through

$$W^2 = \langle r^2 \rangle - \langle r \rangle^2 \tag{1}$$

as the rms fluctuation of the distance r of the perimeter sites from the origin of the cluster growth. Early work[5] claimed this thickness to be proportional to the cluster radius but was based on an incorrect data analysis. Plischke and Rácz[6] first found W to vary as (radius)$^{0.36}$ but then interpreted the systematic decrease of this exponent with cluster size as an indication for an asymptotically logarithmic behavior. Jullien and Botet[7] gave, for flat surfaces of width L, a thickness proportional to $L^{1/2}$ in two dimensions, with possibly a logarithmic variation of W in three dimensions and a saturation to a constant value in four. Later, large Eden clusters were shown to feel the anisotropy of the lattice,[8-10] and the exponent for the surface thickness as a function of radius was seen[8] to increase for large clusters from its minimum near 0.36 towards $\frac{1}{2}$ and then further up, the last value being 0.72 for clusters with 10^7 sites on the square lattice. On the other hand, the same authors[8] found $W \propto L^{1/2}$ for flat surface on strips of widths L up to 724 in two dimensions. This discrepancy was speculated as being due to the anisotropy: Averaged over all directions the distance of the perimeter sites from the origin fluctuates by an amount proportional to the radius, thus $W \propto s^{1/2}$, if the perimeter sites do not form a circle but a more anisotropic structure in the direction of a diamond shape. Averaged only over one direction, as done in a simulation of flat surfaces, a better defined surface thickness is found which increases more weakly than the cluster radius, in agreement with mathematical proofs.[11] However, the anisotropy found in Refs. 7–9 is very weak, and thus for the maximum mass $s=2^{24}$, Ref. 8 could not yet observe the expected asymptotic behavior, surface thickness $\propto$ radius, dominated by the anisotropy. One aim of our paper is to go to larger cluster sizes and to check whether this theoretical expectation is fulfilled. We also want to extend the simulation of flat surfaces in three and four dimensions to larger systems to check if there are surprises hidden beyond previous system sizes.

For the simulation of flat surfaces, many studies[6,7,9,12-15] have also been made for the "dynamic" aspect: How does, as a function of height h over the initially occupied substrate, the surface thickness W increase for fixed strip width L. In such simulations the height h can also be interpreted as a time. The scaling assumption[12]

$$W = L^x f(h/L^z) \tag{2}$$

means that the scaling function f approaches a constant for large arguments, giving $W \propto L^x$ for flat surfaces "in equilibrium," as discussed above. For small arguments of

f, on the other hand, W increases as h^{β} independent of strip width L, with $\beta = x/z$ from standard scaling arguments.[12] Two-dimensional numerical[6,7,9,14] or analytical[13] estimates give $z = \frac{3}{2}$ and thus $\beta = \frac{1}{3}$ if we believe the square-root law $x = \frac{1}{2}$ for the surface thickness at very large heights.

This widespread belief that z is not unity means that geometrically the Eden clusters violate similarity: a strip of width $L = 100$ and height $h = 1000$ will be in "equilibrium," i.e., the surface thickness W no longer increases appreciably with height h. On the other hand, a strip with $L = 10\,000$ and $h = 100\,000$ will still be in the regime where the scaling function $f(h/L^z)$ has a value appreciably below its value at infinity, and the surface is not yet in "equilibrium." This lack of geometric similarity for $z > 1$ not only means that computer simulations are very time consuming for large L but also[14] that the surface thickness W of a round cluster of radius L does not scale with the same exponent as W for flat surfaces of linear dimension L. We want therefore to check whether the trends indicating $z \simeq \frac{3}{2}$ are continued if we use larger strips, and also what is happening for large systems in three and four dimensions. A summary is given in Ref. 16.

Our next section describes the simulation methods on the new Cray Research Cray-2 supercomputer, Sec. III our analysis of the results, and Sec. IV our conclusions.

II. COMPUTATIONAL TECHNIQUE

Our work was done on a Cray-2 supercomputer, which is the largest general-purpose machine presently available for basic research. Its MOS (metal-oxide-semiconductor) main memory contains 268 435 456 64-bit (binary digit) words, it has four independent processors, its FORTRAN compiler performs automatic vectorization not only of trivial statements, and its scalar operations are also fast. Its main drawback for our Monte Carlo simulation seems to be the rather slow access to the main memory, and the not yet perfect code optimization by the Cray FORTRAN compiler. Some of our computations were made on a prototype machine with "only" 16 777 216 words and a single processor. During our work a new compiler became available which corrected an error in the earlier compilers affecting our single-bit storage of the lattice used in some computations.

