# Focusing singularity of the cubic Schrödinger equation

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(Received 3 September 1985)

The cubic Schrödinger equation has singular solutions in two or more space dimensions. The singularities occur at points of intense self-focusing. In this paper we present numerical results that describe the nature of the focusing singularity.

## I. INTRODUCTION

In various physical contexts nonlinear propagation phenomena are modeled by the cubic Schrödinger equation (CSE)

$$i\frac{\partial\psi}{\partial t}+\Delta\psi+|\psi|^2\psi=0,\quad t>0$$
$$\psi(0,\mathbf{x})=\psi_0(\mathbf{x}),\quad \mathbf{x}\in\mathbb{R}^N \tag{1.1}$$

whose main property is that in any dimension $N\geq 2$, solutions may develop a singularity at some finite $t=t_*$. In nonlinear optics $N=2$ and Eq. (1.1) describes propagation of electromagnetic beams in media whose index of refraction increases in proportion to the field amplitude.[1-5] The variable $t$ in this case is distance along the beam and $\mathbf{x}=(x_1,x_2)$ are transverse-beam coordinates. In this context, a singularity corresponds to narrowing of the beam and increase of the field amplitude due to self-focusing. When $N=3$ the cubic Schrödinger equation can be viewed as a limit of the Zakharov system for Langmuir waves in plasmas.[6-9] Here time is represented by the variable $t$ and the singularity is usually called filamentation or collapse. Note that in one dimension, CSE can be solved explicitly by the inverse scattering transform.[10]

The main result of this paper is an accurate description of the nature of the focusing singularity in three dimensions for radially symmetric solutions. This is done by using an adaptive grid defined by a nonlinear change of variables which transforms Eq. (1.1) to a similar one that has no singularities. This enables us to integrate numerically Eq. (1.1) up to times very close to blowup where the maximum initial amplitude has been amplified by factors larger than $10^4$. We find that if the singularity occurs at $t=t_*$ and if $|\mathbf{x}|=r$ then

$$\psi(r,t)\sim\frac{e^{i\theta}\sqrt{C}}{\sqrt{\beta(t_*-t)}}Q\left[\frac{\sqrt{C}r}{\sqrt{\beta(t_*-t)}}\right]\times e^{i(C/\alpha)\ln[C/(t_*-t)]}\,. \tag{1.2}$$

Here $\alpha$, $\beta$, $\theta$, and $C$ are constants that depend on the initial conditions and $Q(\eta)$, $\eta\geq 0$, is the complex-valued solution of the equation

$$\frac{d^2Q}{d\eta^2}+\frac{2}{\eta}\frac{dQ}{d\eta}-Q+iK\frac{d}{d\eta}(\eta Q)+|Q|^2Q=0,\quad \eta>0$$
$$\frac{dQ}{d\eta}(0)=0,\quad Q(\infty)=0,\quad Q(0)\text{ real}\,. \tag{1.3}$$

We find that the constant $K$ in (1.3) is independent of the initial data and has the value $K\sim 0.917$.

There is no mathematical proof that locally the focusing singularity in three dimensions has the self-similar form (1.2) although this has been expected.[6,9] Growth of the peak amplitude according to a $(t_*-t)^{-1/2}$ law has also been observed in direct numerical integration of Eq. (1.1).[11,12] The determination of the constant $K$ in (1.3) and of the logarithmic phase in (1.2) is a consequence of our ability to integrate (1.1) up to times very close to blowup.

At the critical dimension $N=2$, the nature of the focusing singularity is much more difficult to analyze both numerically and analytically. Contradictory predictions have been made based on heuristic considerations and direct numerical integration.[13-18] We summarize our numerical results for this case up to now in Sec. VI.

