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Violations of classical inequalities in quantum optics

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A review is given of correlation experiments in optics which explicitly distinguish between the predictions of classical and quantum theory. In particular, the Cauchy-Schwarz and Bell's inequalities and the phenomena of photon antibunching and squeezing are discussed. The violation of classical inequalities is shown to be associated with the nonexistence of a positive Glauber-Sudarshan P function. An example is given of a dissipative system where a violation of Bell's inequality occurs.

I. INTRODUCTION

The classical theory of electromagnetism predicts a variety of inequalities which may be violated according to quantum mechanics. Single and double beam experiments measuring, in particular, the degree of second-order coherence of light are thus known to provide a means of testing the classical theory of electromagnetism against quantum theory. A review of such nonclassical effects in optics is given in a paper by Loudon.[1] Experiments by Kimble, Dagenais, and Mandel[2–4] and Cresser *et al.*[5] have provided evidence for the nonclassical effect of photon antibunching, thus supporting a quantum theory.

Given that the classical theory of radiation cannot describe all natural phenomena, Bell's theorem[6–10] and inequalities refer to a more general question. Bell's theorem provides a way to test experimentally the prediction of the whole class of local hidden variable theories against the predictions of quantum mechanics. Thus, experiments which demonstrate a violation of Bell's inequality are of wider significance in physics than those which show violation of only the inequalities above. The experiments performed by Aspect *et al.*[11–15] to test Bell's theorem support quantum-mechanical predictions, thus destroying the hypothesis proposed by realists[16] that quantum mechanics has yet to be completed by a local hidden variable (classical) theory. A review of this topic is provided in the paper by Clauser and Shimony.[17]

In this paper we first present a review of experiments designed to test classical radiation theory against the predictions of quantum mechanics. In particular, we examine photon antibunching and violation of the classically predicted Cauchy-Schwarz inequality. The phenomena of squeezing[18] is also discussed. In Sec. III we review Bell's theorem as a stricter test of quantum mechanics. Bell's inequalities are derived for the general case of fields of arbitrary intensity. We discuss the possibility of violation of Bell's inequalities in nonlinear processes such as multimode parametric amplifiers and four-wave mixing, in which the output modes are in some way correlated. The results are extended in Sec. V to a model of a system interacting with its environment, thus including coherent pumping and the effect of dissipation. By examining the steady-state statistics, we show violation of Bell's inequality to be possible in a regime showing strong violation of the Cauchy-Schwarz inequality, thus expanding the applicability of the test of local classical theories versus quantum mechanics to a wider class of systems.

II. CLASSICALLY PREDICTED INEQUALITIES: PHOTON ANTIBUNCHING, CAUCHY-SCHWARZ CRITERION AND SQUEEZING

Experiments measuring the degree of second-order coherence of light are important in providing a means of testing the classical theory of light against quantum theory. We consider a beam of light with stationary and ergodic statistical properties described by a time-dependent cycle averaged classical intensity $I_1(t)$. The classical degree of second-order coherence with time delay τ is written

$$g^{(2)}(\tau)=\langle I_1(t+\tau)I_1(t)\rangle/\langle I_1\rangle^2 \ . \tag{1}$$

In a double beam experiment, one correlates intensities $I_1(t+\tau)$ and $I_2(t)$ of two different beams measured by different detectors

$$g_{12}^{(2)}(\tau)=\langle I_1(t+\tau)I_2(t)\rangle/\langle I_1\rangle\langle I_2\rangle \ . \tag{2}$$

In the classical theory, the zero time delay functions for single-mode beams may be written generally in the form $[I_1(\epsilon_1)=|\epsilon_1|^2, I_2(\epsilon_2)=|\epsilon_2|^2]$

$$\begin{aligned}\langle I_1\rangle&=\int P(\epsilon_1)I_1(\epsilon_1)d^2\epsilon_1 \ ,\\ \langle I_1^2\rangle&=\int P(\epsilon_1)I_1(\epsilon_1)^2d^2\epsilon_1 \ ,\\ \langle I_1I_2\rangle&=\int\int P(\epsilon_1,\epsilon_2)I_1(\epsilon_1)I_2(\epsilon_2)d^2\epsilon_1\,d^2\epsilon_2 \ ,\end{aligned} \tag{3}$$

where the ϵ_i are fluctuating stochastic complex amplitudes describing the fields and $P(\epsilon_i)$ is a probability distribution defined over the entire complex plane. A number of inequalities are derivable from the classical description [Eq. (3)]. First, the variance of the distribution is always positive. Hence,

$$\begin{aligned}&\langle I_1^2\rangle-\langle I_1\rangle^2\geq 0 \ ,\\ &g^{(2)}(0)\geq 1 \ .\end{aligned} \tag{4}$$

Also, the Cauchy-Schwarz inequality is directly applicable:

$$\langle I_1I_2\rangle^2\leq\langle I_1^2\rangle\langle I_2^2\rangle \ . \tag{5}$$

For the case of time delay t between the two beams, we have

$$[g_{12}^{(2)}(t)]^2 \le g_1^{(2)}(0) g_2^{(2)}(0) . \tag{6}$$

In the single beam case, the equation above simplifies to the useful result

$$g^{(2)}(t) \le g^{(2)}(0) . \tag{7}$$

Experimentally, one measures the degree of second-order coherence of a single beam using the apparatus illustrated schematically in Fig. 1. The beam (of intensity $\langle I_A \rangle$, for example) to be studied is split into two parts by a beam splitter (a semitransparent mirror). The intensities ($\langle I_{A1} \rangle$ and $\langle I_{A2} \rangle$, for example) of each arm are measured by two photodetectors whose readings are correlated and averaged electronically with a fixed positive time delay τ. In a classical picture, the incident beam is always split into two identical beams which show identical intensity fluctuations $\langle I_{A1} = I_{A2} \rangle$. Thus, $\langle I_{A1} I_{A2} \rangle$ can be written

$$\begin{aligned} \langle I_{A1} I_{A2} \rangle &= \int P(\epsilon) I_{A1}^2 \, d^2\epsilon \\ &= \tfrac{1}{4} \int P(\epsilon) I_A^2 \, d^2\epsilon \end{aligned} \tag{8}$$

and

$$\langle I_{A1} I_{A2} \rangle / \langle I_{A1} \rangle \langle I_{A2} \rangle$$

is a measure of the function $g^{(2)}(0)$ of the incident beam. Figure 2 shows the experimental arrangement used to measure $g_{12}^{(2)}(0)$ of two beams.

The quantum-mechanical interpretation of second-order coherence is quite different. For a single-mode field, we have

$$g^{(2)}(0) = \langle (a^\dagger)^2 a^2 \rangle / \bar{n}^2 , \quad \bar{n} = \langle a^\dagger a \rangle . \tag{9}$$

$a, a^\dagger$ are boson operators satisfying the commutation relation $[a, a^\dagger] = 1$. The average $\langle \ \rangle$ in this case denotes the quantum expectation value $\langle \hat{O} \rangle = \mathrm{Tr}(\hat{O} \hat{\rho})$ where ρ is the density operator for the light field. Using the boson commutation relation, one may derive

$$\langle (a^\dagger)^2 a^2 \rangle + \langle a^\dagger a \rangle - \langle a^\dagger a \rangle^2 \ge 0$$

and hence the following lower limit on $g^{(2)}(0)$:

$$g^2(0) \ge 1 - \frac{1}{\bar{n}} . \tag{10}$$

Considering a double beam experiment, the Cauchy-Schwarz inequality applied to commuting operators $a^\dagger a$ and $b^\dagger b$ implies

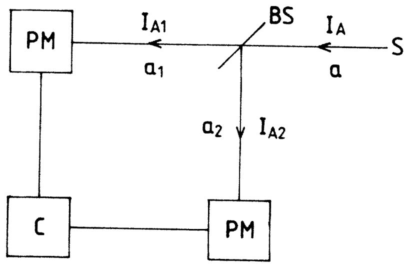

FIG. 1. Test for photon antibunching. BS: beam splitter. PM: photo detector. C: correlator. S: radiation source.

$$| \langle a^\dagger a b^\dagger b \rangle |^2 \le \langle (a^\dagger a)^2 \rangle \langle (b^\dagger b)^2 \rangle .$$

For symmetrical systems such that $\langle (a^\dagger a)^2 \rangle = \langle (b^\dagger b)^2 \rangle$ and $\langle a^\dagger a \rangle = \langle b^\dagger b \rangle = \bar{n}$, this implies the quantum inequality

$$g_{12}^{(2)}(0) \le g^{(2)}(0) + \frac{1}{\bar{n}} . \tag{11}$$

Clearly, the classical inequalities (4) and (6) are too restrictive according to a quantum theory which allows (10) and (11). Light which violates the inequality $g^{(2)}(0) \ge 1$ is said to display the property of photon antibunching.[19] Light fields violating

$$[g_{12}^{(2)}(0)]^2 \le g_1^{(2)}(0) g_2^{(2)}(0)$$

are said to violate the classical Cauchy-Schwarz inequality. Both properties are evidence for the quantum theory of light.

The difference between the predictions of classical and quantum theories with regard to second-order coherence is perhaps best illustrated by the $g^2(0)$ measurement scheme. We adopt a technique developed by Walls[20] and Loudon[1] to describe the system quantum mechanically. Modes a_1 and a_2 correspond to transmission and reflection at the mirror, the original mode a being written $a = (a_1 + a_2)/\sqrt{2}$.

If we consider the mode a to be a single photon number state $|1\rangle$, then the combined mode is a superposition

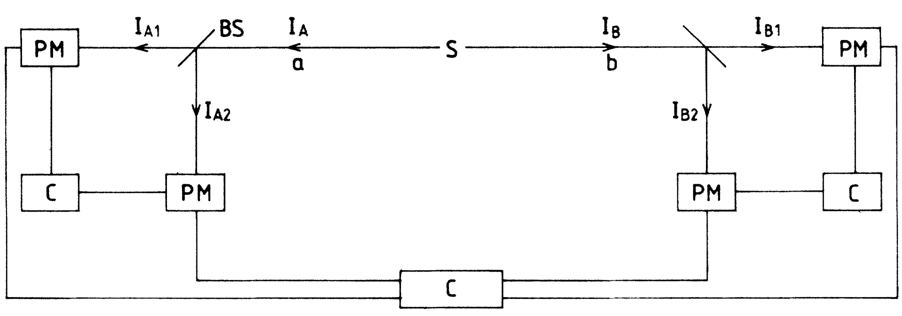

FIG. 2. Test of Cauchy-Schwarz inequality.

$$|\psi\rangle = a^\dagger|0\rangle = \frac{1}{\sqrt{2}}(a_1^\dagger + a_2^\dagger)|0\rangle\,|0\rangle$$
$$= \frac{1}{\sqrt{2}}(|0\rangle\,|1\rangle + |1\rangle\,|0\rangle)\,, \qquad (12)$$

where $|0\rangle$ and $|0\rangle\,|0\rangle$ is the vacuum state for the original mode a and the combined modes a_1 and a_2, respectively. The state (12) incorporates a particulate nature of light. It may be shown, in general, that the measured correlation function

$$\langle a_1^\dagger a_1 a_2^\dagger a_2\rangle / \langle a_1^\dagger a_1\rangle\langle a_2^\dagger a_2\rangle$$

is indeed the

$$g^2(0) = \langle (a^\dagger)^2 a^2\rangle / \langle a^\dagger a\rangle^2$$

of the original beam. There is no justification in a quantum-mechanic interpretation to write the correlation $\langle I_{A1} I_{A2}\rangle$ in terms of an integral form (8) where $I_{A1} = I_{A2}$. In fact, for the idealized single photon number state described above, $\langle I_{A1} I_{A2}\rangle = 0$ implying $I_{A1} = 1$ or 0 and $I_{A2} = 1 - I_{A1}$. Thus, a whole new range of statistics is allowed.