Most of the calculations were done with the standard method used, e.g., in Refs. 8 and 9: A large lattice stores the information about which sites have never been touched by the process, and a separate list keeps the coordinates of the perimeter sites. Thus a lattice point could be stored in a single bit, whereas the perimeter list contains large integers. For every step of the growth process, one selects randomly a member from the perimeter list of momentary length P, occupies it by removing it from the perimeter list, stores the coordinate of the Pth perimeter site in that element of the list where the newly occupied site was stored before, and decreases P by 1. Then for each of the $2d$ neighbors of the newly occupied site (in d dimensions) one checks if it has been touched before; if yes one proceeds to the next neighbor, if no one marks it in the lattice as touched, increases P by 1, and adds this site to the perimeter list.

To simulate flat surfaces we started with a whole line (plane in three dimensions) occupied, instead of a single site for round clusters. For these calculations we saved memory by storing only the growth zone in the computer, as was done already in Ref. 9. Thus when the last added site was close to the upper boundary of the computer memory used, we looked for the perimeter site with the smallest coordinate, determined in which line (of a two-dimensional strip; planes in three dimensions) this empty site lies, and shift the whole structure towards smaller coordinates such that this empty site then lies in the first line. Typically the number of lines over which the perimeter sites are spread is one order of magnitude larger than the thickness W defined above. This spread could be used as another definition of a surface thickness, W'. The fluctuations of W' are much bigger than those of W and no precise data can be extracted. The shifting in the computer memory is very fast since it is handled efficiently by the automatic vectorization. Because of large memories available it occurs only very seldom and uses a negligible fraction of the total CPU (central processing unit) time spent.

Because of these memory savings, we stored for "flat" simulations our lattice as one site per full 64-bit word. For round clusters, where memory shifting was not applied, we used mainly storage in single bits, as in Refs. 8 and 9. Then we gained a factor 64 in memory for a loss of about 30% in speed. This slowing down is much weaker than the factor 3 lost on the Control Data Corporation CDC-176 of Refs. 8 and 9 since the supercomputer has 64-bit words, whereas the CDC-176 has 60-bit words. Finding the connections between bit addresses and word addresses thus involved only fast shifting by 6 bits (factor 64), and no slow divisions and multiplications with 60.

The Eden growth process, as described above, is a *recursive* process, i.e., occupying one site depends upon previously occupied sites. It cannot be vectorized easily. One could simulate many growth processes simultaneously but that would be still quite complicated and use lots of memory (64 times as much). We therefore developed a modified algorithm to vectorize the growth of a single cluster. Cray supercomputers prefer, in contrast to a CDC Cyber-205, a vector length of 64 elements. 64 different perimeter sites were selected "simultaneously" (in the sense of vectorization) to be occupied. Of course, sometimes one will select twice the same site to be occupied and thus make an error. However, the probability of this error occurring is smaller the larger the perimeter P is, and since we are interested here in the limit of cluster size and thus P going to infinity, this systematic error goes to zero. When the neighbors are investigated as described above, we may again make errors of the order of $64/P$, since one site might be neighbor for two of the 64 just-occupied sites. Vectorization requires all 64 growth processes to be simulated in a similar fashion, thus our perimeter lists contained pointers to some ghost lattice of typically several thousand occupied sites (one additional line or plane of sites). These ghost pointers were inserted whenever a site was occupied and thus removed from the

list, or found not to be a fresh perimeter for the newly occupied sites. In principle it would have been sufficient to have all these pointers referring to one single occupied site, e.g., site L in the zeroth line of the strip. But then this element in the computer memory would be investigated very frequently, and those 64 investigators would hinder themselves (memory bank conflicts), thus slowing down the speed. As an analogy, 64 detectives assigned randomly to 1000 different cases will conflict with each other much more rarely than if they all try to solve the same case. In regular intervals, though not necessarily after each group of 64 growth steps, we had to remove the ghost sites from our perimeter lists. We did not attempt to combine vectorization with single bit handling; because of the absence of a suitable vector shift instruction, such vectorization is not possible for Cray machines but would be possible on the Cyber 205. Figure 1 shows, for three-dimensional flat surfaces, how the thickness calculated with the vectorized program approaches, as expected, the correct thickness (from the normal program) if the system size increases.