## II. BASIC PROPERTIES OF THE CUBIC SCHRÖDINGER EQUATION

Regular solutions of (1.1) have two invariants: the $L^2$ norm

$$|\psi(t)|_{L^2}^2=\int|\psi(\mathbf{x},t)|^2d\mathbf{x}=\text{const}, \quad (2.1)$$

and the Hamiltonian

$$H=\int[|\nabla\psi(\mathbf{x},t)|^2-\tfrac{1}{2}|\psi(\mathbf{x},t)|^4]d\mathbf{x}=\text{const}. \quad (2.2)$$

The first conservation law is obtained by multiplying the CSE by $\bar{\psi}$, the complex conjugate of $\psi$, and taking the imaginary part, the second by multiplying by $\partial\psi/\partial t$, and taking the real part. We also have the "variance" identity

$$\frac{1}{8}\frac{d^2}{dt^2}\int|\mathbf{x}|^2|\psi(\mathbf{x},t)|^2d\mathbf{x}=H-\frac{N-2}{4}\int|\psi(\mathbf{x},t)|^4d\mathbf{x}. \quad (2.3)$$

From these identities one can deduce[19] that blowup may only occur at or above the critical dimension $N=2$. Global existence results are given in Refs. 20–22. We give briefly the argument for blowup here.

Suppose that the initial function $\psi_0(x)$ has finite variance $\int|\mathbf{x}|^2|\psi_0(\mathbf{x})|^2d\mathbf{x}$ and that the Hamiltonian $H$ is negative. In two or more dimensions, (2.3) implies that there are constants $C_1$ and $C_2$ such that

$$\int|\mathbf{x}|^2|\psi(\mathbf{x},t)|^2d\mathbf{x}\le 8Ht^2+C_1t+C_2. \quad (2.4)$$

Since $H<0$, there will be a first time $\tilde{t}$ at which the left side becomes zero.[6,7,19,22–24] The uncertainty inequality

$$|f|_{L^2}^2\le\frac{2}{N}|\nabla f|_{L^2}|\mathbf{x}f|_{L^2} \quad (2.5)$$

and (2.1) imply now that

$$\int|\nabla\psi(\mathbf{x},t)|^2d\mathbf{x}\to\infty \quad (2.6)$$

and

$$\sup_{\mathbf{x}}|\psi(\mathbf{x},t)|\to\infty,$$

as $t\to\tilde{t}$. This argument proves that for initial conditions with finite variance, blowup must occur at a finite time if $H$ is negative. Nevertheless, this time need not be $\tilde{t}$: blowup could occur at an earlier time and it generally does. Furthermore, this condition on the initial variance does not seem necessary—singularities were obtained numerically with data that have infinite variance and also when CSE is considered in a periodic domain.[17]

As noticed in Ref. 25, the singularity of the nonlinear Schrödinger equation can be viewed as an extension to continuous systems of the collapse of point masses in the classical $N$-body problem.

## III. NONLINEAR SCALING

In this section we describe the scaling of the nonlinear Schrödinger equation that allows us to integrate up to times very close to the singularity. We restrict attention to radially symmetric solutions $\psi(r,t)$ with $|\mathbf{x}|=r$.

Introduce the variables

$$\xi=\frac{r}{L(t)},\quad \tau=\int_0^t\frac{1}{L^2(s)}ds, \quad (3.1)$$

where $L(t)$ is a positive function that will be specified later. Let $u(\xi,\tau)$ be defined by

$$u(\xi,\tau)=L(t)\psi(r,t), \quad (3.2)$$

where $\psi(r,t)$ solves (1.1). The conservation laws (2.1) and (2.2) imply

$$|u(\tau)|_{L^2}^2=L^{2-N}(\tau)|\psi_0|_{L^2}^2 \quad (3.3)$$

and

$$\int(|\nabla u|^2-\tfrac{1}{2}|u|^4)d\xi=L^{4-N}(\tau)\int(|\nabla\psi_0|^2-\tfrac{1}{2}|\psi_0|^4)d\mathbf{x}. \quad (3.4)$$

It is easily seen that $u(\xi,\tau)$ must satisfy the equation

$$i\frac{\partial u}{\partial\tau}+\frac{\partial^2u}{\partial\xi^2}+\frac{N-1}{\xi}\frac{\partial u}{\partial\xi}-ia(\tau)\frac{\partial}{\partial\xi}(\xi u)+|u|^2u=0. \quad (3.5)$$

The initial condition is $u(\xi,0)=L(0)\psi_0(L(0)\xi)$. The function $a(\tau)$ is given by

$$a(\tau)=L\frac{dL}{dt}=\frac{1}{L}\frac{dL}{d\tau}, \quad (3.6)$$

where we think of $L$ as a function of $t$ or of $\tau$ using (3.1).

