Search for randomness in the kicked quantum rotator

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We reexamine the problem of the δ-kicked rotator in the light of some numerical results which support the thesis of the existence of a singular continuous component in the quasienergy spectrum, in accordance with previously obtained analytical results. In particular, we discuss to what extent a memory of classical chaos may survive in the quantum motion.

I. INTRODUCTION

Deterministic Newtonian systems can exhibit every shade of randomness from simple ergodic flow on a torus to wildly erratic phase-space flow indistinguishable from a truly random process. This inherent randomness exhibited by many Newtonian systems provides hope that classical statistical mechanics can at last be rigorously derived without the use of additional *ad hoc* probability assumptions.

Finite-particle-number, bounded, conservative quantum systems, on the other hand, yield a time evolution which, due to discrete energy spectra, is almost periodic and is therefore ergodic only at best.

Quantum statistical mechanics, which requires randomness in the deterministic Schrödinger flow of the wave function, thus would seem to lie beyond the pale of such systems. Many have suggested that randomness in quantum evolution may be recovered in systems having continuous spectra or even in systems having discrete spectra, provided one considers only time intervals over which a continuum approximation might be valid. However, this possible resolution suffers the defect that a continuous spectrum alone is no guarantee of randomness.

Indeed, a continuous spectrum need imply no more than weak mixing in the quantum flow. In consequence, the question of randomness (chaos) in the time evolution of a quantum system is seen to perhaps involve quite delicate issues, and one is hard pressed not to suspect, at the very least, that quantum mechanics imposes severe limitations on the classical notions of chaos. Regardless, a continuous spectrum is a necessary, even though not a sufficient condition for randomness; thus, it is surely in systems having a continuous spectrum that we must search for randomness, if any, in the quantum flow. In this paper, we present a progress report on the search for a continuous spectrum and randomness in a particular quantum system.

Time-dependent, one-degree-of-freedom Hamiltonian systems are perhaps the simplest which can exhibit continuous spectra and, of these, perhaps the simplest is the "kicked" quantum rotator introduced in Ref. 1. The Hamiltonian for this model is

$$H=p^2/2+\omega^2\cos\theta \sum_{n=-\infty}^{\infty} \delta(t-nT) , \tag{1}$$

where p is the rotator momentum and θ is its angular position, ω^2 is the perturbation strength, and T is the kick period.

The classical equations of motion for Hamiltonian (1) when integrated over a kick period become the mapping equations

$$P_{n+1}=P_n+K\sin\theta_n , \tag{2a}$$

$$\theta_{n+1}=\theta_n+P_{n+1} , \tag{2b}$$

where $K=\omega^2T$, n is the time measured in number of kicks, and P_n is the dimensionless rotator momentum.

Equation (2) is Chirikov's standard map frequently used to study the transition to chaos, the disappearance of the last horizontal Kolmogorov-Arnold-Moser (KAM) curve, and the application of renormalization theory to dynamics. Statistical properties appear in this mapping when the parameter $K \gg 1$. Under this condition, numerical evidence verifies that the P motion obeys a simple, random-walk diffusion equation having the form

$$\overline{P^2} \approx \frac{K^2}{2} n , \tag{3}$$

where $\overline{P^2}$ is the average of the squared rotator momentum and where initially P is taken to be zero. Numerical evidence also establishes that the momentum distribution itself has the time-dependent Gaussian form

$$f(P)=K(\pi n)^{-1/2}\exp(-P^2/K^2 n) \;. \tag{4}$$

It is precisely this type stochastic or random behavior that is being sought in the corresponding quantum system.

II. EARLIER RESULTS ON THE QUANTUM KICKED ROTATOR

The Hamiltonian for the kicked quantum rotator has the form

$$H=-\frac{\hbar^2}{2}\frac{\partial^2}{\partial\theta^2}+\omega^2\cos\theta\sum_n \delta(t-nT) \;. \tag{5}$$

The quantum motion governed by Hamiltonian (5) is described[1] by the following mapping. Given $\psi(\theta)=\sum_{m=-\infty}^{\infty} c_m e^{im\theta}$, then the value of $\psi(\theta)$, called $S\psi(\theta)$, after a free rotation and a kick is given by

$$S\psi(\theta)=e^{-ik\cos\theta}\sum_{m=-\infty}^{\infty} c_m e^{-i[(\tau/2)m^2-m\theta]} \;, \tag{6}$$

where $k=\omega^2/\hbar$ and $\tau=\hbar T$.