The computing time for the nonvectorizable algorithm for flat surfaces, with memory shifting and storage of each lattice point in a full word, was nearly 6 μsec per site added, slightly faster than the corresponding algorithm on the "normal" CDC-176 computer. This can be understood since the execution time is bound by random-access memory references, the memory being about equally fast on both machines. With storage in single bits the supercomputer was twice as fast as the CDC-176. The vectorizable algorithm was, if written in FORTRAN, twice as slow as the nonvectorizable version since random elements of the memory had to be read and stored. The FORTRAN compiler currently does not compile these memory references into gather and scatter machine instructions (which do exist in the hardware). Writing its innermost two loops in assembler code, we increased the speed of the vectorized algorithm to less than 0.8 μsec per step, i.e., gained a factor of 15. However, some of these steps added no sites since a ghost site was selected instead, and thus the effective speed was only about half as large (1.6 μsec per site). Thus the speed gain of assembler-coded vectorization is less useful than the ratio of 0.8 to 6 μsec first suggests. Nevertheless, because of an overall speed gain of a factor of 4, many calculations for larger sizes were done using the vectorized version: most remaining calculations employed the single-bit storage technique. (These speeds hold in two dimensions; for higher dimensions the calculation is 20—40 % slower.)

In summary, the advantage of this supercomputer with its present software for our problem was its large memory, not so much its speed. We thus concentrated on simulating one or a few large systems, instead of hundreds or thousands of intermediate sizes.[8,9] In this way we dramatically reduce the finite-size effect disturbing previous investigations.

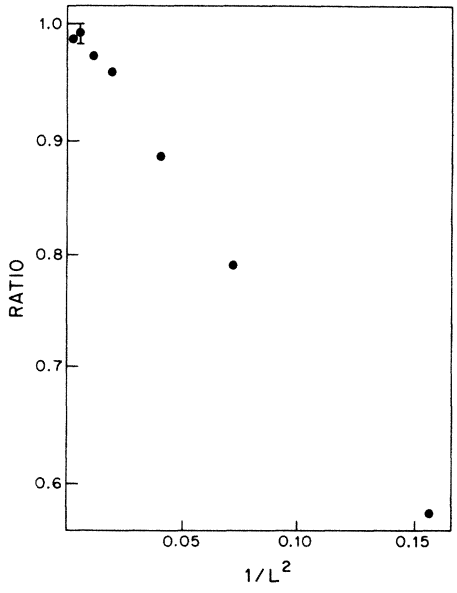

FIG. 1. Ratio of surface thickness as determined from vectorized program, to "true" surface thickness from scalar program, in three dimensions, vs reciprocal number of sites of flat surface. Vectorization is correct only if this ratio approaches unity.

III. RESULTS

A. Round clusters

Figure 2 shows the surface thickness W of our round clusters containing up to $2^{30}=1\,073\,741\,824$ sites, 256 times more than the largest round clusters simulated for Ref. 8. (The quadrants of Ref. 8 correspond to clusters up to 2^{24} sites but do not have the same surface thickness W as the round clusters because of quadrant boundary effects. We now found also their effective exponents to be somewhat different from those of the full clusters.) Table I gives our results for the convenience of future research-

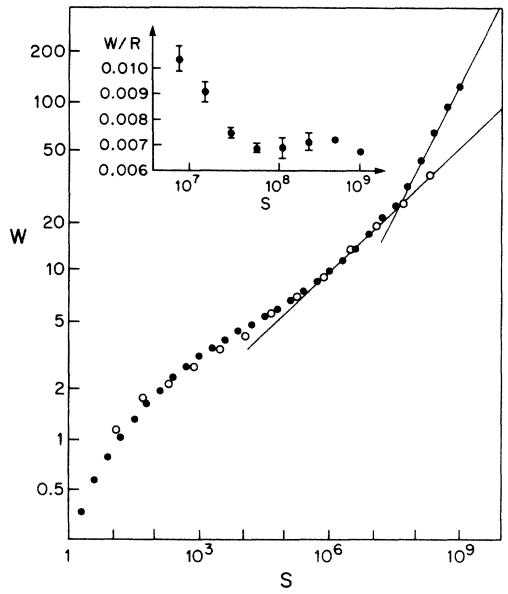

FIG. 2. Surface thickness W for round clusters as a function of number s of sites in the cluster (square lattice). The circles give W for the corresponding flat surfaces in equilibrium. The inset shows the ratio of thickness to radius.

TABLE I. Surface thickness W averaged over the whole circumference of round clusters on the square lattice, as function of cluster size (a "megasite" 1 048 576 occupied lattice sites). (Averages over 1 to 4 runs.)