The simplest choice for $L(t)$ making $|u|$ uniformly bounded is $L(t)=|\psi(0,t)|^{-1}$. This procedure is, however, not suited for numerical computations because of the instability resulting from the local character of the scaling factor. The idea is thus to require the conservation of a suitable integral norm (depending on the space dimension) which as a consequence will control the sup norm.

In three dimensions, we choose $L(t)$ so that the $|\Delta u|_{L^2}$ is constant, independent of $\tau$. In view of (3.2), this is accomplished if we take

$$L^3(t)=\frac{k}{\int_0^\infty|\Delta\psi(r,t)|^2r^2dr} \quad (3.7)$$

with $k$ a constant. The function $a(\tau)$ in (3.5) is obtained by substituting (3.7) in (3.6) and using (1.1) and (3.2). This gives

$$a(\tau)=\frac{2}{3}\frac{1}{k}\text{Im}\int_0^\infty\left[(\Delta\bar{u})^2u^2+4\Delta\bar{u}u\left|\frac{\partial u}{\partial\xi}\right|^2+2(\Delta u)^2\bar{u}\Delta\bar{u}\right]\xi^2d\xi, \quad (3.8)$$

where

$$\Delta=\frac{\partial^2}{\partial\xi^2}+\frac{2}{\xi}\frac{\partial}{\partial\xi}.$$

The original equation (1.1) has been transformed to (3.5) with $a(\tau)$ given by (3.8). This equation has the advantage of being nonsingular. The uniform boundedness of $|u|_{L^\infty}$ cannot be derived using a Sobolev inequality from the conservation of $|\Delta u|_{L^2}$ because of the growth of $|u|_{L^2}$. Nevertheless, our numerical computations show that $|u|_{L^\infty}$ stays bounded.

The large-$\tau$ behavior of $a(\tau)$ reflects the way $L(t)\to 0$ as $t\to t_*$ or $\tau\to\infty$. Suppose, for example, that $a(\tau)$ tends to a constant as $\tau\to\infty$, which as we will see is what happens. Then $L(t)\sim(t_*-t)^{1/2}$.

In two dimensions we arrange that the $L^4$ norm of $u$ be independent of $\tau$ by choosing

$$L^6(t)=\frac{k}{\int_0^\infty |\nabla\psi(r,t)|^4 r\,dr}, \tag{3.9}$$

where $k$ is a constant. The Sobolev inequality[26]

$$|u|_{L^\infty}\le \text{const}\times |u|_{L^2}^{1/3}|\nabla u|_{L^4}^{2/3} \tag{3.10}$$

insures that the maximum of $|u|$ is bounded for all $\tau$. In this case, the function $a(\tau)$ is given by

$$a(\tau)=\frac{2}{3}\frac{1}{K}\operatorname{Im}\int_0^\infty\left[\left(\frac{\partial\bar u}{\partial\xi}\right)^2\left|\frac{\partial u}{\partial\xi}\right|^2 u^2 - \frac{\partial}{\partial\xi}\left|\frac{\partial u}{\partial\xi}\right|^2\frac{\partial u}{\partial\xi}\Delta u\right]\xi\,d\xi . \tag{3.11}$$

The system to be solved now is (3.5) (with $N=2$) and (3.11).