Carmichael and Walls[21] have predicted that the light emitted by a single atom undergoing resonance fluorescence would exhibit the property of photon antibunching. The result was confirmed by Cohen-Tannoudji[22] and Kimble and Mandel.[23] Carmichael and Walls calculated the second-order correlation function $g^{(2)}(\tau)$ in the steady state for the limits of weak and strong driving fields, respectively. For small τ the solutions predict antibunching. Physically, the explanation is that there is a finite time required to reexcite the atom and hence to detect a second photon. Experiments to show this feature have been performed by Kimble, Dagenais, and Mandel.[2] Unfortunately, the effect of number fluctuations in the atomic beam is to preclude direct observation of photon antibunching $[g^2(0) < 1]$. However, violation of the inequality (7) is possible and good agreement has been attained between theory including atomic number fluctuations and experiment. Photon antibunching has been predicted in other optical systems including subsecond harmonic generation[24] and two-photon absorption.[25]

Violation of the Cauchy-Schwarz inequality has been studied by Clauser[26] in the radiation emitted in an atomic two-photon cascade (Fig. 3). Three atomic energy levels are involved. The atom is pumped from the ground state to the upper level, and then emits a photon of frequency ω_1 upon spontaneously decaying to the middle level, and subsequently a photon of frequency ω_2 upon decaying to the ground state. Measurements of the intensity correlations within and between each beam of frequency ω_1 and ω_2 as illustrated in Fig. 2 allows a test of the Cauchy-Schwarz inequalities. As with the atomic fluorescence case, the effect of atomic number fluctuations is to destroy violation of the direct Cauchy-Schwarz inequality (5). However, violation of the inequality (6) is still possible, and experiments performed by Clauser obtain good agreement with the theory. Violation of the Cauchy-Schwarz inequality has also been predicted in the two-photon laser[27] and parametric amplifier.[28]

In quantum optics, quasiprobability distributions are often employed to describe the field in terms of classical c numbers rather than operators. One such representation is the Glauber-Sudarshan P function,[29,30] in which the density operator for the field is expressed in a diagonal coherent state representation as follows:

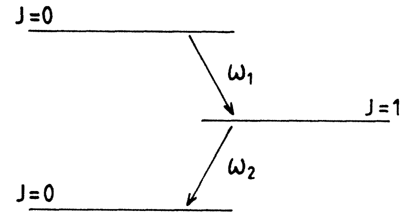

FIG. 3. Atomic two-photon cascade.

$$\rho = \int P(\{\alpha\})|\{\alpha\}\rangle\langle\{\alpha\}|\,d^2\{\alpha\}\,. \qquad (13)$$

$|\{\alpha\}\rangle$ represents a multimode coherent state, the α_i being complex variables and the integration being over the complex planes of the α_i. The advantage of such a P presentation is that the normally ordered correlation functions are expressed in the form of simple integrals

$$\begin{aligned} \langle a^\dagger a\rangle &= \int P(\alpha)|\alpha|^2 d^2\alpha\,, \\ \langle (a^\dagger)^2 a^2\rangle &= \int P(\alpha)|\alpha|^4 d^2\alpha\,, \\ \langle a^\dagger a b^\dagger b\rangle &= \int\int P(\alpha,\beta)|\alpha|^2|\beta|^2 d^2\alpha\, d^2\beta\,. \end{aligned} \qquad (14)$$

Where $P(\alpha)$ exists as a nonsingular positive function, comparison with Eqs. (3) shows that the inequalities $g^2(0) \geq 1$ and the Cauchy-Schwarz inequality (6) are derivable. In this case, the statistics of the field may be described classically. Quantum behavior will occur in a regime where no well-behaved $P(\alpha)$ function exists. In this case other quasiprobability distributions are often used, for example, the Q and Wigner functions. Of particular interest to us is the generalization of the P representation[31,32] as follows:

$$\hat{\rho} = \int P(\{\alpha,\alpha^\dagger\})\frac{|\{\alpha\}\rangle\langle\{\alpha^\dagger\}^*|}{\langle\{\alpha^\dagger\}^*|\{\alpha\}\rangle}\,d\mu(\{\alpha\},\{\alpha^\dagger\}) \qquad (15)$$

in which the correlation functions become

$$\begin{aligned} \langle a^\dagger a\rangle &= \int P\{\alpha,\alpha^\dagger\}\alpha^\dagger\alpha\, d\mu(\{\alpha\},\{\alpha^\dagger\})\,, \\ \langle (a^\dagger)^2 a^2\rangle &= \int P\{\alpha,\alpha^\dagger\}(\alpha^\dagger)^2\alpha^2\, d\mu(\{\alpha\},\{\alpha^\dagger\})\,, \\ \langle a^\dagger a b^\dagger b\rangle &= \int P\{\alpha,\alpha^\dagger,\beta,\beta^\dagger\}\alpha^\dagger\alpha\beta^\dagger\beta\, d\mu(\{\alpha\},\{\alpha^\dagger\})\,. \end{aligned} \qquad (16)$$

$\alpha_i, \alpha_i^\dagger$ are independent complex variables (implying an extra dimensionality) and $d\mu(\{\alpha\},\{\alpha^\dagger\})$ is the integration measure. The variable $\alpha^\dagger\alpha$ is no longer necessarily real and positive, and hence the classical inequalities (4)–(7) are no longer implicit.

Another statistical property to be considered is the fluctuations in the quadrature phases of the field, characterized by the variances $V(X_1)$ and $V(X_2)$ where for a single-mode field

$$X_1 = (a + a^\dagger)/2,\ \ X_2 = (a - a^\dagger)/2i\,. \qquad (17)$$

The Heisenberg uncertainty principle predicts

$$[V(X_1)V(X_2)]^{1/2} \geq \tfrac{1}{4}\,.$$

Hence, the value $\frac{1}{4}$ is the minimum uncertainty product, and reflects quantum noise. A coherent state is a minimum uncertainty state with $V(X_1)=V(X_2)=\frac{1}{4}$. The phenomena of squeezing[18] occurs when the variance in one of the quadratures is reduced below that of a coherent state, i.e., $V(X_1)<\frac{1}{4}$. Squeezing refers to a reduction in quantum noise which is not present in classical radiation theory [where $V(X_1)\geq 0$] and thus is not a direct test of quantum versus classical theories. However, there is a sense in which squeezing is a quantum property. We write in the Glauber-Sudarshan representation

$$\begin{aligned} V(X_1) &= \tfrac{1}{4}+\tfrac{1}{4}(2\langle a^\dagger,a\rangle+\langle a,a\rangle+\langle a^\dagger,a^\dagger\rangle) \\ &= \tfrac{1}{4}\left[1+\int P(\alpha)[\langle(\alpha+\alpha^*)^2\rangle-\langle\alpha+\alpha^*\rangle^2]d^2\alpha\right], \end{aligned} \tag{18}$$

where

$$\langle a_i,a_j\rangle=\langle a_ia_j\rangle-\langle a_i\rangle\langle a_j\rangle .$$

Squeezing is possible only when $P(\alpha)$ becomes negative. Such a property is associated with quantum light fields. Squeezed states of light have been of recent interest as having potential applications in quantum nondemolition measurement theory[33] and in optical communications.[34,35]

A simple example of a system which may exhibit photon antibunching, violation of the Cauchy-Schwarz inequality (6), and squeezing is the nondegenerate parametric amplifier, described by the interaction Hamiltonian:

$$H=i\hbar g(ab-a^\dagger b^\dagger) . \tag{19}$$

Time-dependent solutions are

$$\begin{aligned} a &= a(0)\cosh(|g|t)-\frac{ig}{|g|}b^\dagger(0)\sinh(|g|t) , \\ b &= b(0)\cosh(|g|t)-\frac{ig}{|g|}a^\dagger(0)\sinh(|g|t) . \end{aligned} \tag{20}$$

Photon antibunching is possible if the initial state for the field is a highly excited coherent state.

To examine squeezing, for simplicity we consider the field to be initially in the vacuum state. One finds $\langle a^2\rangle=\langle b^2\rangle=\langle a\rangle=\langle b\rangle=0$ and thus no squeezing is exhibited in the modes a or b. The element $\langle ab\rangle$, however, is nonzero. Upon examining the combined mode $e=(a+e^{-i\phi}b)/\sqrt{2}$, one finds

$$\begin{aligned} V(X_{1,2}) &= \tfrac{1}{4}+\tfrac{1}{4}[\langle a^\dagger a\rangle+\langle b^\dagger b\rangle \\ &\qquad \pm(e^{-i\phi}\langle ab\rangle+e^{i\phi}\langle a^\dagger b^\dagger\rangle)] . \end{aligned} \tag{21}$$

For the optimal choice of phase ϕ, and since $\langle a^\dagger a\rangle=\langle b^\dagger b\rangle$,

$$V(X_1)=\tfrac{1}{4}+\tfrac{1}{2}(\langle a^\dagger a\rangle-|\langle ab\rangle|) . \tag{22}$$

The rotation of modes and selection of a phase shift has been necessary to capture the nonzero phase element $\langle ab\rangle$. Squeezing is thus obtained only when $|\langle ab\rangle|>\langle a^\dagger a\rangle$. The classical theory represents the single modes a and b as stochastic amplitudes ϵ_1 and ϵ_2, respectively, as described in Eq. (3). Consideration of

$$\langle(\epsilon_1+\lambda\epsilon_2^*)(\epsilon_1^*+\lambda^*\epsilon_2)\rangle\geq 0$$

and

$$\lambda=\bar{\lambda}(\mathrm{Re}\langle\epsilon_1\epsilon_2\rangle+i\,\mathrm{Im}\langle\epsilon_1\epsilon_2\rangle)/|\langle\epsilon_1\epsilon_2\rangle|$$

proves a Cauchy-Schwarz inequality

$$|\langle\epsilon_1\epsilon_2\rangle|^2\leq\langle|\epsilon_1|^2\rangle\langle|\epsilon_2|^2\rangle .$$

Quantum mechanically $\langle(a+\lambda b^\dagger)(a^\dagger+\lambda^* b)\rangle$ implies the inequality (since $\langle a^2\rangle=\langle ab^\dagger\rangle=\cdots=0$)

$$|\langle ab\rangle|^2\leq\langle a^\dagger a\rangle^2+\langle a^\dagger a\rangle . \tag{23}$$

Thus, the maximum value of the difference $(\langle ab\rangle-\langle a^\dagger a\rangle)$ is $\frac{1}{2}$ as $\langle a^\dagger a\rangle$ becomes large. This corresponds to perfect squeezing. In fact, the solution (20) implies satisfaction of the identity

$$|\langle ab\rangle|^2=\langle a^\dagger a\rangle^2+\langle a^\dagger a\rangle \tag{24}$$

and the variance $V(X_1)=\frac{1}{4}e^{-2|g|t}$ shows perfect squeezing in the long-time limit.

The violation of the classical inequality $|\langle ab\rangle|^2\leq\langle a^\dagger a\rangle^2$ is not a directly measurable violation of a Cauchy-Schwarz inequality since $\langle ab\rangle$ is not an observable. However, the Cauchy-Schwarz inequality (6) $\langle a^\dagger ab^\dagger b\rangle\leq\langle(a^\dagger)^2a^2\rangle$, predicted by classical radiation theory is also violated. In fact, the quantity $a^\dagger a-b^\dagger b$ is a constant of motion and the modes a and b are maximally correlated at all times,

$$\begin{aligned} &\langle a^\dagger ab^\dagger b\rangle=\langle(a^\dagger)^2a^2\rangle+\langle a^\dagger a\rangle , \\ &\langle a^\dagger a\rangle=\langle b^\dagger b\rangle , \end{aligned} \tag{25}$$

holding for all g and implying the maximum violation of the Cauchy-Schwarz inequalities according to the quantum result (11).

Quantum models of the degenerate parametric amplifier in an optical cavity, including pump and dissipation terms have been studied by Drummond *et al.*,[24] Milburn and Walls,[36] and Lugiato and Strini.[37] Analyses of the nondegenerate parametric amplifier in a cavity have been given by Graham,[38] who used a Wigner representation, and McNeil and Gardiner,[28] who used a generalized P representation. The features of squeezing and antibunching and violation of the Cauchy-Schwarz inequality are shown to be still present in these models. In fact, Graham[39] has derived the following forms of the identities (24) and (25) for the case of nonzero and equal dissipation from each mode a and b in a steady state

$$\begin{aligned} &\text{(i)}\ \ \langle a^\dagger ab^\dagger b\rangle=\langle(a^\dagger)^2a^2\rangle+\tfrac{1}{2}\langle a^\dagger a\rangle , \\ &\text{(ii)}\ \ |\langle ab\rangle|^2=\langle a^\dagger a\rangle^2+\tfrac{1}{2}\langle a^\dagger a\rangle , \end{aligned} \tag{26}$$

showing directly violation of Cauchy-Schwarz inequalities and squeezing to still be possible, although there has been a reduction in the cross correlations between a and b below the maximum allowed value.