"Megasites"	W
1	10.0±0.4
2	11.2±0.5
4	13.6±0.7
8	16.9±0.9
16	21.1±0.9
32	24.5±0.7
64	31.8±0.8
128	45.2±2.7
256	66.1±2.5
512	92.5±2.1
1024	125.2

ers. We see a complex behavior but starting from about 50 000 000 sites (i.e., above the previous limits) the thickness seems roughly proportional to the square root of the cluster mass s, i.e., proportional to the cluster radius. The inset shows the ratio of thickness to cluster radius; for the largest clusters the radius, defined as the average distance of the perimeter sites from the origin, is 18 488 lattice constants. From the last decade in cluster sizes we determine the surface thickness as

$$W = 0.002\,65 s^{0.52\pm0.02} . \quad (3)$$

We have also looked at the anisotropy of the clusters as in Ref. 8 and found again that the perimeter sites on the lattice axes are slightly farther away from the origin than those along the lattice diagonal, the degree of anisotropy being 2%, the same as in Ref. 8 for their largest clusters. This weak anisotropy explains why we needed clusters containing more than 50 000 000 sites to see the asymptotic behavior: The thickness W of flat surfaces, to be discussed below, is for clusters below 50 000 000 sites larger or about equal to the W of round clusters, if we identify cluster radius and strip width L. This identification was questioned in Refs. 13 and 14 but seems to work nevertheless. Figure 2 shows these values too, and the two data sets separate near 50 000 000 sites. Thus, as speculated in Ref. 8 without direct evidence, the surface thickness of very large clusters is dominated by the anisotropy of the cluster shape, whereas the more intrinsic surface thickness, as measured on flat surfaces, is smaller asymptotically. Thus in the limit of extremely large clusters, the surface thickness is negligibly small compared to the cluster radius but the clusters are not exactly circular. This result is compatible with mathematical theorems[11] about the perimeter distribution.

B. Rectangular clusters

For flat surfaces oriented along the lattice direction we found in the square lattice for strip widths L between 512 and 8192,

$$W \propto L^{0.511\pm0.025} , \quad (4)$$

thus confirming the exponent $\frac{1}{2}$ found earlier[7,8] for L near 512: No change in the trend here. Thus asymptotically the thickness of flat surfaces seems to grow as $0.42L^{1/2}$ on the square lattice. Again, Table II gives details of our results; purely empirically we can fit most of these data and those of Ref. 6 on

$$W = 0.42L^{1/2}(1 + 18/L - 31/L^{3/2} + \cdots) , \quad (5)$$

but we do not assert that this fit gives a reliable estimate of the correction exponents.

The thickness of the surface layer in a simple cubic lattice for three dimensions and a hypercubic lattice for four dimensions is plotted in Fig. 3 in the form of W^2 versus $\log_2(L)$, with L up to 1024 in three and 128 in four dimensions, for flat surfaces of linear dimension L with again the orientation in a lattice direction. We see that the four-dimensional thickness increases first faster, then

TABLE II. Equilibrium surface thickness W for flat surfaces of linear extent L in d dimensions (of which one can be regarded as time).

L	d	Thickness
512	2	9.38±0.02
1024	2	13.6 ±0.2
2048	2	18.9 ±1.3
4096	2	26.1 ±0.5
8192	2	38.7 ±0.5
2	3	1.96±0.01
3	3	2.60±0.04
4	3	2.90±0.03
6	3	3.21±0.03
8	3	3.41±0.02
16	3	3.75±0.01
24	3	3.95±0.01
32	3	4.09±0.01
48	3	4.32±0.02
64	3	4.52±0.02
96	3	4.82±0.03
128	3	5.12±0.03
192	3	5.49±0.10
256	3	6.07±0.10
384	3	6.68±0.10
512	3	7.17±0.15
768	3	7.21±0.4
1024	3	8.8 ±0.3
2	4	2.43±0.16
3	4	3.60±0.15
4	4	4.07±0.14
6	4	4.42±0.14
8	4	4.45±0.17
12	4	4.81±0.12
16	4	4.75±0.02
24	4	4.90±0.03
32	4	5.00±0.01
48	4	5.08±0.01
64	4	5.20±0.03
96	4	5.3 ±0.1
128	4	5.5 ±0.1