In the two-dimensional case we also used the scaling that makes $|\nabla u|_{L^2}^2$ independent of $\tau$ by taking

$$L^2(t)=\frac{c}{\int_0^\infty |\nabla\psi(r,t)|^2 r\,dr} . \tag{3.12}$$

The corresponding $a(\tau)$ is now

$$a(\tau)=-\frac{1}{C}\operatorname{Im}\int_0^\infty \bar u^2\left(\frac{\partial u}{\partial\xi}\right)^2\xi\,d\xi . \tag{3.13}$$

With this choice of scale factor, the maximum of $|u|$ is no longer bounded for all $\tau$. However, it grows very slowly with $\tau$ and in practice the scaling (3.12) is as efficient as (3.9).

In the two-dimensional case the decay of $a(\tau)$ to zero as $\tau\to\infty$ determines the behavior of $L(t)$ on $t\to t_*$ and hence the nature of the blowup. If, for example, $a(\tau)\sim -C_1\tau^{-\gamma}$, $\gamma>0$, with $C_1$ some constant, then

$$L\sim C_2(t_*-t)^{1/2}\Big/\ln^\alpha\left[\frac{1}{t_*-t}\right],\quad t\to t_* \tag{3.14}$$

where $\alpha=\gamma/2(1-\gamma)$ and $C_2$ is another constant. However, our numerical calculations so far are not adequate to accurately determine the large-$\tau$ behavior of $a(\tau)$. As we discuss in Sec. VI, we observe that $a(\tau)$ goes to zero much slower than a power, which suggests that $L(t)$ behaves more like $(t_*-t)^{1/2}g(t_*-t)$ where $g(t_*-t)$ changes more slowly than $\ln[1/t_*-t)]$ to a negative power.

## IV. NUMERICAL METHOD

Equation (1.1) is solved numerically by using a pseudospectral method for the space variable and a second-order Adams-Bashforth, Crank-Nicolson scheme for the time stepping, similar to one of the schemes used in Ref. 17:

$$i\frac{u^{n+1}-u^n}{\delta\tau}+\tfrac{1}{2}\Delta(u^{n+1}+u^n) +i\left[\tfrac{3}{2}a^n\frac{\partial}{\partial\xi}(u^n\xi)-\tfrac{1}{2}a^{n-1}\frac{\partial}{\partial\xi}(u^{n-1}\xi)\right] +\tfrac{3}{2}|u^n|^2u^n-\tfrac{1}{2}|u^{n-1}|^2u^{n-1}=0 . \tag{4.1}$$

To implement a collocation method, the spatial domain $[0,\infty)$ is mapped on the interval $(-1,+1]$ by the transformation

$$z=\frac{l-\xi}{l+\xi}, \tag{4.2}$$

where $l$ is an adjustable parameter. We used $l=64$ in three dimensions and $l=128$ in two dimensions. The functions are then expanded in a series of Chebyshev polynomials. We retain 65 polynomials in three dimensions and 129 in two dimensions. Half of the collocation points are between 0 and $l$. The time step is $\delta\tau=2\times10^{-3}$, in three dimensions and $\delta\tau=10^{-3}$ in two dimensions.

The explicit part of Eq. (4.1) is computed by collocation: multiplications are done in physical space and differentiation in Chebyshev space. Concerning the implicit part, we first construct an approximation of the operator

$$L=\frac{i}{\delta\tau}+\frac{1}{2}\Delta \tag{4.3}$$

from a collocation approximation of the derivative operator $D$. The boundary conditions $(\partial/\partial\xi)u^{n+1}(0,\tau)=u^{n+1}(\infty,\tau)=0$ are taken into account by replacing the first line of the $L$ matrix by the first line of the $D$ matrix and the last one by $(0,0,\dots,1)$. The resulting matrix $L_B$ is inverted once and stored. Time stepping is made (in physical space) by multiplying $L_B^{-1}$ with the $N$ vector consisting of the explicit part of Eq. (4.1) when the first and last components have been replaced by zero to impose the boundary conditions.