Due to the periodicity of mapping (6), it is convenient to redefine $\tau=\tau/4\pi$ and consider values of τ lying only in the interval $[0,1]$. Note that the classical K is such that here $K=4\pi k\tau$.

The simple mapping of Eq. (6) is especially interesting because, as we discuss later, it can yield quantum motion having a continuous spectrum as required for randomness. We thus now turn to the details of our progress report on the search for randomness (beyond $\psi^*\psi$) in quantum mechanics.

In all studies of the kicked quantum rotator, two central features of the motion are observed. First for rational values of τ in the kick period equation $T=4\pi\tau/\hbar$, the rotator energy resonantly increases asymptotically as t^2. This is a strictly quantum phenomenon and does not occur in the corresponding classical system. This quantum peculiarity arises because the unperturbed quantum motion, as opposed to the classical, has the same period independent of initial state. Second, for suitable values of system parameters, the quantum motion can mimic the classical diffusive energy growth yielding

$$\langle E\rangle=\sum_m \frac{m^2}{2}|c_m|^2 \simeq \frac{k^2}{4}t \;.$$

However, in the quantum motion this linear energy growth persists only up to a break time t_B. Empirically it appears that $t_B\to\infty$ as $\hbar\to 0$ or equivalently $k\to\infty$, for constant $K=4\pi k\tau$. For times greater than the break time, the quantum energy appears to enter a steady-state oscillatory regime. In short, the kicked quantum rotator apparently introduces a limitation to classical diffusion and randomness. Break time also relates to the spectral properties of the driven quantum motion, a point to which we now turn.

III. THE LOCALIZATION PHENOMENON

Recall that the wave-function solution for a periodically driven quantum system can always be written

$$\psi(t)=P(t)e^{iGt}\psi(0) \;, \tag{7}$$

where P is a periodic unitary operator and G is a self-adjoint operator.

The character of the time evolution for ψ is known once the spectrum of G, called the quasienergy (abbreviated q.e., hereafter) spectrum is known. As we shall show later, the quasienergy spectrum for our model can be pure point only when the rotator energy remains bounded for all times, in which case the initial wave packet remains localized in momentum space. Thus the very existence of a break time after which energy growth stops completely would ensure that the quasienergy spectrum is pure point and that the full quantum motion is almost periodic.

Chirikov, Izrailev, and Shepelyansky[2] have advanced an heuristic argument which yields a quantitative estimate for this quantum limitation of diffusion due to the break-time phenomenon. In essence, this argument is the following: Assume that the q.e. spectrum is pure point since otherwise there will be no finite break time. By the quantum theory of measurement, this pure-point character of the spectrum can become apparent after only a certain time t^*. Prior to this time the system will behave as if its spectrum were continuous and hence the limitation on diffusion will occur no later than t^*. Chirikov *et al.* therefore identify t^* with the break time t_B in order of magnitude.

To estimate t^*, recall that the total number of q.e. levels is infinite and that all these levels are located within a bounded interval. In addition, recall that each particular state ψ of the system can be obtained as a superposition of quasienergy eigenstates; while, generally speaking, the superposition involves an infinite number of q.e. states, we may assume that for an initially localized packet, the finite time evolution of ψ is dominated by a finite number in N_ψ of them. This effective number N_ψ is also the number of eigenfrequencies actually occurring in the time evolution of ψ. Therefore the spectrum of the motion corresponding to state ψ will consist of N frequencies with average spacings about N_ψ^{-1}. It is now clear that $t^*\approx N_\psi$ and we must estimate N_ψ. First, we assume that N_ψ roughly coincides with the number of unperturbed eigenstates significantly involved in the motion. Next, we identify N_ψ with $\Delta P(t^*)$, that is, with the spread in momentum achieved at time t^*. Experimentally we have $\Delta P(t^*)\sim k(t^*)^{1/2}$ (for $k>1$, $K>1$). Since $N_\psi\sim t^*\sim\Delta P(t^*)$, we finally get that, in order of magnitude, $t^*\sim k^2$. In other words, both the break time and the localization length N_ψ are of the order of the classical diffusion coefficient.