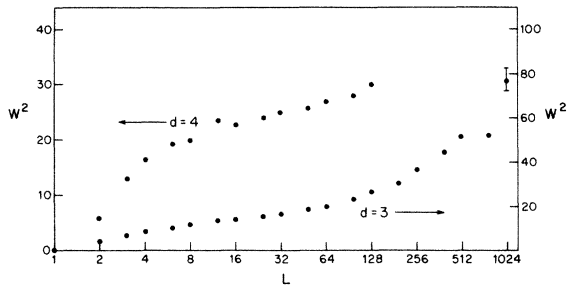

FIG. 3. Squared equilibrium width vs system size in three and four dimensions. Simple arguments would give a straight line in three dimensions, and a constant in four.

slower with L than the three-dimensional one. Thus it is at first sight plausible that the four-dimensional thickness approaches a constant and that in three dimensions it increases logarithmically. This result would agree with earlier hopes from much smaller L. However, the last points for the largest systems do no longer follow that trend but are above the trend, strongly in three and slightly in four dimensions. Perhaps also in four dimensions the surface is becoming "rough," i.e., has a thickness going to infinity in infinite systems. Our systems are not large enough to allow a quantitative analysis; they merely shatter the hope that four-dimensional Eden clusters are simple in their structure already for small sizes.

These hopes were based on simple capillary-wave theory which predicts a constant width for dimensionality above three. In three dimensions it predicts a squared width proportional to $\log_2(L)$. We see from Fig. 3 already that the behavior is more complicated, at least in the size range covered here. We leave it to the reader to speculate, with the help of Table II, about the asymptotic variation of the thickness in three and four dimensions; the statistical errors increase with increasing system size and make such speculations rather unreliable.

Since in three dimensions for $L=1024$ we used about 210 megawords of Cray-2 memory in a 9-h run, it will not be easy to go to much larger sizes within reasonable computer time. The data of Jullien and Botet fitted much better the expected logarithmic law since they were restricted to sizes smaller than those for which we obtained the upturn in our data. We see here again how new advances in computational power give qualitatively different results from previous attempts.

We verified the computational credibility of the width for the largest systems by using two different, independently developed computer programs, and also using two different types of random-number generators.

C. Nonequilibrium results

The Eden process is a nonequilibrium process. Nevertheless the above properties of the surface thickness over flat substrates can be regarded asymptotically as equilibrium properties if we imagine an observer to rise together with the rising surface and if we go to large enough heights. In this sense the height $h=t$ over the original substrate is a measure of the time, and instead of talking about d-dimensional Eden clusters one has $d-1$ space and one time dimension. In this well-known interpretation, the behavior of flat surfaces for "small" height corresponds to the approach towards equilibrium. Small here means that the heights have to be much larger than the lattice constant but much smaller than the asymptotic limit of the thickness for "time" going to infinity at fixed system width L. In particular one can ask with what power of the time does the surface thickness increase in this intermediate region, i.e., what is the value of the exponent β in

$$W \propto h^{\beta}, \quad 1 \ll W(h) \ll W(\infty) . \tag{6}$$

Previous work in two dimensions found β near $\frac{1}{3}$ [often expressed as $z=1/(2\beta)$ near $\frac{3}{2}$ in dynamical scaling[12]] but Hirsch and Wolf[9] pointed out that from numerical data, $\frac{1}{3}$ is only a lower bound for β. Kardar *et al.*[13] found $z=\frac{3}{2}$ in two dimensions from a general field theory of surfaces.

Figure 4 shows a rich behavior of the effective exponent $\beta=d\ln W/d\ln h$ as a function of h, including partial occupation of the first layer with $h \ll 1$. Even more complicated than the recent results of Family[15] in a different model we see here a maximum, a minimum, and an in-between near saddle point. Somewhat surprisingly, for the triangular lattice the behavior is much simpler. We hope that theoretical work will explain the behavior at short times by a perturbative treatment. However, the crucial feature in two dimensions is the asymptotic increase of the effective exponent β for large times: Our last data points are near $\frac{1}{3}$ but the slope of the curve is positive. Table III gives the effective exponents β as function of height, from several runs for $L=1\,048\,576$. Figure 5 tries to answer the question if β rises to a value above $\frac{1}{3}$ ($z<\frac{3}{2}$). We see that the ratio $W/h^{1/3}$ seems to approach a constant which moreover is reassuringly close to unity.