The global precision of the calculation can be controlled in various ways. In two dimensions, $|u|_{L^2}^2$ is an invariant. We thus check its conservation together with that of the norm that the scaling makes constant. Table I displays the case corresponding to an initial condition $\psi_0(r)=4/(1+r^{2})$ when the scaling (3.9) is used. We observe the $|\nabla u(\tau)|_{L^4}$ is conserved well until the end of the computation ($\tau=120$). In contrast, $|u|_{L^2}$ abruptly de-

TABLE I. Behavior of the invariants in two dimensions for Lorentzian initial conditions.

| $\tau$ | $\lvert u\rvert_{L^2}^2$ | $\lvert\nabla u\rvert_{L^4}^4$ |
|---|---|---|
| 0 | 8.0 | $0.380\,962\,38\times10^{-1}$ |
| 20 | 7.999 977 1 | $0.380\,952\,37\times10^{-1}$ |
| 40 | 7.999 977 2 | $0.380\,952\,37\times10^{-1}$ |
| 60 | 8.000 023 8 | $0.380\,952\,37\times10^{-1}$ |
| 80 | 8.009 603 6 | $0.380\,952\,37\times10^{-1}$ |
| 100 | 6.002 097 7 | $0.380\,952\,37\times10^{-1}$ |
| 120 | 2.398 769 7 | $0.380\,952\,38\times10^{-1}$ |

TABLE II. Scaling factor $L(\tau)$ and invariant $|\Delta u|_{L^2}$ in three dimensions for Lorentzian initial conditions.

| $\tau$ | $L$ computed from (3.3) | $L$ computed from (3.4) | $\|\Delta u\|^2_{L^2}$ |
|---|---|---|---|
| 0 | 0.353 556 | 0.353 553 | 3.748 68 |
| 1 | $0.794\,562\times10^{-1}$ | $0.794\,502\times10^{-1}$ | 3.748 66 |
| 2 | $0.235\,292\times10^{-1}$ | $0.235\,257\times10^{-1}$ | 3.748 69 |
| 3 | $0.749\,444\times10^{-1}$ | $0.749\,466\times10^{-1}$ | 3.748 70 |
| 4 | $0.253\,568\times10^{-2}$ | $0.253\,311\times10^{-2}$ | 3.748 71 |
| 5 | $0.880\,091\times10^{-3}$ | $0.877\,596\times10^{-3}$ | 3.748 71 |
| 6 | $0.309\,466\times10^{-3}$ | $0.306\,966\times10^{-3}$ | 3.748 71 |
| 7 | $0.110\,497\times10^{-3}$ | $0.107\,778\times10^{-3}$ | 3.748 71 |
| 8 | $0.370\,935\times10^{-4}$ | $0.377\,005\times10^{-2}$ | 3.748 73 |

teriorates around $\tau=90$. This suggest that the large-$\xi$ behavior of $u$ becomes important. This does not necessarily imply that at this time $u(\xi,\tau)$ is computed poorly at small or moderate values of $\xi$, since the function of which $a$ is the integral has a relatively faster decay at infinity.

In three dimensions, the only conserved quantity is $|\Delta u|_{L^2}$ when (3.7) is prescribed. However, the scaling factor $L(\tau)$ can be obtained either from (3.3) or (3.4). Comparison of the resulting values of $L$ are presented in Table II for the initial condition $\psi_0(r)=6\sqrt{2}/(1+r^2)$. At the time where we stopped the computation, the agreement is still rather good, even though for a situation where $|u(\tau)|_{L^2}$ diverges with $\tau$ and $\int_0^\infty(|\nabla u|^2-\frac{1}{4}|u|^4)\xi^2 d\xi$ tends to zero as the difference of two nearly equal quantities. Higher resolution, with more points at large distance, allows integration over longer times.