The preparation of correlated states of photons implying violation of the classical Cauchy-Schwarz inequality are thus important in providing a test of quantum versus classical theories. As pointed out by Graham[39] the idealized lossless Hamiltonian (19) produces a maximally correlated pure quantum state in which a measurement of the photon number in a mode a determines completely and precisely the photon number in mode b. Similar highly correlated states (spin or polarization states) have been important in discussions of the completeness of quantum mechanics and the reader is referred to the Einstein-Podolsky-Rosen paradox.[16] Particularly important to the question of the completeness of quantum theory is that such states allow a violation of another well-known classical inequality: Bell's inequality.[6] We are concerned with the violation of Bell-type inequalities in highly correlated states such as produced by a nondegenerate parametric amplifier and also the effect of dissipation on any violation observed.

III. BELL'S INEQUALITY

The previous section reviewed evidence that the classical theory of electromagnetism cannot adequately explain results of all second-order correlation experiments. The quantum-mechanical interpretation of the violation of the classical theory of radiation inequalities is considered evidence for the particle nature of light. However, it is the view of many physicists that quantum mechanics is an incomplete theory and the reader is referred to discussions of the Einstein-Podolsky-Rosen paradox.[10,16] Bell's theorem[6-8] refers to a very general question. Can the quantum theory describing the radiation field be completed by any local hidden variable theory?

A hidden variable theory is defined as any physical theory which postulates the existence of states $|\lambda\rangle$ in terms of which the expectation value of a quantum-mechanical observable O is written

$$\langle O\rangle=\int \rho(\lambda)O(\lambda)d\lambda\ . \tag{27}$$

The $\rho(\lambda)$ denotes a normalizable probability distribution over the states $|\lambda\rangle$, with hidden variables λ defining the state $|\lambda\rangle$. The integration over λ may be continuous or discrete, and there is no restriction on the number of variables λ. $O(\lambda)$ is the expected value of O, given that the system is in the state $|\lambda\rangle$ specified by the variables λ.

Now consider the modification (Fig. 4) of the experimental arrangement of Fig. 2 used to test the Cauchy-Schwarz inequality. The 50%-50% beam splitters are replaced by double-beam polarizers A and B. The polarizer $A(B)$ transmits light polarized along an axis of $\theta(\phi)$ to the reference x axis in the x-y plane, and reflects light polarized in the orthogonal direction, in the x-y plane. Photomultipliers detect intensities $\langle I_{A+}\rangle$, $\langle I_{A-}\rangle$, $\langle I_{B+}\rangle$, and $\langle I_{B-}\rangle$ as illustrated, and correlators measure $\langle I_{A+}I_{B+}\rangle$, etc. The functions $\langle I_{A+}\rangle$, $\langle I_{A+}I_{B+}\rangle$, etc. depend on the angles θ, ϕ of the polarizers, as well as the properties λ of the incident light. The function $\rho(\lambda)$ is independent of θ and ϕ. A hidden variable theory writes, in general,

$$\langle I_{Ai}I_{Bj}\rangle_{\theta\phi}=\int \rho(\lambda)I_{Ai}(\lambda,\theta,\phi)I_{Bj}(\lambda,\theta,\phi)d\lambda\ , \tag{28}$$

where we denote, for example, $I_{A+}(\lambda,\theta,\phi)$ the expected values of the intensity at A_+ given the state $|\lambda\rangle$.

It would seem reasonable to assume[6-9,17]

$$\begin{aligned} I_{Ai}(\lambda,\theta,\phi)&=I_{Ai}(\lambda,\theta)\ ,\\ I_{Bj}(\lambda,\theta,\phi)&=I_{Bj}(\lambda,\phi)\ . \end{aligned} \tag{29}$$

This assumption is referred to as the locality assumption and says that, for a given λ, the results at B cannot depend on the value θ the experimenter chooses to select at A. It prescribes that the measured value of a quantity in one system is not casually affected by what one chooses to measure on the other system. A hidden variable theory satisfying (29) is called a local hidden variable theory. The assumptions (28) and (29) for a realistic theory seem quite reasonable and necessary. We note that we are no longer making any assumptions [of the type (8)] that the field is wavelike. However, it is well known that these postulates predict inequalities (Bell's theorem) which clearly contradict the predictions of quantum mechanics.

We consider the following expectation value:

$$\begin{aligned} E(\theta,\phi)&=\frac{\langle I_{A+}I_{B+}\rangle+\langle I_{A-}I_{B-}\rangle-\langle I_{A+}I_{B-}\rangle-\langle I_{A-}I_{B+}\rangle}{\langle I_{A+}I_{B+}\rangle+\langle I_{A-}I_{B-}\rangle+\langle I_{A+}I_{B-}\rangle+\langle I_{A-}I_{B+}\rangle}\\ &=\frac{\langle (I_{A+}-I_{A-})(I_{B+}-I_{B-})\rangle}{\langle (I_{A+}+I_{A-})(I_{B+}+I_{B-})\rangle}\ . \end{aligned} \tag{30}$$

Quantum mechanically, the expectation value is written

$$E(\theta,\phi)=\frac{\langle :(c_+^\dagger c_+-c_-^\dagger c_-)(d_+^\dagger d_+-d_-^\dagger d_-):\rangle}{\langle :(c_+^\dagger c_++c_-^\dagger c_-)(d_+^\dagger d_++d_-^\dagger d_-):\rangle}\ , \tag{31}$$

where : : denotes normal ordering and the detected modes c_+,c_- and d_+,d_- correspond to the fields at A_+,A_- and B_+,B_-, respectively.

Assuming the existence of a local hidden variable theory, one may write the observables $E(\theta,\phi)$ as follows:

$$\begin{aligned} E(\theta,\phi)=\frac{1}{D}\int \rho(\lambda)&[I_{A+}(\lambda,\theta)-I_{A-}(\lambda,\theta)]\\ &\times[I_{B+}(\lambda,\phi)-I_{B-}(\lambda,\phi)]d\lambda\ , \end{aligned} \tag{32}$$

where

$$\begin{aligned} D=\int \rho(\lambda)&[I_{A+}(\lambda,\theta)+I_{A-}(\lambda,\theta)]\\ &\times[I_{B+}(\lambda,\phi)+I_{B-}(\lambda,\phi)]d\lambda\ . \end{aligned}$$

The total intensity through each polarizer is written

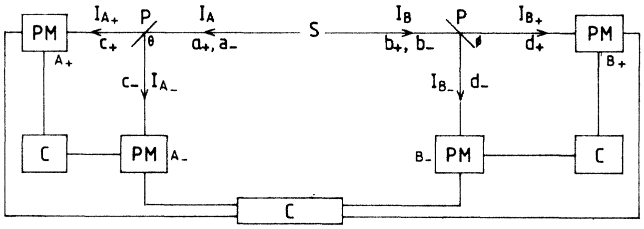

FIG. 4. Test for violation of Bell's inequalities. P: polarizer/beam splitter.

$$\begin{aligned} I_A(\lambda) &= I_{A+}(\lambda,\theta)+I_{A-}(\lambda,\theta)\,, \\ I_B(\lambda) &= I_{B+}(\lambda,\phi)+I_{B-}(\lambda,\phi)\,, \end{aligned} \tag{33}$$

and $I_A(\lambda)$ and $I_B(\lambda)$ are, in principle, independent of θ,ϕ and, in fact, correspond to the intensity of the light at A and B with polarizers removed. Thus,

$$D=\int \rho(\lambda)I_A(\lambda)I_B(\lambda)d\lambda\,. \tag{34}$$

We write

$$\begin{aligned} \bar{S}_A(\lambda,\theta) &= \frac{I_{A+}(\lambda,\theta)-I_{A-}(\lambda,\theta)}{I_A(\lambda)}\,, \\ \bar{S}_B(\lambda,\phi) &= \frac{I_{B+}(\lambda,\phi)-I_{B-}(\lambda,\phi)}{I_B(\lambda)} \end{aligned} \tag{35}$$

and rearrange as follows:

$$E(\theta,\phi)=\frac{1}{D}\int f(\lambda)\bar{S}_A(\lambda,\theta)\bar{S}_B(\lambda,\phi)d\lambda\,, \qquad D=\int f(\lambda)d\lambda\,, \tag{36}$$

where $f(\lambda)=\rho(\lambda)I_A(\lambda)I_B(\lambda)$. The functions are bounded at

$$|\bar{S}_A(\theta)|\,,\quad |\bar{S}_B(\phi)|\le 1\,. \tag{37}$$

Bell's original proof[6] derives an inequality assuming a local deterministic [$|S_A(\lambda,\theta)|=1$, $|S_B(\lambda,\phi)|=1$ for all θ,ϕ] theory, since in the idealized case that Bell considered initially of a single photon in each direction a and b and ideal or maximum correlation between the polarization of each photon for all $\theta=\phi$ [i.e., $S_A(\lambda,\theta)=S_B(\lambda,\theta)$ for all θ], it is possible to prove determinism is necessary. The necessity of a deterministic theory [$I_{A+}(\lambda,\theta)=1,0$ and $I_{A-}(\lambda,\theta)=1-I_{A+}(\lambda,\theta)$] was discussed in Sec. II for the case of a single photon incident on a beam splitter and also follows since $\langle I_{A+}(\lambda,\theta)I_{A-}(\lambda,\theta)\rangle=0$ for all θ. In fact, considering the set up in Fig. 2 to test violation of the classical Cauchy-Schwarz inequality, the maximum correlation [Eq. (25)] allowed by quantum theory between modes a and b also implies a deterministic theory (i.e., the existence of a pure quantum state). However, such perfectly correlated pure quantum states are unlikely to exist in a real experiment. We have already seen, with reference to the Cauchy-Schwarz inequality experiment, that the effect of dissipation on the system is to reduce the intensity correlation [Eq. (26)] between modes below the maximum value. It was shown that the maximum correlation was not necessary to still show quantum statistics and a violation of the classical Cauchy-Schwarz inequality. Similarly, the assumption of determinism is not necessary to derive other forms of Bell's inequality which are violated by certain quantum states, and we proceed as in Bell's second 1971 proof,[8,17] allowing for nonperfect correlation in polarization. We also do not restrict attention to the single photon [$I_A(\lambda)=I_B(\lambda)=1$] case, but allow $I_A(\lambda),I_B(\lambda)>0$.

Thus, proceeding as in Bell's 1971 proof,

$$\begin{aligned} E(\theta,\phi)-E(\theta,\phi') &= \frac{1}{D}\int d\lambda\, f(\lambda)\bar{S}_A(\theta,\lambda)\bar{S}_B(\phi,\lambda)[1\pm\bar{S}_A(\theta',\lambda)\bar{S}_B(\phi',\lambda)] \\ &\quad -\frac{1}{D}\int d\lambda\, f(\lambda)\bar{S}_A(\theta,\lambda)\bar{S}_B(\phi',\lambda)[1\pm\bar{S}_A(\theta',\lambda)\bar{S}_B(\phi,\lambda)]\,, \\ |E(\theta,\phi)-E(\theta,\phi')| &\le \frac{1}{D}\int d\lambda\, f(\lambda)[1\pm\bar{S}_A(\theta',\lambda)\bar{S}_B(\phi',\lambda)]+\frac{1}{D}\int d\lambda\, f(\lambda)[1\pm\bar{S}_A(\theta',\lambda)\bar{S}_B(\phi,\lambda)] \\ &= 2\pm\frac{1}{D}\int d\lambda\, f(\lambda)[\bar{S}_A(\theta',\lambda)\bar{S}_B(\phi',\lambda)+\bar{S}_A(\theta',\lambda)\bar{S}_B(\phi,\lambda)] \\ &= 2\pm[E(\theta',\phi')+E(\theta',\phi)]\,. \end{aligned} \tag{38}$$

Thus,

$$-2\le B\le 2\,, \tag{39}$$

where

$$B=E(\theta,\phi)-E(\theta,\phi')+E(\theta',\phi')+E(\theta',\phi)$$

which is a Bell's inequality, termed the Bell-Clauser-Horne-Shimony-Holt (Bell-CHSH) inequality.