Localization of quantum wave packets in cases where classical mechanics would predict a diffusive behavior is a well-known phenomenon in solid-state physics also.

Indeed, the (time-independent) Hamiltonian of a particle in a random potential on a line can be proved to have a pure-point spectrum with eigenfunctions exponentially localized in space; this fact is known as Anderson localization.

An intimate connection between the localization in momenta which occurs in the kicked rotator and Anderson localization was discovered by Fishmann *et al.*[3] They showed that the equations which must be satisfied by the q.e. eigenfunctions of the rotator, written in the momentum space, look like the Schrödinger equation for the eigenstates of Anderson's model. Whether this similarity is close enough that Anderson's result can be invoked in this case is a point which turns out to depend on the "degree of randomness" of the number sequence $\xi_n = tg[\lambda - \tau(n^2/2)]$. This is a very delicate question; at first sight one would say that the answer might depend strongly on the arithmetic properties of τ. Indeed, we will see in the next section that the spectral type of the rotator depends on such properties of τ.

Nevertheless, for typical irrational values of τ the picture of Anderson localization in momentum space fits very well the rotator problem also. In Fig. 1 we show the shape of the squared modulus of the wave function in momentum space, after the break time t^* and for the initial δ-like state.

The exponential behavior is apparent here; also, the localization length is in excellent agreement with the estimate given above by Chirikov *et al.*[2] This is surprising because Anderson's result is concerned with the *asymptotics* for $n \to \infty$ of *eigenfunctions* of q.e.; here, we find very good exponential decay even for *small n* and for a *nonstationary state.* (Actually, in the next section we will show that the asymptotics for $n \to \infty$ should give in this case a *faster than exponential decay.*) Evidently, some mechanism must be working here that requires further investigation.

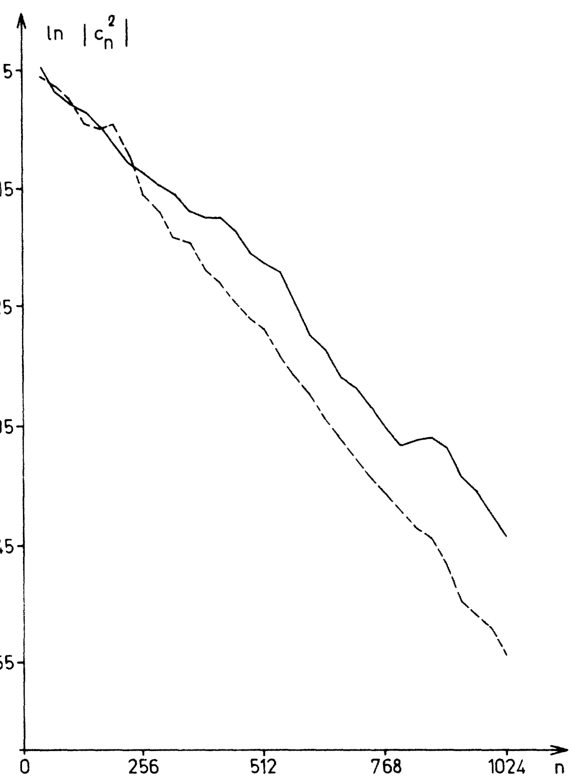

FIG. 1. Distribution function $|c_n|$ after 50 000 iterations of the quantum mapping for the case $k=10$ and $\tau=[0,25,1,1,1,\dots]$ (dashed line), $\tau=[0,25,100,200,400,800,\dots]$ (solid line). Notice the fairly good exponential localization in momentum space over several orders of magnitude.

IV. QUASIENERGY SPECTRUM AND ENERGY GROWTH

In the preceding section we have seen that there is a close relationship between the behavior in time of the rotator energy and the nature of its q.e. spectrum. Actually, this is a rather general feature for periodically driven quantum systems; the nature of the q.e. spectrum has a quite general and definite connection with energy growth (or localization) which can, to some extent, be analyzed without making reference to the particular model under investigation. In this section, we shall outline some basic facts about this relationship. In general, the q.e. spectrum of the operator G in Eq. (7) can be decomposed into its absolutely continuous, singular continuous, and pure-point parts; *a priori*, any or all of these parts may occur.