## V. RESULTS IN THREE DIMENSIONS

Computations have been performed with initial conditions $\psi_0(r)=6\sqrt{2}e^{-r^2}$ (referred to as G3) and $\psi_0(r)=6\sqrt{2}/(1+r^2)$ (referred to as L3). The Gaussian initial function has finite variance but the Lorentzian does not. Both solutions display a similar behavior. We find that the amplitude tends to a fixed profile (Fig. 1) while the phase remains $\tau$-dependent. Figure 2 shows that for large

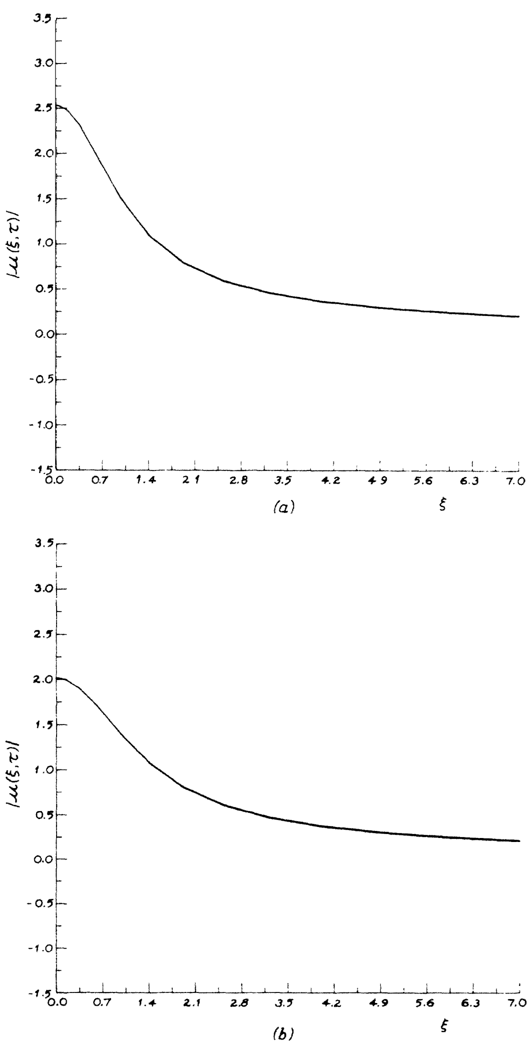



FIG. 1. Amplitude $|u(\xi,\tau)|$ vs $\xi$ in three dimensions. (a) G3 initial conditions for $\tau=3.2$ to 6 in steps of 0.4. (b) L3 initial conditions for $\tau=4.5$ to 8 in steps of 0.5

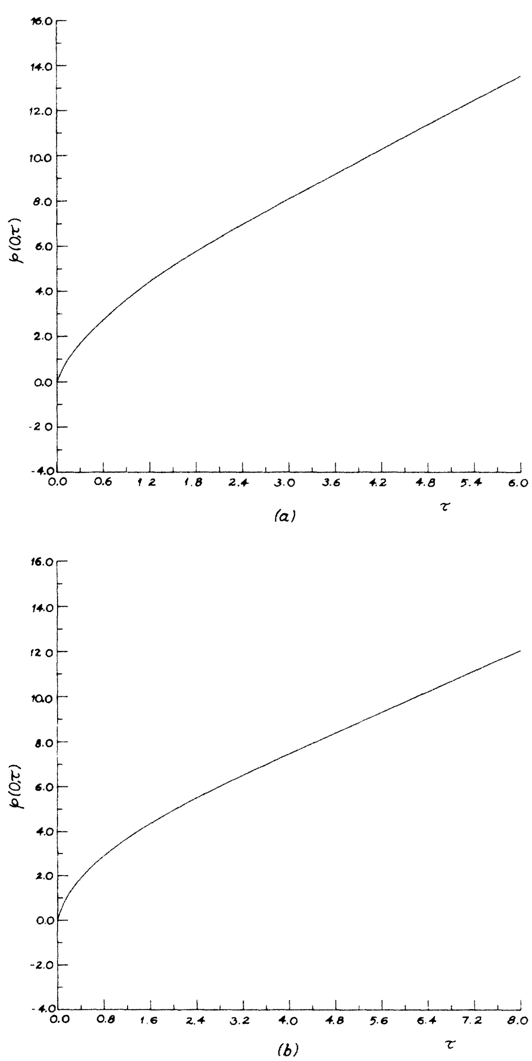



FIG. 2. Phase at the origin vs $\tau$ in three dimensions. (a) G3 initial conditions. (b) L3 initial conditions.