The inequality derived above assumes a general local stochastic hidden variable theory. There has been, however, another assumption [Eqs. (33) and (34)] made which needs testing in a real experiment where ideal conditions may not hold. The assumption has been made that the intensities $I_A(\lambda)$ and $I_B(\lambda)$ [i.e., the function $f(\lambda)=\rho(\lambda)I_A(\lambda)I_B(\lambda)$ and the integral D] are independent of the polarizer angles θ and ϕ. Clauser and Horne[40] derived a modified form of Bell's inequality which does not require this auxiliary assumption. The inequality refers to an experiment using single channel polarizers which allow detection of the mode transmitted in the direction of the

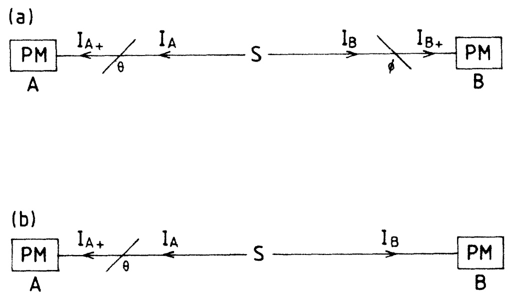

FIG. 5. Test of Clauser-Horne inequalities. (a) Measurement of $G(\theta,\phi)$. (b) Measurement of $r_A(\theta)$.

polarizer axis only.

Consider the experimental arrangement illustrated in Fig. 5. The expected value $\langle I_{A+}I_{B+}\rangle_{\theta\phi}$ is written in terms of a local hidden variable theory

$$G(\theta,\phi)=\langle I_{A+}I_{B+}\rangle_{\theta\phi}=\int d\lambda\,\rho(\lambda)I_{A+}(\lambda,\theta)I_{B+}(\lambda,\phi)\ . \tag{40}$$

The total intensities measured at A and B, respectively, without the polarizers present are independent of θ and ϕ. Hence, $I_A(\lambda)$ and $I_B(\lambda)$ are independent of θ and ϕ and we have $I_A(\lambda)\geq I_{A+}(\lambda,\theta)$ and $I_B(\lambda)\geq I_{B+}(\lambda,\phi)$.

Thus,

$$G(\theta,\phi)=\langle I_{A+}I_{B+}\rangle_{\theta\phi}=\int d\lambda\,f(\lambda)\frac{I_{A+}(\lambda,\theta)I_{B+}(\lambda,\phi)}{I_A(\lambda)I_B(\lambda)}\ , \tag{41}$$

where $f(\lambda)=\rho(\lambda)I_A(\lambda)I_B(\lambda)$. Now the expected value of the intensity products $\langle I_{A+}I_B\rangle_\theta$ and $\langle I_AI_{B+}\rangle_\phi$ [Fig. 5(b)] are

$$\begin{aligned} r_A(\theta)=\langle I_{A+}I_B\rangle_\theta&=\int d\lambda\,\rho(\lambda)I_{A+}(\lambda,\theta)I_B(\lambda)\\ &=\int d\lambda\,f(\lambda)\frac{I_{A+}(\lambda,\theta)}{I_A(\lambda)}\ ,\\ r_B(\phi)=\langle I_AI_{B+}\rangle_\phi&=\int d\lambda\,f(\lambda)\frac{I_{B+}(\lambda,\phi)}{I_B(\lambda)}\ . \end{aligned} \tag{42}$$

The following lemma was used in the Clauser-Horne proof.

Lemma

$$\text{if } 0\leq x,\ x'\leq X \text{ and } 0\leq y,\ y'\leq Y \tag{43}$$

then

$$0\geq xy-xy'+x'y+x'y'+x'Y-yX\geq -XY\ .$$

Hence defining

$$x=\frac{I_{A+}(\lambda,\theta)}{I_A(\lambda)},\quad x'=\frac{I_{A+}(\lambda,\theta')}{I_A(\lambda)}\ ,$$
$$y=\frac{I_{B+}(\lambda,\phi)}{I_B(\lambda)},\quad y'=\frac{I_{B+}(\lambda,\phi')}{I_B(\lambda)}\ ,$$

the lemma ($X=Y=1$), after multiplying by $f(\lambda)$ and integrating the left-hand side inequality, predicts the Clauser-Horne inequality

$$G(\theta,\phi)-G(\theta,\phi')+G(\theta',\phi')+G(\theta',\phi)-r_A(\theta')-r_B(\phi)\leq 0\ . \tag{44}$$

The Bell-type inequalities above arise from the assumptions of any local hidden variable theory, and in this provide a stronger test of quantum mechanics than the inequalities considered previously. One example of a system, suggested by Clauser *et al.*,[41] predicted by quantum mechanics to violate the Bell inequalities is the $J=0\to J=1\to J=0$ two-photon cascade in the singlet states of an alkaline earth (Fig. 3). Laser radiation pumps the atom into an excited $J=0$ state, from which the atom may decay to the ground state via emission of two photons. Violation of the Cauchy-Schwarz inequality for the emitted radiation has been discussed in Sec. II. Because of parity and angular momentum conservation, it is easy to show that there is a strong correlation in the polarization of emitted photons. Conservation of momentum demands that the photons move in opposite directions, along the z axis. In terms of a linear polarization basis we write in quantum mechanics the idealized combined two-photon state as

$$|\psi\rangle=\frac{1}{\sqrt{2}}(a_+^\dagger b_+^\dagger+a_-^\dagger b_-^\dagger)|0\rangle\ . \tag{45}$$

a_+ and a_- are boson operators for modes travelling in the z direction, with polarization vectors along the x and y axes respectively. Similarly b_+ and b_- represent operators for the mode propagating in the $-z$ direction. The detected modes at A_+,A_-,B_+,B_- are orthogonal transformations c_+,c_-,d_+,d_-, respectively, of the modes a_+,a_-,b_+,b_-:

$$\begin{aligned} c_+&=a_+\cos\theta+a_-\sin\theta,\quad d_+=b_+\cos\phi+b_-\sin\phi\ ,\\ c_-&=-a_+\sin\theta+a_-\cos\theta,\quad d_-=-b_+\sin\phi+b_-\cos\phi\ . \end{aligned} \tag{46}$$

θ is the angle between the polarizer axis of A and the x axis. Calculation of the joint correlation functions is straightforward.

$$\begin{aligned} \langle c_+^\dagger c_+d_+^\dagger d_+\rangle&=\cos^2\psi\ ,\\ \langle c_+^\dagger c_+d_-^\dagger d_-\rangle&=\sin^2\psi\ , \end{aligned} \tag{47}$$

where $\psi=\theta-\phi$. Calculation shows

$$E(\theta,\phi)=\cos(2\psi) \tag{48}$$

and on selecting

$$\psi=\phi-\theta=\theta'-\phi=\theta'-\phi'=\tfrac{1}{3}(\theta-\phi') \tag{49}$$

one sees

$$B=3\cos(2\psi)-\cos(6\psi)\ . \tag{50}$$

With $\psi=22.5°$, $B=2\sqrt{2}$ showing clear violation of the inequality (39), $|B|\leq 2$. One may show similarly a violation of the Clauser-Horne inequality (44).

Recent experiments by Aspect *et al.*[11–15] have directly verified the quantum-mechanical predictions (with no corrections for features such as atomic number fluctuations necessary), thus providing the strictest test yet of quantum mechanics.

We are interested in other states of the radiation field produced by nonlinear interactions which give violation of Bell's inequalities. We point out[42,43] in the first instance that such states cannot be described in terms of a nonsingular positive $P(\alpha)$ representation. Consider a test of Bell's inequality involving four modes of the field: a_+, a_-, b_+, b_-, for example, and one considers the rotated modes c_+, c_-, d_+, d_- of Eq. (46). Denoting the c numbers for the a_+, a_-, b_+, b_- modes as $\alpha_+, \alpha_-, \beta_+, \beta_-$, respectively, we define the following c numbers for c_+, c_-, d_+, and d_-:

$$\gamma_+ = \alpha_+\cos\theta + \alpha_-\sin\theta, \quad \delta_+ = \beta_+\cos\phi + \beta_-\sin\phi ,$$
$$\gamma_- = -\alpha_+\sin\theta + \alpha_-\cos\theta, \quad \delta_- = -\beta_+\sin\phi + \beta_-\cos\phi . \tag{51}$$

Considering the expression (31), we may write [denoting $\underline{\alpha} = (\alpha_+, \alpha_-, \beta_+, \beta_-)$] in the Glauber-Sudarshan P representation:

$$\begin{aligned} E(\theta,\phi) &= \frac{1}{X}\int P(\underline{\alpha})(|\gamma_+|^2 - |\gamma_-|^2)(|\delta_+|^2 - |\delta_-|^2)d^2\underline{\alpha} \\ &= \frac{1}{X}\int P(\underline{\alpha})(|\alpha_+|^2 + |\alpha_-|^2)(|\beta_+|^2 + |\beta_-|^2)\bar{S}(\gamma)\bar{S}(\delta)d^2\underline{\alpha} , \end{aligned} \tag{52}$$

where

$$X = \int P(\underline{\alpha})(|\alpha_+|^2 + |\alpha_-|^2)(|\beta_+|^2 + |\beta_-|^2)d^2\underline{\alpha}$$

and

$$\bar{S}(\gamma) = \frac{|\gamma_+|^2 - |\gamma_-|^2}{|\alpha_+|^2 + |\alpha_-|^2}, \quad \bar{S}(\delta) = \frac{|\delta_+|^2 - |\delta_-|^2}{|\beta_+|^2 + |\beta_-|^2} .$$

Since $\gamma_\pm$ is a function of θ and not ϕ, and $\delta_\pm$ is a function of ϕ and not θ, the Glauber-Sudarshan representation is a local one.

The form of Eq. (52) is identical to that for $E(\theta,\phi)$ introduced in Eq. (36). Hence, one can prove the Bell inequality provided the $P(\alpha)$ function is positive and normalizable. For fields possessing a singular or negative $P(\alpha)$ function, the proof breaks down and violation of the Bell inequalities is possible.

Alternatively, one could use a generalized P representation (15). In this case the form of the expression for $E(\theta,\phi)$ is the same, but replacing γ^* with $\gamma^\dagger$, an independent complex variable. Thus, one can no longer place the bounds on $\bar{S}(\gamma)$ and $\bar{S}(\delta)$ and violation of Bell's inequality is possible. A solution for the positive P function, in which $P(\{\alpha,\alpha^\dagger\})$ is always positive and the integration domain is over the entire $\alpha_i, \alpha_i^\dagger$ complex planes,[31] has been derived for the pure quantum state (45) by Drummond.[43]

IV. MODEL INTERACTION HAMILTONIANS

In Sec. II we considered the Hamiltonian

$$H = \hbar(\kappa ab + \kappa^* a^\dagger b^\dagger) \tag{53}$$

and showed how a system modelled by (53) will generate statistics violating, in particular, the Cauchy-Schwarz inequality, as well as showing other features such as squeezing and antibunching. We are now interested in similar types of Hamiltonians describing nonlinear interactions which produce a correlated state of photons and show a violation of Bell's inequalities.

A. Four-mode example

The four-mode field is worthy of consideration since Bell's test may easily be formulated in this case. We may illustrate Bell's inequality by a discussion of the generalization of (53) to two pairs of coupled modes

$$H = \hbar(\kappa a_+^\dagger b_+^\dagger + \kappa a_-^\dagger b_-^\dagger - \kappa^* a_+ b_+ - \kappa^* a_- b_-) , \tag{54}$$

where $\kappa = g\varepsilon_1\varepsilon_2$. For example, a_+ and b_+ may represent two photons of the same linear polarization (along the x axis, for example) but travelling in opposite directions (along the z axis), while a_- and b_- represent photons also travelling in opposite directions along the z axis but with orthogonal polarization (y axis). The frequency and phase matched Hamiltonian may thus describe a system of three-level atoms with selection rules of the type mentioned in Sec. III interacting with two counter propagating pump fields ε_1 and ε_2 (assumed classical) in a typical four-wave mixing scheme.