We shall now investigate the effects of these components on the long-time behavior of the kinetic energy. We have for the kinetic energy

$$E(t) = \sum_{m=-\infty}^{\infty} \frac{m^2}{2} |c_m(t)|^2 \tag{8a}$$

and

$$c_m(nT) = \frac{1}{\sqrt{2\pi}} [e^{im\theta}, \psi(nT)] , \tag{8b}$$

where $\psi(nT) = S^n \psi(0)$ is the wave function at time $t = nT$.

Suppose that $\psi(0)$ lies in the subspace of absolute continuity of S, i.e., that the motion has a pure absolutely continuous spectrum.

As is well known,[4] this implies that $c_m(nT) \to 0$ as $n \to \infty$, and this in turn implies that $E(nT) \to \infty$ as $n \to \infty$, since for all n, $\sum_m |c_m(nT)|^2 = 1$.

Instead, if ψ lies in the pure-point subspace of S, each $c_m(nT)$ displays an almost periodic behavior, resulting in a similar behavior in $E(nT)$.[5]

The third possibility, that motion exhibits a singular continuous spectrum, seems not to be emphasized in much of the literature. In this case, the autocorrelation of ψ given by Eq. (8b) may not tend to zero as $n \to \infty$; however, it will do so when time averaged. Therefore, one would have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} |c_m^2(jT)| = 0 ,$$

but not necessarily $c_m^2(jT) \to 0$. As a consequence, the energy $E(nT)$ would display an erratic behavior, but such that

$$\frac{1}{n} \sum_{j=1}^{n} E(jT) \to \infty \quad \text{as } n \to \infty .$$

In general, both a continuous and a pure-point spectrum can coexist. Generic state vectors ψ decompose accordingly in their projections onto the corresponding subspaces.

That part of the wave packet ψ belonging to these continuous subspaces will keep "spreading" in momentum space; therefore, the presence of a continuous part in the spectrum will always cause the kinetic energy to grow to infinity, in time average at least. In any case, $\langle E \rangle_{nT}^{1/2}$ is a measure of the "spread" of the distribution of the $\langle |c_m|^2 \rangle_{nT}$; therefore, the "height" c_{nT} of the same distribution satisfies $c_{nT} \sim \langle E \rangle_{nT}^{-1/2}$. Thus the rate of decay of the $\langle |c_m|^2 \rangle_{nT}$ has a direct bearing on the rate of increase of $\langle E \rangle_{nT}$. For example, in the case of a singular continuous spectrum, nc_{nT} will be unbounded so that $n \langle E \rangle_{nT}^{-1/2}$ will be unbounded as well. No such limitation is apparent in the case of an absolutely continuous spectrum. This argument is absolutely not rigorous; indeed, providing the needed rigor may not be easy. Nonetheless, our argument does indicate that a singular continuous spectrum may cause a slower rate of energy increase than an absolutely continuous spectrum.

In regard to the resonant motion of the kicked quantum rotator we have specific information on the q.e. spectrum. In particular, it has been shown that, at resonance, the q.e. spectrum has an absolutely continuous component. Moreover, it has been shown[6,7] that the growth of rotator energy with time t is proportional to t^2. But perhaps the most remarkable feature of resonance is that its quadratic growth of energy is the fastest possible.

This follows from an important feature of the dynamics of the rotator; namely, it preserves analyticity. In other words, starting with an analytic wave function at time $t=0$, the wave function must remain analytic for all times. In consequence, by the theory of Fourier series, the c_m distribution in momentum space will at all times have an exponentially or even faster decaying tail. It now takes only a small additional effort to sharpen this result. Indeed, we have previously shown[8] that, if all c_m at time $n\tau$ satisfy the inequality $|c_m(n\tau)| \le Ae^{-\beta|m|}$ for some positive constants A,β, then c_m at time $(n+1)\tau$ will satisfy $|c_m(n+1)\tau)| \le A'e^{-\beta|m|}$, where $A'=qA$ and q is a constant independent of n.