This approach to a constant does not yet prove $\beta=\frac{1}{3}$ since finite-size effects have to be taken into account: For a fixed width L, the effective exponent β must go to zero for sufficiently long times since then the thickness W reaches its equilibrium value. This equilibrium value is extrapolated to be about 430 in our case, from the results of Sec. III B. Finite-size effects therefore always force β to have a maximum as function of time; as pointed out by

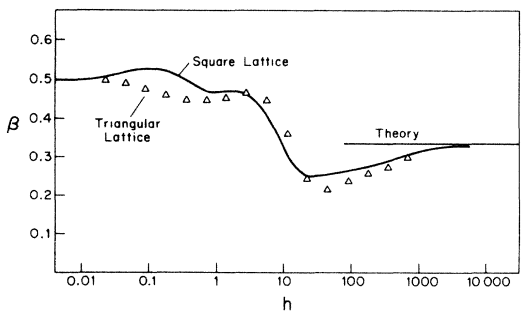

FIG. 4. Dynamic exponent $\beta=d(\ln W)/d(\ln h)$ vs time h in square and triangular lattice.

TABLE III. Increase for long times of the dynamic exponent β on the square lattice with $L=1\,048\,576$, found from N independent runs. Statistical errors are of the order of 0.01. For $L=262\,144$ the results from $N=8$ runs are given in parentheses, with about the same errors.

Time interval	β	N	
128–256	0.275	8	(0.288)
256–512	0.282	7	(0.294)
512–1024	0.311	4	(0.315)
1024–2048	0.319	3	(0.317)
2048–4096	0.324	2	(0.333)
4096–8192	0.326	1	(0.316)

Hirsch and Wolf,[9] the numerical estimates are thus only lower bounds for β. Our largest thicknesses observed here were about 20, which is not many orders of magnitude smaller than 430. Thus, we made also runs with smaller sizes $L=262\,144$ and $L=65\,536$ and found there, within their larger scattering, the exponent β again to be about $\frac{1}{3}$. For $L=8192$, on the other hand, it was significantly lower. Thus we think our data for $L=1\,048\,526$, limited to heights (times) less than 10^4, are not yet influenced measurably by the finite L value and thus show our data to be consistent with the theoretical prediction of Kardar *et al.*[13] Of course, it is possible that for even larger times another maximum is found and that β then falls again.

In three and four dimensions our analogous studies were much more complicated. The reason is simple: In all dimensions at the end of the simulations the surface thickness W is of the order of ten lattice constants. In two dimensions, for $L=10^6$ this value is much smaller than the equilibrium thickness near 400. The same number of sites in the substrate plane corresponds to $L=1000$ in three and $L=100$ in four dimensions. Then also the asymptotic thickness is of order 10 and thus no longer

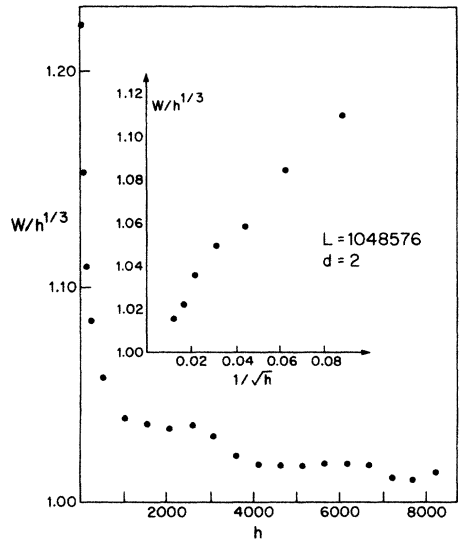

FIG. 5. Variation of thickness W, divided by the theoretically predicted power of time h, with time or, in the inset, with $(\text{time})^{-1/2}$, for the square lattice.

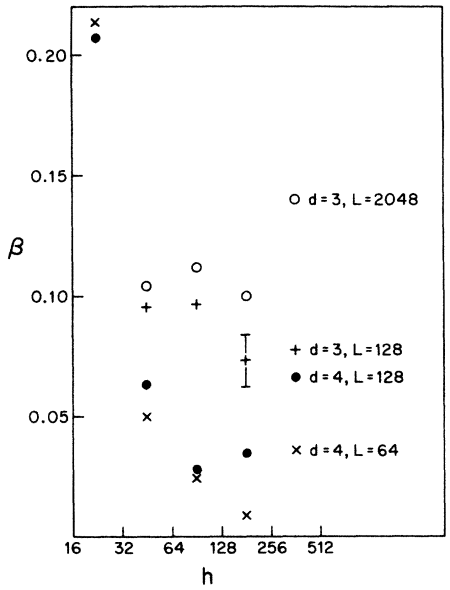

FIG. 6. Dynamic exponent β for intermediate times in three and four dimensions.