The time-dependent solutions to the equations of motion are

$$a_\pm = a_\pm(0)\cosh(|\kappa|t) \frac{-i\kappa}{|\kappa|} b_\pm^\dagger(0)\sinh(|\kappa|t) ,$$
$$b_\pm = b_\pm(0)\cosh(|\kappa|t) \frac{-i\kappa}{|\kappa|} a_\pm^\dagger(0)\sinh(|\kappa|t) . \tag{55}$$

Next consider the following orthogonal rotation of modes as may be produced by two beam splitters with transmittivities $\cos\theta$ and $\cos\phi$:

$$c_+ = a_+\cos\theta + a_-\sin\theta, \quad c_- = -a_+\sin\theta + a_-\cos\theta ,$$
$$d_+ = b_+\cos\phi + b_-\sin\phi, \quad d_- = -b_+\sin\phi + b_-\cos\phi . \tag{56}$$

In the example above, the beam splitters correspond to the double beam polarizers discussed in Sec. III and illustrated in Fig. 4. Alternatively, a_+ and a_- (b_+ and b_-) could represent modes impinging at right angles on opposite sides of a semitransparent mirror beam splitter A (and B), the modes a_+ and a_- (b_+ and b_-) being distinguished by different propagation directions as opposed to polarization.

Placing photomultipliers behind the polarizers, one is able to measure the photon number correlation functions and calculate $E(\theta,\phi)$:

$$E(\theta,\phi) = \frac{\langle (c_+^\dagger c_+ - c_-^\dagger c_-)(d_+^\dagger d_+ - d_-^\dagger d_-)\rangle}{\langle (a_+^\dagger a_+ + a_-^\dagger a_-)(b_+^\dagger b_+ + b_-^\dagger b_0)\rangle} . \tag{57}$$

For $\theta=\phi$, $E(\theta,\phi)$ corresponds to a measure C of the correlation in polarization of the emitted photons, simplified by symmetry between a and b to

$$C=\frac{\langle a_+^\dagger a_+ b_+^\dagger b_+\rangle-\langle a_+^\dagger a_+ b_-^\dagger b_-\rangle}{\langle a_+^\dagger a_+ b_+^\dagger b_+\rangle+\langle a_+^\dagger a_+ b_-^\dagger b_-\rangle}=E(\theta,\theta)\ . \tag{58}$$

For the pure quantum state (45) this correlation is perfect, i.e., $C=1$. The function $E(\theta,\phi)$ for fields of the type (45) and also (54) with an initial state a vacuum is independent of the particular reference axes xy chosen for the linear polarization basis [i.e., $E(\theta,\phi)$ depends only on the difference $\psi=\theta-\phi$]. This result follows since the Hamiltonian (54) is invariant under the transformation (56) (with $\theta=\phi$) to a different linear polarization basis. Given that the nonsymmetrical functions such as $\langle a_+^\dagger a_+ b_+^\dagger b_-\rangle$ are zero, this result is equivalent to the following identity:

$$\langle a_-^\dagger b_-^\dagger a_+ b_+\rangle=\langle a_+^\dagger a_+ b_+^\dagger b_+\rangle-\langle a_+^\dagger a_+ b_-^\dagger b_-\rangle\ . \tag{59}$$

Calculation then shows

$$\begin{aligned}\langle c_+^\dagger c_+ d_+^\dagger d_+\rangle&=C_0\cos^2\psi+C_1\sin^2\psi\ ,\\ \langle c_+^\dagger c_+^\dagger d_-^\dagger d_-^\dagger\rangle&=C_0\sin^2\psi+C_1\cos^2\psi\ ,\end{aligned} \tag{60}$$

where $C_0=\langle a_+^\dagger a_+ b_+^\dagger b_+\rangle$ and $C_1=\langle a_+^\dagger a_+ b_-^\dagger b_-\rangle$. Finally quantum mechanics predicts

$$E(\theta,\phi)=C\cos 2\psi,\quad C=\frac{C_0-C_1}{C_0+C_1}\ . \tag{61}$$

When one considers

$$B=E(\theta,\phi)+E(\theta',\phi')-E(\theta,\phi')+E(\theta',\phi)$$

one finds with the choice of angles (49) that

$$B=2\sqrt{2}C\ .$$

Thus, fields of the type (45) and (54) (with the initial state a vacuum), invariant under the transformation (56) with $\theta=\phi$ to a second orthogonal basis, will show a violation of Bell's inequality provided the correlation C of polarization between modes a and b is sufficiently large $C>0.707$. The idealized field (45) produced from two-photon spontaneous emission from a single atom has maximum correlation $C=1$ and shows the maximum violation of Bell's inequalities. The four-mode parametric amplifier (54) (initial state a vacuum) has the solution

$$C=\frac{1}{1+2\tanh^2(|\kappa|t)}\ . \tag{62}$$

Thus, one obtains a violation of Bell's inequality for small $|\kappa|t$. As $|\kappa|t\to\infty$, $C\to\frac{1}{3}$ and the violation is destroyed.

When dealing with multiparticle fields, the correlation C decreases even though one can still in some states have perfect correlation between certain pairs of photons. Drummond[43] considered cooperative spontaneous emission of N photon pairs from N atoms $[|\psi\rangle=(a_+^\dagger b_+^\dagger+a_-^\dagger b_-^\dagger)^N|0\rangle]$ and showed that violation of modified Bell inequalities involving higher-order correlations is still possible. We remark that for the four-mode parametric amplifier (54) discussed above, such higher-order Bell inequalities are still only violated in the small $|\kappa|t$ regime.

There are at least two important features of the quantum states discussed above which allow violation of Bell's inequalities. First, we have strong violation of the classical Cauchy-Schwarz inequality, i.e., the modes a_+ and b_+ (a_- and b_-) are maximally correlated:

$$\langle a_+^\dagger a_+ b_+^\dagger b_+\rangle=\langle (a_+^\dagger)^2 a_+^2\rangle+\langle a_+^\dagger a_+\rangle\ .$$

For example, considering the idealized state (45) in which one photon is emitted into each mode, we have $\langle a_+^\dagger a_+ b_+^\dagger b_+\rangle=\langle a_+^\dagger a_+\rangle$ implying perfect correlation between the detected polarization of each mode. Importantly, the correlation is perfect for all $\theta=\phi$. No classical wave field can explain such perfect correlation for all angles $\theta=\phi$. The idealized state

$$|\psi\rangle=\frac{1}{\sqrt{2}}(a_+^\dagger b_+^\dagger+a_-^\dagger b_-^\dagger)|0\rangle$$

is written in quantum mechanics as a superposition of states. Although there is no intrinsic phase in the system, the element $\langle a_+^\dagger b_+^\dagger a_- b_-\rangle$ is nonzero due to quantum interference between states $a_+^\dagger b_+^\dagger|0\rangle$ and $a_-^\dagger b_-^\dagger|0\rangle$. This is a second important feature of states showing violation of Bell's inequality. To illustrate the quantum interference, we introduce phase shifts between the modes a_+ and a_- and b_+ and b_- as follows:

$$\begin{aligned}c_+&=a_+\cos\theta+a_-\sin\theta e^{i\psi_1}\ ,\\ d_+&=b_+\cos\phi+b_-\sin\phi e^{i\psi_2}\ ,\\ c_-&=-a_+\sin\theta+a_-\cos\theta e^{i\psi_1}\ ,\\ d_-&=-b_+\sin\phi+b_-\cos\phi e^{i\psi_2}\ .\end{aligned} \tag{63}$$

Then calculation shows for states of the type discussed above where nonsymmetrical elements are zero

$$\begin{aligned}E(\theta,\phi)=C[&\cos(2\theta)\cos(2\phi)\\ &+\sin(2\theta)\sin(2\phi)\cos(\psi_1+\psi_2)]\ .\end{aligned} \tag{64}$$

The optimal situation is $\psi_1+\psi_2=0$. For $\psi_1+\psi_2=\pi/2$ one obtains complete factorization with respect to θ and ϕ and violation of Bell's inequality is not possible.

B. Two-mode example

The states shown above to violate Bell's inequalities showed two strong features: maximum intensity correlations between linearly polarized modes propagating in opposite directions, thus giving a strong violation of the classical Cauchy-Schwarz inequality; and the demonstration of quantum interference. The predictions of local hidden variable theories via Bell's inequalities are formulated in terms of correlations of four beams. We are thus introduced to the possibility of such experiments performed on two-mode output systems showing strong violation of the classical Cauchy-Schwarz inequality, with appropriate use of beam splitters to provide the quantum interference.

Consider the idealized state

$$\psi = a^\dagger b^\dagger |0\rangle , \tag{65}$$

where $|0\rangle$ symbolizes the joint vacuum state for mode a and b. Such a state models for example the two-photon cascade where a single excited atom emits a photon of mode a and of mode b. We consider also the two-mode Hamiltonian studied earlier

$$H = \hbar(\kappa^* ab + \kappa a^\dagger b^\dagger) . \tag{66}$$

The time-dependent solutions for (66) are given by

$$\begin{aligned} a &= a(0)\cosh(|\kappa|t) \frac{-i\kappa}{|\kappa|} b^\dagger(0)\sinh(|\kappa|t) , \\ b &= b(0)\cosh(|\kappa|t) \frac{-i\kappa}{|\kappa|} a^\dagger(0)\sinh(|\kappa|t) . \end{aligned} \tag{67}$$

Both systems (65) and (66) prepare correlated pairs of photons and show violation of the Cauchy-Schwarz inequality. Violation of the Bell inequalities may be shown as illustrated in Fig. 6 by splitting each of the two output modes a and b with a 50%-50% beam splitter. The modes are then recombined (a with b) by the second beam splitters with variable transmittivities given by x and w. Thus, the final detected modes are defined

$$\begin{aligned} c_1 &= ax + by, \quad d_1 = aw - bv , \\ c_2 &= -ay + bx, \quad d_2 = av + bw , \end{aligned} \tag{68}$$

where $x^2+y^2=1$ and $w^2+v^2=1$. For convenience we express $x=\cos\theta$, $y=\sin\theta$, $w=\cos\phi$, $v=\sin\phi$. Calculation shows

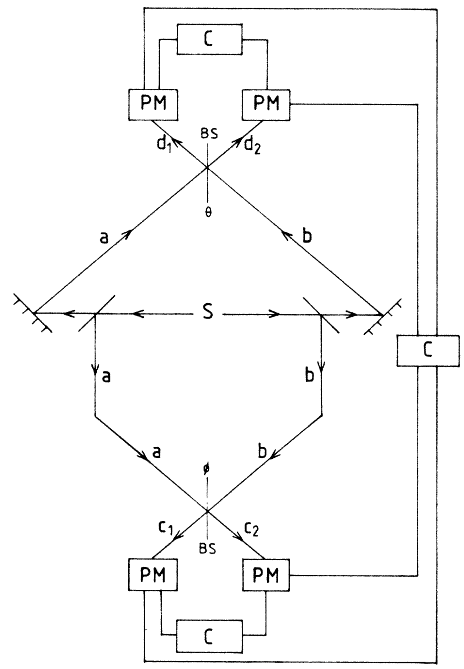

FIG. 6. Test of Bell's inequalities: two-mode source.

$$\begin{aligned} \langle c_1^\dagger c_1 d_1^\dagger d_1 \rangle &= \langle a^\dagger a b^\dagger b \rangle \sin^2(\theta-\phi) \\ &\quad + \langle (a^\dagger)^2 a^2 \rangle (\cos^2\theta \cos^2\phi + \sin^2\theta \sin^2\phi) , \\ \langle c_1^\dagger c_1 d_2^\dagger d_2 \rangle &= \langle a^\dagger a b^\dagger b \rangle \cos^2(\theta-\phi) \\ &\quad + \langle (a^\dagger)^2 a^2 \rangle (\cos^2\theta \sin^2\phi + \sin^2\theta \cos^2\phi) \end{aligned} \tag{69}$$

since nonsymmetrical terms such as $\langle (a^\dagger)^2 ab \rangle$ are zero for states of the type (65) and (66) we are considering. Defining

$$E(\theta,\phi) = \frac{\langle :(c_1^\dagger c_1 - c_2^\dagger c_2)(d_1^\dagger d_1 - d_2^\dagger d_2): \rangle}{\langle :(c_1^\dagger c_1 + c_2^\dagger c_2)(d_1^\dagger d_1 + d_2^\dagger d_2): \rangle} , \tag{70}$$

we find

$$E(\theta,\phi) = \frac{\langle a^\dagger a b^\dagger b \rangle \cos[2(\theta-\phi)] + \langle (a^\dagger)^2 a^2 \rangle \cos(2\theta)\cos(2\phi)}{\langle a^\dagger a b^\dagger b \rangle + \langle (a^\dagger)^2 a^2 \rangle} . \tag{71}$$