In other words, the fact that $|c_m|=O(e^{-\beta|m|})$ as $|m| \to \infty$ at some time implies that $|c_m|=O(e^{-\beta|m|})$ at all subsequent times. On the other hand, this result implies that, if $c_m=\delta_{mm_0}$ at time 0, then the c_m cannot have an exponential tail at any other time, for in this case, $|c_m|=O(e^{-\beta|m|})$ for whatever $\beta>0$.

Moreover, from $A=qA$, we see that the distribution $|c_m|^2$ cannot spread in time faster than linearly; therefore, the energy $\sum_m m^2|c_m|^2$ cannot grow faster than αn^2, where α is some constant.

V. TRANSITION FROM DISCRETE TO CONTINUOUS SPECTRUM: ANALYTICAL AND NUMERICAL RESULTS

In regard to randomness, or its lack, perhaps the most crucial issue regarding the kicked quantum rotator is whether, and under what circumstances, it has a point and/or a continuous spectrum. Thus far, numerical experiments have proved powerless to decide this issue. In Fig. 2 we present results of a typical experiment with irrational τ, using an initial δ-function excitation with $k=10$, $K>1$. The energy undergoes a diffusivelike growth proportional to the break time t_B followed by a seemingly stationary plateau of small oscillations. However, since the energy must eventually become unbounded (continuous spectrum) or else exhibit a recurrent near re-

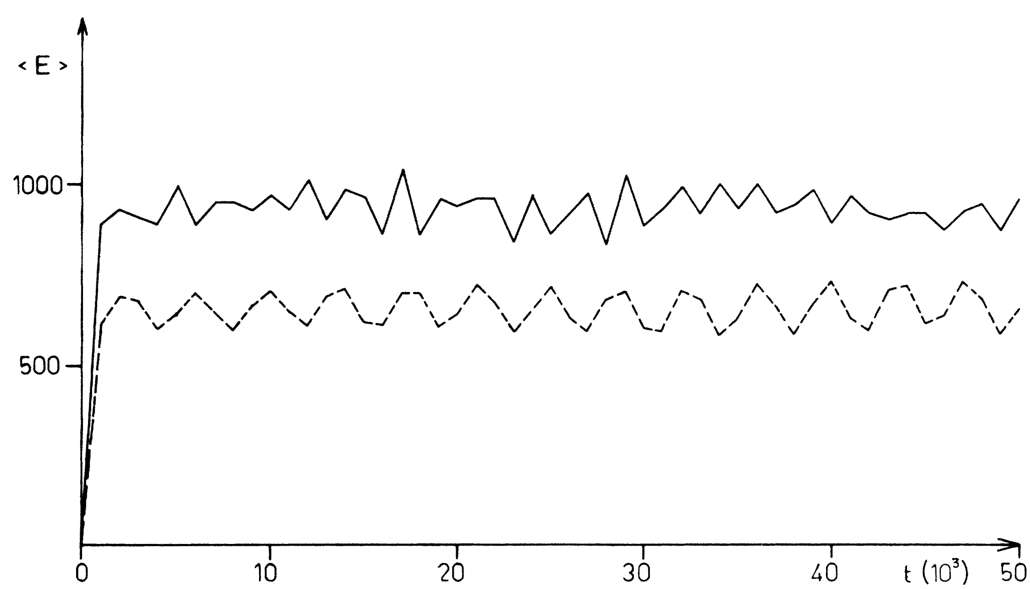

FIG. 2. Average energy $\langle E \rangle$ versus time for the quantum kicked rotator with $k=10$ and $\tau=[0,25,1,1,1,\ldots]$ (dashed line), $\tau=[0,25,100,200,400,800,\ldots]$ (solid line). Integration up to 50000 iterations of the quantum mapping reveals that after the initial increase, the average energy oscillates around a fixed value.

turn to initial state (discrete spectrum), the energy must eventually move away, either up or down, from the value shown in Fig. 2. Unfortunately, even though we have pushed the integration time interval near the limit of currently available super computers, we have been unable to decide which alternative actually occurs in this case. In consequence, we have been driven to analysis.