much larger than the time-dependent width. Necessarily the effective[9] β approaches zero if the thickness approaches its asymptotic limit. Thus our data for three and four dimensions seem reliable only for rather short times; we used our largest systems here by storing only the growth zone, and its lattice sites in single bits. ($L=8192$ in three and $L=256$ in four dimensions.) We see from Fig. 6 that the behavior is qualitatively similar to two dimensions for heights (times) up to about 500. The effective exponent sharply decreases with increasing height between 4 and 100. Then for the largest systems it increased again. Thus we have again a minimum as in two dimensions, the value at the minimum being even lower for $d=3$ and $d=4$ than for $d=2$. Longer times would have cost too much computer time but presumably would have

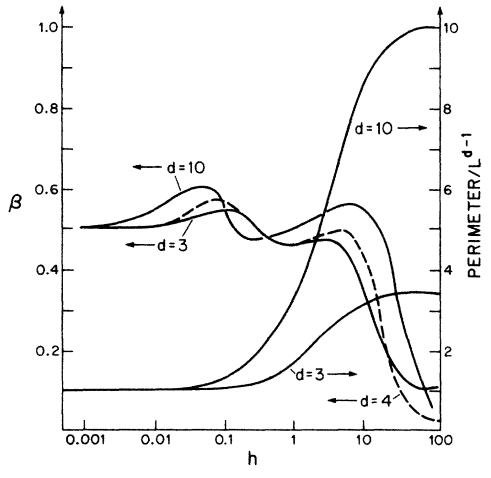

FIG. 7. Short-time variation of dynamic exponent β with time in three and more dimensions (left). Right scale shows the corresponding variation of the perimeter, normalized by the number of sites in the flat surface.

not been very helpful since then finite-size effects would take over. Thus we do not know whether after the minimum the three- and four-dimensional exponent β rises to its two-dimensional value $\frac{1}{3}$ (as speculated for $d=3$ in Ref. 13), or whether it approaches another value. The only reliable information seems that for heights up to a few hundred, before finite-size effects dominate, the three- and four-dimensional flat surfaces behave similar to the two-dimensional ones.

For short times (partially filled planes) this conclusion is confirmed by Fig. 7. Again the effective exponent β starts for very short times at $\beta=\frac{1}{2}$ (this value can easily be shown to be exact for t going to zero), reaches a maximum, falls down to a value slightly below $\frac{1}{2}$, and rises again slightly. This increase is stronger for $d=3$ and 4 than for $d=2$; in the latter case the behavior was nearly that of a saddle point. Figure 7 also gives information on the approach to equilibrium for the perimeter.

D. High dimensions

Since already in four dimensions we had difficulties in establishing that the asymptotic behavior is not that of smaller systems (last points in Fig. 3), in higher dimensions we would be unable to find such effects reliably. Also the upturn in the dynamic exponent β was just visible in four dimensions (Fig. 6) and cannot be expected to be tested reliably for higher d. Thus we concentrated on very short times for higher $d=10$ and $d=12$, using $L=4$. (For $L=2$ we got similar results; thus in this region finite-size effects are not important.) We see in Fig. 7 a behavior similar to that in lower dimensions.

For somewhat larger times, the exponent β in ten dimensions for $L=4$ drops sharply to zero; it is about 0.19, 0.04, and 0.02 in the time intervals 32–64, 64–128, and 128–256, respectively. For this small L value one cannot expect to see an upturn in β for longer times even if there is one.

IV. DISCUSSION

For two-dimensional round clusters we found roughly what we expected: a surface thickness proportional to the cluster radius since the cluster is anisotropic. As long as the cluster is of moderate size, this effect is not yet visible in the surface thickness which is then dominated by that of a corresponding flat surface, and therefore the asymptotic behavior sets in only at 10^8 sites. How can we interpret the complicated behavior found here for flat surfaces?