The derivation of the Bell inequality

$$-2 \le E(\theta,\phi) - E(\theta,\phi') + E(\theta',\phi) + E(\theta',\phi') \le 2$$

satisfied by local hidden variable theories and concerning the correlations of the four detected beams is directly applicable. We select the following values for the parameters

$$\theta=0, \quad \phi=\pi/8, \quad \theta'=\phi/4, \quad \phi'=3\pi/8$$

and find

$$\begin{aligned} E(\theta,\phi) - E(\theta,\phi') + E(\theta',\phi') + E(\theta',\phi) \\ = \frac{-\langle a^\dagger a b^\dagger b \rangle 2\sqrt{2}}{\langle a^\dagger a b^\dagger b \rangle + \langle (a^\dagger)^2 a^2 \rangle} . \end{aligned} \tag{72}$$

Thus, violation of Bell's inequality is attained where

$$C' = \frac{\langle a^\dagger a b^\dagger b \rangle}{\langle a^\dagger a b^\dagger b \rangle + \langle (a^\dagger)^2 a^2 \rangle} \ge 0.707 . \tag{73}$$

Clearly a strong violation of the Cauchy-Schwarz inequality in the prepared state will imply violation of the Bell inequality. The state (65) has $\langle (a^\dagger)^2 a^2 \rangle$ zero and hence $C'=1$, thus exhibiting the maximum violation possible. The state (66) satisfies $\langle a^\dagger a b^\dagger b \rangle = \langle (a^\dagger)^2 a^2 \rangle + \langle a^\dagger a \rangle$ and hence it is seen that the Bell inequality is violated in the region of small $\langle a^\dagger a \rangle$ or for small $|\kappa| t$.

Highly correlated pairs of photons may be produced by parametric down conversion in a nonlinear dielectric. Early experiments were conducted by Burnham and Weinberg.[44] Recent experiments by Friberg, Hong, and Mandel[45] have produced photon pairs in a potassium dihydrogen phosphate crystal with a correlation time of the order of 100 psec.

V. VIOLATION OF BELL'S INEQUALITY IN DISSIPATIVE SYSTEMS

The states demonstrated in Sec. IV to violate Bell's inequalities are pure quantum states with maximum intensity correlations between two modes a and b. In reality, a quantum system will always interact with its macroscopic environment introducing such effects as dissipation. For example, a more realistic description (discussed in Sec. II) of the two-mode parametric amplifier (65) would be

$$H = ih(ca^\dagger b^\dagger - c^\dagger ab) + ih\varepsilon(c^\dagger - c) + a^\dagger\Gamma_a + a\Gamma_a^\dagger + b^\dagger\Gamma_b + b\Gamma^\dagger , \tag{74}$$

where c is a quantized pump mode, pumped by the classical laser field ε, and Γ_a, Γ_b are reservoir operators modelling the environment and introducing a dissipation rate κ of field modes a and b. A strong violation of the classical Cauchy-Schwarz inequality has been shown to be still possible in the steady state for small $\langle a^\dagger a\rangle$.[28,39] In fact, as discussed above, the identity $\langle a^\dagger a b^\dagger b\rangle = \langle (a^\dagger)^2 a^2\rangle + \frac{1}{2}\langle a^\dagger a\rangle$ holds in the steady state.[39] The test of Bell's inequality formulated above for two-mode systems is directly applicable and the condition (73) on the intensity correlations for violation of Bell's inequality can still be satisfied, for small $\langle a^\dagger a\rangle$.

In this section, we consider the four mode system described by the Hamiltonian (54) with two pairs of coupled modes. We include damping of each mode, and also coherent pumping of the pump mode, which we now quantize in this more complete model. We illustrate how the quasiprobability distributions discussed in Sec. II may be used to describe the system in terms of c numbers rather than operators. By examining the steady-state statistics, we aim to show violation of Bell's inequalities, thus expanding the applicability of the test of local hidden variable theories to systems including dissipation. This result has been presented by Reid and Walls[42] in summarized form.

The model Hamiltonian we wish to study is, in the interaction picture:

$$\begin{aligned} H &= \sum_{i=1}^{3} H_i , \\ H_1 &= i\hbar\kappa'\varepsilon_p c(a_+^\dagger b_+^\dagger + a_- b_-) \\ &\quad - ih\kappa'^*\varepsilon_p^{\dagger *} c^\dagger(a_+ b_+ + a_- b_-) , \\ H_2 &= i\hbar\varepsilon(c^\dagger - c) , \\ H_3 &= \sum_{i=1}^{5} (a_i\Gamma_i^\dagger + a_i^\dagger\Gamma_i) . \end{aligned} \tag{75}$$

H_1 describes the interaction of a suitable atomic medium with the fields. For example, a_+ and a_- could be cavity modes of frequency ω, propagating in the same direction z, but with perpendicular polarization vectors (aligned along the x and y axes, for example). b_+, b_- are modes of the same frequency ω but propagating in the $-z$ direction, and with polarization vectors defined similarly to the a modes.

The pump modes ε_p and c also have propagation vectors in opposite directions, so that the phase and frequency matched Hamiltonian H_1 describes a four-wave mixing process. Mode c is pumped by an external laser driving field of amplitude ε, and this process is modelled by the Hamiltonian H_2. The intensity of the driving mode ε_p is assumed to be sufficiently large and undepleted that we may replace the mode operator by the classical laser amplitude, the complex number ε_p. The atom-field interaction H_1 may thus describe a four-wave mixing via a two-photon atomic process, in which a selection rule dictates that the interaction proceeds via modes of the same polarization. We point out, however, that the solutions to be presented may be applied to any physical system described by a Hamiltonian of the form (75). Irreversible damping of the cavity modes $c, a_\pm, b_\pm$ is described by the Hamiltonian H_3. For convenience, we have denoted the system modes a_+, a_-, b_+, b_-, c by a_i $(i = 1,2,3,4,5)$, respectively. Γ_i represent reservoir operators interacting with each mode a_i.

Standard techniques[46] are employed to eliminate the heat baths and derive the equation of motion for the system density operator ρ. The master equation is

$$\frac{\partial\rho}{\partial t} = \frac{1}{ih}[H_1 + H_2, \rho] + \sum_{i=1}^{5} \gamma_i(2a_i\rho a_i^\dagger - a_i^\dagger a_i\rho - \rho a_i^\dagger a_i) + \sum_{i=1}^{5} 2\gamma_i n_i^{\text{th}}(a_i\rho a_i^\dagger - \rho a_i a_i^\dagger - a_i^\dagger a_i\rho + a_i^\dagger\rho a_i) , \tag{76}$$

where γ_i are the mode damping rates, and n_i^{th} are the mean numbers of thermal photons in the heat baths Γ_i.

The operator equations may be converted into a c-number:[31] classical-type Fokker-Planck equation by expanding the density operator $\hat{\rho}$ in terms of the generalized Glauber P representation, developed by Drummond and Gardiner,[31]

$$\rho = \int P(\alpha_i, \alpha_i^\dagger)\frac{|\{\alpha_i\}\rangle\langle\{\alpha_i^\dagger\}^*|}{\{\alpha_i^\dagger\}^*\{\alpha_i\}} du(\{\alpha_i\},\{\alpha_i^\dagger\}) , \tag{77}$$

where α_i $(i = 1,2,3,4,5)$ correspond to modes a_i $(i = 1,2,3,4,5)$. We denote $\alpha_1 = \alpha_+, \alpha_2 = \alpha_-, \alpha_3 = \beta_+, \alpha_4 = \beta_-, \alpha_5 = \eta$ for ready identification. $|\{\alpha_i\}\rangle$ is the five-mode coherent state $|\alpha_+\rangle|\alpha_-\rangle|\beta_+\rangle|\beta_-\rangle|\eta\rangle$. P is the quasiprobability function defined over the integration domain, yet to be determined, and $d\mu(\{\alpha_i\},\{\alpha_i^\dagger\})$ is the integration measure. The stochastic variables α_i and $\alpha_i^\dagger$ are independent complex variables, no longer complex conjugate as in the usual Glauber P representation. Using operator rules, identical in form to those used in the usual Glauber formulation, one may derive the following Fokker-Planck equation:

$$\frac{\partial P}{\partial t}=\Bigg\{\Bigg[\Bigg[-\frac{\partial}{\partial\eta}(\varepsilon-\gamma_5\eta-\kappa\alpha_+\beta_+-\kappa\alpha_-\beta_-)-\frac{\partial}{\partial\alpha_+}(-\gamma_1\alpha_++\kappa\eta\beta_+^\dagger)-\frac{\partial}{\partial\alpha_-}(-\gamma_2\alpha_-+\kappa\eta\beta_-^\dagger)$$
$$-\frac{\partial}{\partial\beta_+}(-\gamma_3\beta_++\kappa\eta\alpha_+^\dagger)-\frac{\partial}{\partial\beta_-}(-\gamma_4\beta_-+\kappa\eta\alpha_-^\dagger)\Bigg]+\text{c.c.}\Bigg]$$
$$+\Bigg[\Bigg[\frac{\partial^2}{\partial\alpha_+\partial\beta_+}\kappa\eta+\frac{\partial^2}{\partial\alpha_-\partial\beta_-}\kappa\eta\Bigg]+\text{c.c.}\Bigg]+\gamma_1 n_1^{\text{th}}\frac{\partial^2}{\partial\alpha_+\partial\alpha_+^\dagger}+\gamma_2 n_2^{\text{th}}\frac{\partial^2}{\partial\alpha_-\partial\alpha_-^\dagger}+\gamma_3 n_3^{\text{th}}\frac{\partial^2}{\partial\beta_+\partial\beta_+^\dagger}$$
$$+\gamma_4 n_4^{\text{th}}\frac{\partial^2}{\partial\beta_-\partial\beta_-^\dagger}+\gamma_5 n_5^{\text{th}}\frac{\partial^2}{\partial\eta\,\partial\eta^\dagger}\Bigg\}P\,, \tag{78}$$

where c.c. means the repeat of terms of immediately preceding, but interchanging $\alpha_i \leftrightarrow \alpha_i^+$, $\kappa \leftrightarrow \kappa^*$, and $\varepsilon \leftrightarrow \varepsilon^*$, and we define $\kappa=\kappa'\varepsilon_p$.

In a semiclassical theory (one in which the electromagnetic field is considered classical) the double derivative noise terms of the form $(\partial^2/\partial\alpha_+\partial\beta_+)\kappa\eta$ are absent. These terms are a consequence of the noncommutativity of boson field operators in quantum electromagnetic theory and are referred to as quantum noise terms. It is the quantum noise terms which allow quantum effects (such as violation of Bell inequalities) in the output statistics, and which force us to use a quantum quasiprobability distribution such as the generalized P representation. The work of Drummond and Gardiner[31] shows how the quantum noise terms, in the normal Glauber P representation, destroy the positive-definite nature of the diffusion matrix $\underset{\approx}{D}$. However, in the extra dimensional P representation, one obtains a positive definite diffusion matrix.

It is worthwhile examining the deterministic or semiclassical behavior of the system. This refers to the behavior in the presence of zero noise, that is, where the stochastic nature of the variables may be ignored ($\langle\alpha_i\rangle=\alpha_i$). The steady-state ($\dot\alpha_i=0$) solution for the output field intensities is

$$|\alpha_+|^2+|\alpha_-|^2=\begin{cases}0, & \varepsilon<\varepsilon_{\text{thr}}\\ \dfrac{1}{\kappa}(\varepsilon-\varepsilon_{\text{thr}}), & \varepsilon\ge\varepsilon_{\text{thr}}\end{cases} \tag{79}$$

where $\varepsilon_{\text{thr}}=\gamma_5\gamma/\kappa$ and we have considered the case of symmetrical damping: $\gamma_1=\gamma_2=\gamma_3=\gamma_4=\gamma$. It is seen that there is abrupt transition behavior at the threshold driving intensity ε_{thr} (Fig. 7).