One's first attempt at analysis might naturally begin with perturbation theory since the unperturbed system ($k=0$) has an especially simple motion, recurrent with a pure-point spectrum. One therefore reasonably expects the weakly perturbed system to also have a pure-point spectrum. From our earlier discussions, however, we know this to be false, no matter the smallness of k, since rational values of τ yield a continuous component in the spectrum. Moreover, detailed investigations reveal that quantum perturbation analysis is here plagued with the same small denominators which appear in the KAM theory of classical mechanics. This similarity to KAM theory again reinforces the expectation that the number theoretic properties of τ may be crucial.

Because of our central concern with the possibility of finding a set of irrational τ values for which the spectrum is continuous, we have, following the lead of KAM theory, examined intervals of τ values centered on irrationals especially well approximated by rationals. In essence, since each rational τ value yields a continuous spectrum, we ask, if by choosing irrational values of τ suitably close to rationals, the continuous spectrum might not survive the perturbation, rational to irrational. Indeed, we have been able to prove[8] that there is a class of irrational τ values for which the q.e. spectrum is continuous.[9] This proof definitively establishes that an irrational, "nonresonant" τ value does not automatically imply a point q.e. spectrum, contrary to earlier belief. However, our rigorous analysis leaves two open questions: What is the nature of this continuous spectrum, and how big is this class or irrationals—specifically whether they are numerically detectable or not? In order to answer the first question, a delicate analysis of the spectrum in resonance is needed but is not yet available. However, even if it were possible to prove that the continuous nonresonant spectrum is singular, we would not then expect behavior in the quantum flow any more "chaotic" than that which occurs in classical weak mixing.

The remaining problem is now whether this spectral peculiarity and its dependence on system parameters can be detected by numerical experiments, as well as how it affects the energy dependence on time.

In order to address these questions, we outline the mathematical arguments used to prove the existence of a continuous spectrum for irrational τ values. As we have seen, for rational τ, the energy grows like t^2, as $t\rightarrow\infty$. The important feature to exploit now is the fact that the time evolution of the rotator depends on τ in a continuous way. This means that, if a given initial state evolves under two different rotator dynamics, associated with slightly different values τ,τ' of the external period, the resulting evolutions will remain close to each other; the longer the time, the smaller the difference $|\tau-\tau'|$. Suppose then that we seek an appropriate τ in the form of a continued fraction $(a_1,a_2,a_3,\ldots)$, i.e.,

$$\tau=a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cdots}}\ ,$$

with principal convergents $\tau_n=p_n/q_n$. The energy $E(t)$ will remain close to the behavior typical for τ_1 during a time interval which depends on the size of $|\tau-\tau_1|$. If $|\tau-\tau_1|$ is small enough, $E(t)$ will also follow to some extent the t^2 increase associated with the rational τ_1 value. At some later time the irrational character of τ would emerge, causing the growth of $E(t)$ to slow; typically, $E(t)$ would then enter an oscillatory regime.

Nevertheless, the second convergent is a better rational approximation to τ than τ_1. Therefore, as it departs from the resonant increase associated with τ_1, $E(t)$ would be still following the resonant trend corresponding to τ_2; this trend continues over a period which is longer as $|\tau-\tau_2|$ is smaller, i.e., a_3 greater. Certainly, by suitably choosing a_3, we can arrange things so that $E(t)$ follows the τ_2 resonance sufficiently long so that it takes part in the corresponding quadratic increase. Can this process be continued indefinitely? In other words, is it possible, by carefully selecting the a_n's to keep $E(t)$ "jumping" from one resonance to another, thus taking up some increment from each? In this case, we would have a nonterminating sequence $(a_1,a_2,\ldots,)$, i.e., an *irrational* τ such that $E(t)$ would be unbounded in time.