If we ignore the small maxima and minima for short times, which we observed for $d=2$–12, we may approximate the dynamic exponent β in that region up to heights (times) of a few lattice constants by its theoretical value $\beta=\frac{1}{2}$ for short times. Then we can distinguish between three time regions. First β stays at its short-time value, then it falls off sharply, and after about 10^2 layers filled it increases slowly. For $d=2$ this increase stopped at $\beta=\frac{1}{3}$, for d above 4 we could not see this increase due to our finite system size. The perimeter, on the other hand, i.e., the total number of empty neighbors of occupied sites, reached its asymptotic value much earlier, as noted already in Ref. 9; roughly it has reached equilibrium when the effective exponent β falls off to its minimum (Fig. 7). Thus we think that there are two reasons why the surface thickness is much larger than unity: In the first time period, the perimeter grows from its initial size to its equilibrium size several times larger. As a result the width increases, roughly with the square root of the height (time), from zero to some intrinsic width. It stays there (at about 4 lattice constants in two dimensions, at about 5 in four dimensions, and in between in three) during the second time interval mentioned above, as Fig. 8 illustrates. During that time the growth exponent β is therefore close to zero. Later, in the third period, fluctuations of this intrinsic surface make the total surface thickness dependent on the system size, as predicted by capillary-wave theory. In two dimensions we saw that in this third "fluctuation" regime the thickness increases as $(\text{time})^{1/3}$ whereas in three and four dimensions we only observed the beginning of this third phase, and for higher dimensions we could not even see that. For sufficiently large times at a fixed system size L, the surface thickness W will again reach a plateau, the equilibrium thickness of Sec. III B. For $d=2$, 3, and 4, the third phase, the growing of the surface thickness due to fluctuations in the "intrinsic" surface, starts after a height of the order of 10^2 (provided the system is large enough). Thus the asymptotic behavior can only be seen if the observed thickness is larger than the intrinsic thickness of 4–5 lattice constants.

In two dimensions we could easily reach larger thicknesses and thus draw conclusions about the asymptotic behavior. In higher dimensions our thicknesses were always below 10, and such conclusions are unreliable.

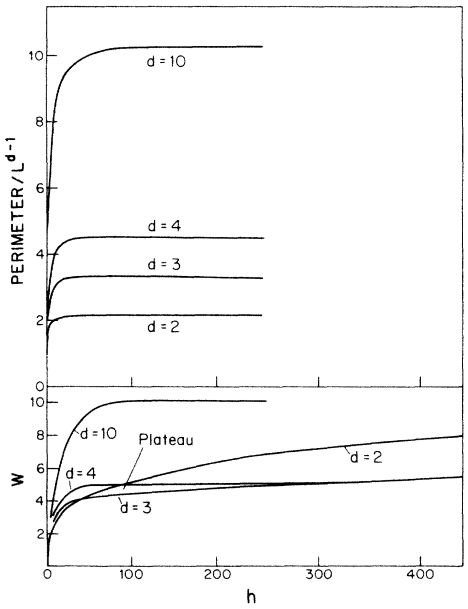

FIG. 8. Variation of thickness W with time h in various dimensions, indicating the initial build-up of the intrinsic surface thickness, followed by the much slower increase due to long-wavelength fluctuations.

Earlier work has only seen the intrinsic surface, not the fluctuation regime.

We now also have an explanation for the equilibrium behavior of Fig. 3. For small system sizes, the equilibrium thickness is of the order of the intrinsic thickness, i.e., about 5. Only if the linear dimension L is larger than about 10^2 do we see if surface fluctuations push the thickness to larger values. Thus in a plot[16] of thickness versus L the curves for $d=2$, 3, and 4 all go through about the same point $L=100$, $W=5$; only for larger L the new growth is seen. Equilibrium asymptotic behavior can thus only be observed for linear dimensions L far above 100. We saw it in $d=2$, found it hampered by statistical fluctuations in $d=3$, and just saw its beginning in $d=4$. Equilibrium studies in higher dimensions thus also are expected to require much more than 100^d occupied sites, a challenge for future supercomputers.

In summary, large-scale computations were made to look for the asymptotic properties of the seemingly so-simple Eden model. In two dimensions we confirmed the trend visible before that the thickness of a flat surface increases as the square root of its linear dimension. For round clusters we found for the first time what was speculated before, that the surface width, averaged over all directions, varies proportional to the radius. The asymptotic dynamic exponent $z=1/(2\beta)$ for $d=2$ seems to agree with both a theory[13] and previous numerical results for much smaller systems. For small and intermediate times, maxima and minima are seen in the variation of the effective dynamic exponent. In higher dimensions we found that earlier trends are not continued if we go to larger system sizes, and we offered an explanation that linear dimensions L above 100 are needed to study the fully developed equilibrium of the surface fluctuations.

From the computational point of view we found applications for the huge memory of the Cray-2 but would have liked to have also a corresponding increase in speed, which was lacking in our simulations. The Eden model is simple to define but very difficult to study numerically in its asymptotic regime. Also its Monte Carlo simulation cannot easily be vectorized for the Cray-2, and thus this study utilized mainly its huge memory, not its vector speed.

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