At this stage, we assume the pump mode c is heavily damped ($\gamma_5 \gg \gamma$), so that it may be adiabatically eliminated from the equations. The simplest procedure is to convert Eq. (78) to its equivalent Langevin equation using Ito rules, and then to assume $\dot\eta=\dot\eta^\dagger=0$ and solve for the amplitude η. To simplify further, all thermal noise is ignored ($n_i^{\text{th}}=0$), a valid assumption at low temperatures.

The final Fokker-Planck equation is

$$\frac{\partial P}{\partial t}=\Bigg[\Bigg[-\frac{\partial}{\partial\alpha_+}(-\gamma\alpha_++\kappa\bar\eta\beta_+^\dagger)-\frac{\partial}{\partial\alpha_-}(-\gamma\alpha_-+\kappa\bar\eta\beta_-^\dagger)-\frac{\partial}{\partial\beta_+}(-\gamma\beta_++\kappa\bar\eta\alpha_+^\dagger)-\frac{\partial}{\partial\beta_-}(-\gamma\beta_-+\kappa\bar\eta\alpha_-^\dagger)$$
$$+\frac{\partial^2}{\partial\alpha_+\partial\beta_-}\kappa\bar\eta+\frac{\partial^2}{\partial\alpha_-\partial\beta_-}\kappa\bar\eta\Bigg]+\text{c.c.}\Bigg]P \tag{80}$$

with $\bar\eta=(\varepsilon-\kappa\alpha_+\beta_+)/\gamma_5$. The steady-state solution to (80) can be found by the standard method of potentials.[47] The potential solution is

$$P=N[(\varepsilon-\kappa\alpha_+\beta_+-\kappa\alpha_-\beta_-)(\varepsilon^*-\kappa\alpha_+^\dagger\beta_+^\dagger-\kappa\alpha_-^\dagger\beta_-^\dagger)]^q$$
$$\times\exp(2\alpha_+^\dagger\alpha_++2\alpha_-^\dagger\alpha_-+2\beta_+^\dagger\beta_++2\beta_-^\dagger\beta_-)\,, \tag{81}$$

where

$$q=\frac{2\gamma\gamma_5}{\kappa^2}-1=2\left[\frac{\varepsilon_{\text{thr}}}{\kappa}\right]-1$$

and N is a normalization constant, yet to be determined. If we had used the usual Glauber P function, the solution for P would be identical in form to (81), but replacing $\alpha_i^\dagger$

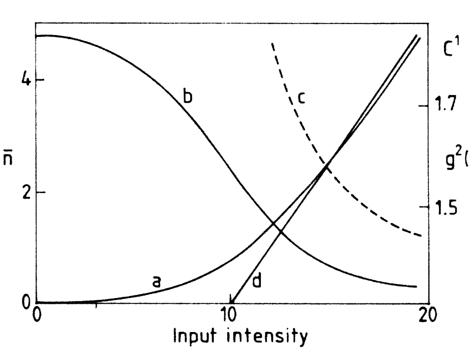

FIG. 7. Violation of Cauchy-Schwarz inequality vs input pump intensity $|\varepsilon/\kappa|^2$, $q=19$. (a) $\bar n=\langle a_+^\dagger a_+\rangle$. (b) $g^{(2)}(0)$. (c) $c^1=\langle a_+^\dagger a_+ b_+^\dagger b_+\rangle/\langle a_+^\dagger a_+\rangle\langle b_+^\dagger b_+\rangle$. (d) semiclassical intensity.

with α_i^* (the complex conjugate) and with the integration of variables being defined over the entire complex planes. It is seen that the $2|\alpha_+|^2, \ldots$ terms would make such a Glauber P solution divergent and unnormalizable.

The statistical properties of the light are provided by the moments of the quasiprobability distribution. Normally ordered correlation functions are written, in the P representations

$$\begin{aligned}\langle (a_+^\dagger)^i (a_-^\dagger)^j (b_+^\dagger)^m (b_-^\dagger)^n a_+^i a_-^j b_+^m b_-^n \rangle &= I_{ijmn} \\ &= \int P(\underline{\alpha})(\alpha_+^\dagger)^i \alpha_+^{\dagger i} (\alpha_-^\dagger)^j \alpha_-^j (\beta_+^\dagger)^m \beta_+^m (\beta_-^\dagger)^n \beta_-^n \, d\underline{\alpha} ,\end{aligned} \tag{82}$$

where $d\underline{\alpha} = d\alpha_+ d\alpha_+^\dagger d\alpha_- d\alpha_-^\dagger d\beta_+ d\beta_+^\dagger d\beta_- d\beta_-^\dagger$. The normalization constant N is deduced from the zeroth moment ($i=m=j=n=0$): $I_{0000} = N^{-1}$. The higher correlations, for example the mean intensity $\langle a_+^\dagger a_+ \rangle$, are then written as the ratio of two moments:

$$\langle a_+^\dagger a_+ \rangle = I_{1000} / I_{0000} = N I_{1000} .$$

A method for calculating the required integrals is given in the Appendix. The intensity $\langle a_+^\dagger a_+ \rangle$ of photons of each mode in the cavity is ($N' = N_0 N$) and the second-order correlation function for each mode is

$$\begin{aligned}\langle a_+^\dagger a \rangle &= \frac{2N' \left| \frac{\varepsilon}{\kappa} \right|^2}{(q+3)^2} \sum_{m=0}^{\infty} \frac{z^m}{(q+4)_m^2} \, {}_1F_2(2; q+m+4, q+m+2; z) , \\ \langle (a_+^\dagger)^2 a_+^2 \rangle &= \frac{2N' \left| \frac{\varepsilon}{\kappa} \right|^2 z}{(q+4)^2 (q+3)^2} \sum_{m=0}^{\infty} \frac{z^m}{(q+5)_m^2} \, {}_1F_2(3; q+m+5, q+m+5; z) .\end{aligned} \tag{83}$$

Joint mode correlations are

$$\begin{aligned}\langle a_+^\dagger a_+ b_+^\dagger b_+ \rangle &= \frac{N' \left| \frac{\varepsilon}{\kappa} \right|^2}{(q+3)^2} \sum_{m=0}^{\infty} \frac{z^m (m+1)^2}{(q+4)_m^2} \, {}_1F_2(1; q+m+4, q+m+4; z) , \\ \langle a_+^\dagger a_+ b_-^\dagger b_- \rangle &= \langle a_+^\dagger a_+ a_-^\dagger a_- \rangle \\ &= \frac{N' \left| \frac{\varepsilon}{\kappa} \right|^2 z}{(q+4)^2 (q+3)^2} \sum_{m=0}^{\infty} \frac{z^m (m+1)}{(q+5)_m^2} \, {}_1F_2(2; q+m+5, q+m+5; z) ,\end{aligned} \tag{84}$$

where $z = |2\varepsilon/\kappa|^2$.

Calculations reveal certain functions to be zero, e.g.,

$$\langle a_i \rangle = \langle a_i^2 \rangle = \cdots = \langle a_+ b_- \rangle = \langle a_+^\dagger b_-^\dagger a_- b_+ \rangle = 0 .$$

However, the values of other nondiagonal elements such as $\langle a_+ b_+ \rangle$ and $\langle a_+^\dagger b_+^\dagger a_- b_- \rangle$ are nonzero

$$\langle a_+ b_+ \rangle = N' \left| \frac{\varepsilon}{\kappa} \right| (q+2) \sum_{m=0}^{\infty} \frac{z^m (m+1)}{(q+m+2)(q+m+1)} \, {}_1F_2(1; q+m+2, q+m+3; z) \tag{85}$$

and

$$\langle a_+^\dagger b_+^\dagger a_- b_- \rangle = \frac{N' \left| \frac{\varepsilon}{\kappa} \right|^2}{(q+4)^2 (q+3)^2} \sum_{m=0}^{\infty} \frac{z^m (m+1)}{(q+4)_m^2} \, {}_1F_2(2; q+m+4, q+m+4; z) .$$

These nonzero correlations, although not observables themselves, are important in the calculation of the quantum properties of squeezing and violation of Bell's inequalities, respectively.

The expressions (83)–(85) have been numerically computed, and results are shown plotted in Figs. 7 and 8. The mean intensity $\langle a_+^\dagger a_+ \rangle$ (Fig. 7) behaves in a similar fashion to the semiclassical mean, except that there is a small nonzero intensity below threshold. At and above threshold, an interesting property is apparent. The effect of the quantum fluctuations is to decrease the mean intensity above threshold, since $\langle a_+^\dagger a_+ \rangle$ becomes less than the semiclassical mean, by $\sim \frac{1}{8}$. This behavior is also exhibited in both nondegenerate and degenerate parametric am-

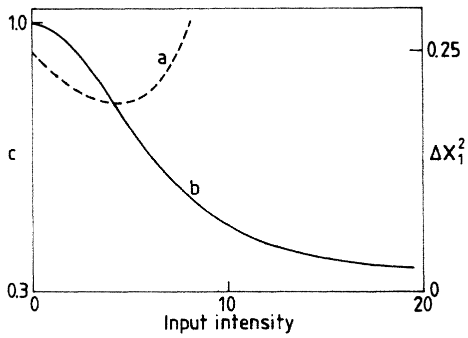

FIG. 8. Squeezing and violation of Bell's inequality vs input pump intensity $|\varepsilon/\kappa|^2$, $q=19$. (a) ΔX_1^2. (b) C.

plification.[28,36]

The second-order correlation

$$g^{(2)}(0)=\langle (a_+^\dagger)^2 a_+^2\rangle/\langle a_+^\dagger a_+\rangle^2$$

is also plotted for a particular value of $q=2(\varepsilon_{\rm thr}/\kappa)-1$. A small amount of antibunching is predicted below threshold, but only for small values of q. In the limit of zero driving field ε, $g^{(2)}(0)$ is written

$$g^{(2)}(0)\to\frac{2(q+3)^2}{(q+4)^2} \quad \text{as } z\to 0\ . \tag{86}$$

Thus, we obtain $g^{(2)}(0)<1$ for q in the domain $(-1,\sqrt{2}-2)$, and the corresponding minimum value for $g^{(2)}(0)$ as $q\to-1$ is $\frac{8}{9}$. Thus this system would not be a good experimental test for the feature of antibunching.

The property of squeezing is present in the field statistics. This is to be expected since the Hamiltonian (75) resembles that of the parametric amplifier (74). Consider the following mode combination $e=(a_+ + b_+)/\sqrt{2} = X_1+iX_2$. The variance ΔX_2^2 is plotted in Fig. 8. Squeezing is obtained below threshold, but to a much lesser degree than in the nondegenerate parametric amplifier.

The correlation between intensities of the coupled modes a_+,b_+, however, is as strong as in the usual parametric amplifier [the identity 26(i) holding]. Thus, the classical Cauchy-Schwarz inequality

$$C^1=\langle a_+^\dagger a_+ b_+^\dagger b_+\rangle/\langle a_+^\dagger a_+\rangle\langle b_+^\dagger b_+\rangle \le g^{(2)}(0)$$

is violated for all values of input intensity (Fig. 7).

To illustrate violations of Bell's inequality the rotated modes (56) are considered. Calculation of $E(\theta,\phi)$ [Eq. (57)] shows $E(\theta,\phi)=C\cos 2(\theta-\phi)$ where C has been defined according to (61). The selection of the angles (49) gives the result as before: $B=2\sqrt{2}C$, and hence violation of the Bell inequality $|B|\le 2$ is possible for $|C|\le 0.707$. Figure 8 reveals that for a region below threshold, corresponding to a strong violation of the Cauchy-Schwarz inequality, Bell's inequality is violated.