It is not easy to answer this question along the qualitative lines just sketched. However, as we have already quoted, a careful analysis[8] yields the exact result, that there are indeed irrational values of τ with this property. These are irrational numbers that are so rapidly approximated by rationals that the asymptotic divergence of $E(t)$, which is typical for rational cases, "carries over" to the limit. In order to find numerical evidence for these facts, one must somehow discriminate between rational and irrational numbers—which is obviously very delicate. However, the advantage of the outlined strategy is that it can be used in effective numerical experiments. In Fig. 3 we show results of one such experiment. One can clearly see the resonant behavior of $E(t)$ corresponding to $\tau=(0,1)$ and to $\tau=(0,1,70)$. In the first case, for irrationals of type $(0,1,70,a_4,\ldots)$, a parting of the ways from (0,1) occurs rather early, but then $E(t)$ shares a more or less extended phase of quadratic increase with the resonance (0,1,70), depending on the choice of a_4. Whatever choice of a_4 one may take, $E(t)$ would then eventually display the resonance $(0,1,70,a_4)$; however, it is not possible to reach this point with the available computers. This indicates that the duration of the oscillatory "plateau" between subsequent resonances is rapidly increasing. The results in Fig. 3 are a sample of the behavior of $E(t)$ corresponding to the continuous irrational or nonresonant spectrum.

Therefore, in light of Fig. 3 and of the discussion above, the following picture emerges for the energy growth. This growth occurs by "jumps" separated by plateaus of exponentially increasing duration. Each jump is associated with the bandlike structure of the resonant spectrum corresponding to one particular convergent τ_n,

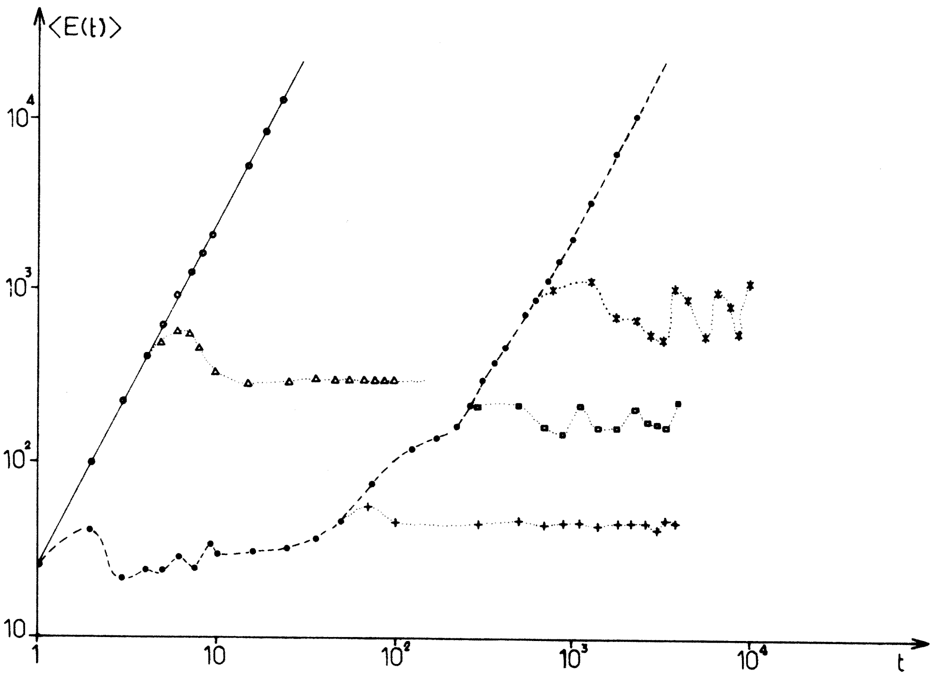

FIG. 3. Average energy $\langle E \rangle$ versus time for the case $k=10$ and for several values of τ. ($\bigcirc$) $\tau=[0,1]$; ($\triangle$) $\tau=[0,1,1000]$; ($\bullet$) $\tau=[0,1,70]$; ($*$) $\tau=[0,1,70,10\,000]$; ($\square$) $\tau=[0,1,70,1000]$; ($+$) $\tau=[0,1,70,50]$. The solid straight line represents the analytical result $\langle E(t) \rangle = (k^2/4)t^2$ for the case $\tau=1$. Notice also the asymptotic t^2 increase for the resonant case $\tau=[0,1,70]$ (dashed line).

and transition into the subsequent "plateau" occurs as soon as the structure of the actual spectrum on a finer scale comes into play.

This peculiar behavior of $E(t)$ is suggestive of a singular continuous character in the spectrum. We now present independent evidence supporting this conclusion. In Fig. 4, we exhibit the behavior of $E(t)$ when a certain *a priori* singular continuous structure (associated in this case with a Cantor set of zero Lebesgue measure) is imposed on the q.e. spectrum. In other words we assume that, as a result of some unspecified external time periodic perturbation, the Floquet operator becomes a unitary operator with a singular continuous spectrum prescribed in advance. However fictitious, this procedure yields results in remarkable qualitative agreement with the above picture.