$\tau$ the phase at the origin $\phi(0,\tau)$ behaves like $C\tau$. A mean-square fit gives $C \approx 1.81$ for the Gaussian initial conditions and $C \approx 1.140$ for the Lorentzian initial conditions. In Fig. 3, $\phi(\xi,\tau)-C\tau$ is plotted for different values of $\tau$. We see that this quantity is asymptotically independent of $\tau$. The scaling factor $L$ is obtained from one of the equations (3.3) and (3.4). We observe in Fig. 4 that $\ln L \sim \alpha\tau$, $\alpha \approx 1.66$ for G3 initial conditions and $\alpha \approx 1.04$ for L3 initial conditions. From Eq. (3.1) we can compute $t$ and see in Fig. 5 that $L^2 \sim \beta(t_* - t)$ with $\beta = 3.3$ for the Gaussian initial condition and $\beta = 2.1$ for the Lorentzian initial conditions. As a consequence, $a(\tau)$ should tend to a constant $-A$. This is seen in Fig. 6. The limit value is $A \approx 1.66$ for the Gaussian initial condition and $A \approx 1.045$ for the Lorentzian initial condition. Note that the maximum integration time $\tau = 6$ for the Gaussian initial condition corresponds to $t = 0.034\,301\,966\ldots$ and to an amplitude at the origin $|\psi(0,t)| = 2.485 \times 10^4$. The maximum integration time is $\tau = 8$ for Lorentzian initial condition and corresponds to $t = 0.035\,619\,6\ldots$ with an amplitude at the origin $|\psi(0,t)| = 5.38 \times 10^4$.

The above observations indicate that $L(t)$ scales like a power law with an exponent $\frac{1}{2}$. Furthermore, the $\tau$ dependence of $u(\xi,\tau)$ is reduced to a phase factor $e^{iC\tau}$. We thus write

$$u(\xi,\tau) = S(\xi)e^{i(\theta + C\tau)} \quad \text{for } \tau \to \infty \tag{5.1}$$