VI. CONCLUSION

We have presented a review of experiments in optics designed to test quantum mechanics against classical theories. First discussed was the measurement on a single beam (for example, a single mode a) of the second-order correlation function $[\langle (a^\dagger)^2 a^2\rangle]$. The quantum property of photon antibunching cannot be explained by the classical (wave) theory of light. The double-beam (for example, two modes a and b) experiment allows measurements of the joint second-order correlation function $(\langle a^\dagger a b^\dagger b\rangle)$ between modes and also the second-order correlation function $[\langle (a^\dagger)^2 a^2\rangle]$ of each mode. The classical wave theory of light predicts a Cauchy-Schwarz inequality $[\,|\langle a^\dagger a b^\dagger b\rangle|^2 \le \langle (a^\dagger)^2 a^2\rangle\langle (b^\dagger)^2 b^2\rangle]$ which can be violated in quantum theory; for example, in two-photon atomic cascade experiments. The two-mode parametric amplifier is discussed as a system which exhibits violation of the Cauchy-Schwarz inequality $[\,|\langle a^\dagger a b^\dagger b\rangle|^2 \le \langle (a^\dagger)^2 a^2\rangle\langle (b^\dagger)^2 b^2\rangle]$.

Correlations between the photons emitted in the two lateral components of the fluorescence triplet in resonance fluorescence from a two-level atom have also been shown to violate a Cauchy-Schwarz inequality.[49] Recently proposals have been made to generate subpoissonian light using selective deletion from the correlated two-photon pair produced in atomic cascades[50] or parametric down conversion.[51]

The results of the photon counting experiments showing photon antibunching and violation of the Cauchy-Schwarz inequality is not confusing in physical terms if one adopts a particle view of light. However, quantum mechanics is deeper than this, and experiments designed to incorporate both phase and particle features of the field prove more interesting. Bell's theorem is discussed and provides a means of testing quantum mechanics against all possible classical or realistic theories (including particle theories) which are not in conflict with relativity (local). In a first description, the Bell experiment is introduced as a modification of the Cauchy-Schwarz inequality experiment, with the spatially separated 50%-50% beam splitters being replaced with polarizers. The inequalities are formulated in terms of four output modes [for example, two, (a_+,a_-) and (b_+,b_-), in each beam], with arbitrary intensity allowed. The polarizers rotate (for example, at one polarizer, $c_+=a_+\cos\theta+a_-\sin\theta$, $c_-=-a_+\sin\theta+a_-\cos\theta$, where θ is the polarizer angle) the modes (e.g., a_+,a_-) of each beam to new detected modes (c_+,c_-). Thus, the detected joint correlation functions of the two beams can incorporate phase elements of the original field. States which we predict to violate Bell inequalities show strong violation of the Cauchy-Schwarz inequality (particle like) and also quantum interference.

An alternative example of a Bell experiment is also given. We consider a two-mode source (a and b) with 50%-50% beam splitters to split the system into four beams (a_1,a_2,b_1,b_2). The modes are then combined in pairs (a_1 and b_1, a_2 and b_2) with two spatially separated beam splitters (or "rotators") of variable transmittivities.

The Einstein-Podolsky-Rosen paradox has been shown[39] to apply to such an experiment where the two-mode source exhibits the maximum intensity joint correlation $\langle a^\dagger a b^\dagger b\rangle$ allowed by quantum mechanics. We show explicitly that violation of Bell's inequalities is possible for sources such as two-photon cascade or a two-mode parametric amplifier which exhibit a strong violation of Cauchy-Schwarz inequality. The result is an interesting new example of a system predicted to violate Bell's inequalities. In this case, one does not have a quantum superposition state as an original source. Quantum interference effects are provided by the beam splitters.

Lastly, we illustrate that a pure quantum state with perfect or maximum correlation (according to quantum mechanics) is not essential to disprove classical theory. We show explicitly violation of Bell's inequalities to be possible in the steady state in a regime of small photon number for a system including coherent pumping and where dissipation into the environment is modelled via interaction with a reservoir.

ACKNOWLEDGMENTS

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APPENDIX

We thus wish to calculate integrals of the type

$$I_{0000}=\int(\varepsilon-\kappa\alpha_+\beta_+-\kappa\alpha_-\beta_-)^q(\varepsilon^*-\kappa\alpha_+^\dagger\beta_+^\dagger-\kappa\alpha_-^\dagger\beta_-^\dagger)^q\exp(2\alpha_+^\dagger\alpha_++2\alpha_-^\dagger\alpha_-+2\beta_-^\dagger\beta_-+2\beta_+^\dagger\beta_+)d\underline{\alpha}\ . \tag{A1}$$

Substituting

$$u_1=\alpha_+,\ \ v_1=\alpha_+^\dagger,\ \ u_2=\alpha_-,\ \ v_2=\alpha_-^\dagger,\ \ t_1=\frac{\kappa\alpha_+\beta_+}{\varepsilon},\ \ \omega_1=\frac{\kappa\alpha_+^\dagger\beta_+^\dagger}{\varepsilon^*},\ \ t_2=\frac{\kappa\alpha_-\beta_-}{\varepsilon},\ \ \omega_2=\frac{\kappa\alpha_-^\dagger\beta_-^\dagger}{\varepsilon^*}\ , \tag{A2}$$

we find

$$I_{0000}=\varepsilon^{2q}\left[\frac{\varepsilon}{\kappa}\right]\int(1-t_1-t_2)^q(1-\omega_1-\omega_2)^q u_1^{-1}v_1^{-1}u_2^{-1}v_2^{-1}\exp\left[2u_1v_1+2u_2v_2+2\left[\frac{\varepsilon}{\kappa}\right]\frac{t_1\omega_1}{u_1v_1}+2\left[\frac{\varepsilon}{\kappa}\right]\frac{t_2\omega_2}{u_2v_2}\right]d\underline{\mathbf{u}} \tag{A3}$$

where $d\underline{\mathbf{u}}=du_1\,du_2\,dv_1\,dv_2\,dt_1\,dt_2\,d\omega_1\,d\omega_2$.

We make the further substitutions

$$x=1-t_2,\ \ g=1-\omega_2,\ \ y=\frac{t_1}{1-t_2},\ \ h=\frac{\omega_1}{1-\omega_2} \tag{A4}$$

to find

$$I_{0000}=\varepsilon^{2q}\left|\frac{\varepsilon}{\kappa}\right|^4\int u_1^{-1}v_1^{-1}u_2^{-1}v_2^{-1}e^{2u_1v_1}e^{2u_2v_2}2x^{q+1}(1-y)^q g^{q+1}(1-h)^q\exp\left[2\left|\frac{\varepsilon}{\kappa}\right|^2\frac{ghxy}{u_1v_1}+2\left|\frac{\varepsilon}{\kappa}\right|^2\frac{(1-g)(1-x)}{u_2v_2}\right]d\mathbf{v}\ . \tag{A5}$$

where $d\mathbf{v}=du_1\,du_2\,dv_1\,dv_2\,dx\,dg\,dy\,dh$. The next step is to expand the exponential:

$$\exp\left[2\left|\frac{\varepsilon}{\kappa}\right|^2\frac{ghxy}{u_1v_1}\right]=\sum_{m=0}^{\infty}\frac{2^m\left|\frac{\varepsilon}{\kappa}\right|^{2m}}{m!}\frac{g^mh^mx^my^m}{u_1^mv_1^m}\ , \tag{A6}$$

$$I_{0000}=\varepsilon^{2q}\left|\frac{\varepsilon}{\kappa}\right|^4\sum_{m=0}^{\infty}\frac{2^m\left|\frac{\varepsilon}{\kappa}\right|^{2m}}{m!}\int u_1^{-m-1}v_1^{-m-1}u_2^{-1}v_2^{-1}e^{2u_2v_2}e^{2u_1v_1}$$

$$\times x^{q+m+1}(1-y)^qy^mg^{q+m+1}(1-h)^qh^m\exp\left[2\left|\frac{\varepsilon}{\kappa}\right|^2\frac{(1-g)(1-x)}{u_2v_2}\right]d\underline{\mathbf{v}}\ . \tag{A7}$$

Consider the integrals in u_1 and v_1:

$$I_{u_1v_1}=\int\int u_1^{-m-1}v_1^{-m-1}e^{2u_1v_1}du_1\,dv_1$$

$$=\sum_{k=0}^{\infty}\frac{2^k}{k!}\int u_1^{-m-1+k}du_1\int v_1^{-m-1+k}dv_1\ . \tag{A8}$$

With the integration contours in u_1,v_1 closed circles about the origin, since[47]

$$\int u_1^{-x}\, du_1 = 2\pi i \delta(x-1)\ , \tag{A9}$$

we have

$$I_{u_1 v_1} = -4\pi^2 \frac{2^m}{m!}\ . \tag{A10}$$

The integrals in y and h are

$$I_{yh} = \int (1-y)^q y^m\, dy \int (1-h)^q h^m\, dh\ , \tag{A11}$$

which are readily seen to be the contour integrals defining the beta functions.[48] Thus,

$$I_{yh} = \frac{[\Gamma(q+1)]^2[\Gamma(m+1)]^2}{\Gamma(m+q+2)^2}\ , \tag{A12}$$

where Γ is the gamma function.

Upon expanding the second exponential, we obtain

$$I_{0000} = -4\pi^2\varepsilon^{2q}\left|\frac{\varepsilon}{\kappa}\right|^4 [\Gamma(q+1)]^2 \sum_{m=0}^{\infty} \frac{\left|\frac{2\varepsilon}{\kappa}\right|^{2m}[\Gamma(m+1)]^2}{(m!)^2\Gamma(m+q+2)^2} \sum_{n=0}^{\infty}\frac{2^n}{n!}\left|\frac{\varepsilon}{\kappa}\right|^{2n} \int\int\int\int u_2^{-1-n} v_2^{-1-n} e^{2u_2 v_2} x^{q+m+1} \times g^{q+m+1}(1-g)^n(1-x)^n \times du_2\, dv_2\, dx\, dg\ . \tag{A13}$$

The integrals in u_2, v_2 are defined similarly to u_1, v_1:

$$\int\int u_2^{-1-n} v_2^{-1-n} e^{2u_2 v_2}\, du_2\, dv_2 = -4\pi^2\frac{2^n}{n!}\ . \tag{A14}$$

The integrals in x and g are similar to those in y and h.

$$\int x^{q+m+1}(1-x)dx \int g^{q+m+1}(1-g)^n dg = \frac{[\Gamma(1+m+2)]^2[\Gamma(n+1)]^2}{[\Gamma(q+m+n+3)]^2}\ . \tag{A15}$$

The final expression for the zeroth moment is ($z=|2\varepsilon/\kappa|^2$)

$$I_{0000} = 16\pi^4\varepsilon^{2q}\left|\frac{\varepsilon}{\kappa}\right|^4 \Gamma(q+1)^2 \sum_{m=0}^{\infty}\frac{z^m[\Gamma(m+1)]^2[\Gamma(q+m+2)]^2}{(m!)^2[\Gamma(m+q+2)]^2}\sum_{n=0}^{\infty}\frac{z^n[\Gamma(n+1)]^2}{(n!)^2[\Gamma(q+m+n+3)]^2}\ . \tag{A16}$$

This expression may be simplified. Recalling the definition of the hypergeometric function

$${}_1F_2(a;b,c;z) = 1 + \frac{az}{1!b} + \frac{a(a+1)z^2}{2!b(b+1)c(c+1)} + \cdots\ , \tag{A17}$$

the last series can be abbreviated

$$\sum_{n=0}^{\infty}\frac{[\Gamma(n+1)]^2}{(n!)^2[\Gamma(q+m+n+3)]^2} = \frac{1}{[\Gamma(q+m+3)]^2}\, {}_1F_2(1;q+m+3,q+m+3;z)\ . \tag{A18}$$

Thus,

$$I_{0000} = N^{-1} = N_0 \sum_{m=0}^{\infty} z^m\, {}_1F_2(1;q+m+3,q+m+3;z)/[(q+3)_m^2]\ , \tag{A19}$$

where the Pochhammer symbol is defined as $(x)_k = x(x+1)\cdots(x+k-1)$ and $(x)_0=1$ and

$$N_0 = 16\pi^4\epsilon^{2q}\left|\frac{\epsilon}{\kappa}\right|^4 \Big/ (q+2)^2(q+1)^2\ .$$

Higher moments may be calculated similarly.

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