The possibility that the spectrum of the kicked rotator may exhibit, for irrational τ, some hierarchical, self-similar structure, has been carefully numerically investi-

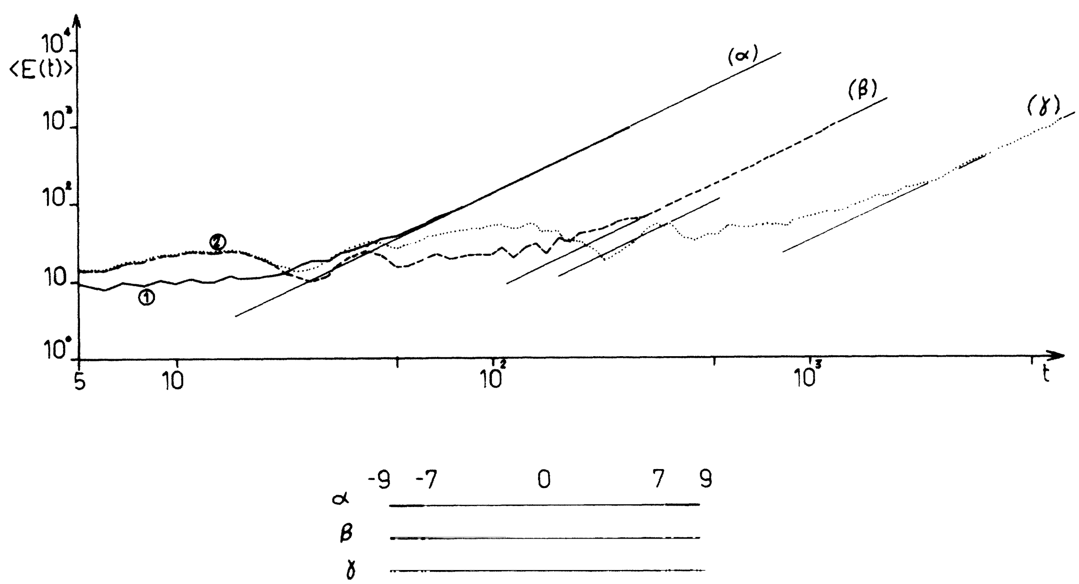

FIG. 4. Plot of $\langle E(t) \rangle$ versus t, in the case where the q.e. spectrum is a Cantor set generated by iterated deletion of intervals, as illustrated. The straight lines show the quadratic increases corresponding to the band structure obtained at steps α,β,γ in the construction of this Cantor set.

gated by Dorizzi, Grammaticos, and Pomeau,[10] with a negative answer. In these computations, regularity in the q.e. spectrum was sought by approximating the golden mean value of τ by ratios of successive Fibonacci numbers. However, for such a "strong" irrational τ, the continuous spectrum may be even lacking. Moreover, it has been proven by Bellissard[11] that the spectrum for irrational τ has no gaps. While this result does not preclude that the spectrum may be singular, it excludes the possibility that it may be a Cantor set.

VI. CONCLUSIONS

The relevance of numerical and analytical results presented in the preceding section to the deeper questions raised in this paper depends on the practicality of detecting the continuous (singular) nonresonant spectrum. In turn, this problem is closely related to the nature of the set of values of τ which yield this type of spectrum. At present, we know only that they make up a dense G-δ set,[11] i.e., a "big" set in the topological sense. However, we cannot yet exclude their having zero (Lebesgue) measure.

Insofar as the effects of a singular continuous spectrum become apparent only over a long time scale and for "strange" parameter values, this kicked rotator behavior may be remindful of Arnold's diffusion. However, some important questions must still be answered before predicting a kind of quantum chaos on account of it. For, in the first place, we have not yet been able to determine its dependence on the perturbation strength k. Second, we should remember that, if the degree of quantum stochasticity is to be assessed according to the same standards as in the classical theory (and this might well be a problem in itself), then only the very poor stochasticity connected with weak mixing might be guaranteed in this problem.

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