where $\theta$ is a constant. Substituting in (3.3), we obtain

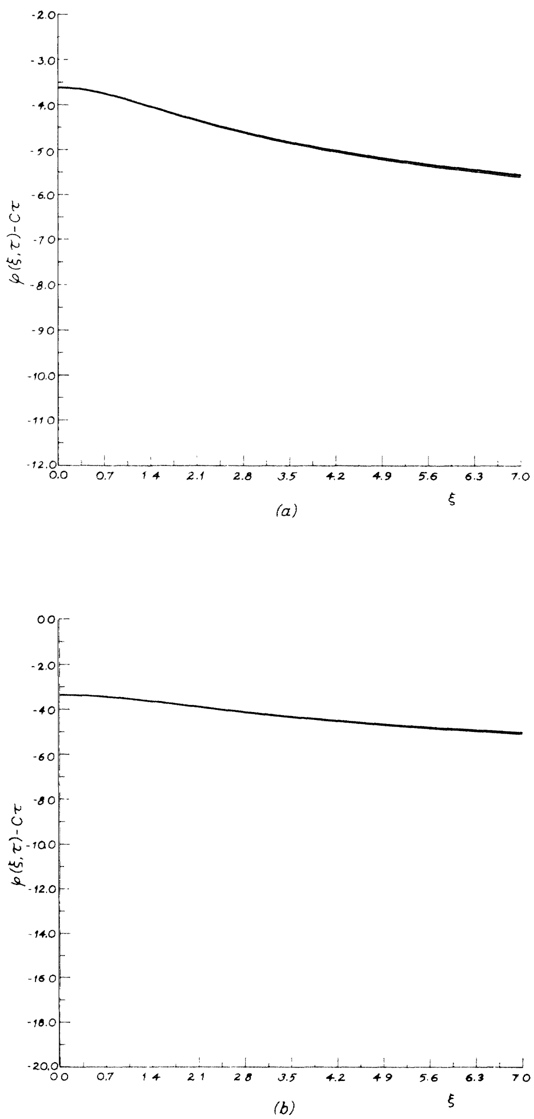



FIG. 3. $\phi(\xi,\tau)-C\tau$ vs $\xi$ in three dimensions. (a) G3 initial conditions for $\tau = 3$ to 6 in steps of 0.6 and $C = 1.81$. (b) L3 initial conditions for $\tau = 4$ to 8 in steps of 0.5 and $C = 1.14$.

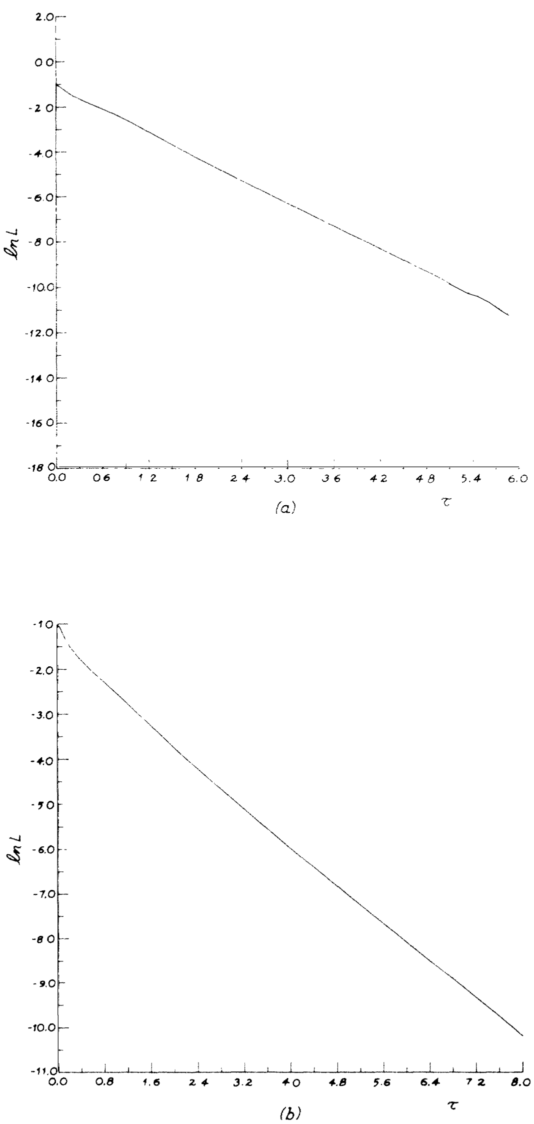



FIG. 4. Scaling factor $L$ vs $\tau$ in three dimensions. (a) G3 initial conditions. (b) L3 initial conditions.

$$\Delta_\xi S - CS + iA\frac{\partial}{\partial \xi}(\xi S) + |S|^2 S = 0\ . \tag{5.2}$$

After the rescaling

$$\eta = \sqrt{C}\,\xi,\ \ Q(\eta) = \frac{1}{\sqrt{C}}S(\xi)\,, \tag{5.3}$$

Eq. (5.1) takes the form

$$\Delta_\eta Q - Q + iK\frac{\partial}{\partial \eta}(\eta Q) + |Q|^2 Q = 0 \tag{5.4}$$

with $K = A/C$. Figure 7 clearly shows that functions $Q$ constructed from the two different initial conditions are identical. The ratio $K$ has also the same value $K \approx 0.917$. We thus conclude that in the primitive variables, the solution has the asymptotic representation

$$\psi(r,t) \sim \frac{e^{i\theta}\sqrt{C}}{[\beta(t_* - t)]^{1/2}} Q\left[\frac{\sqrt{C}r}{[\beta(t_* - t)]^{1/2}}\right] \times \exp\left[i\frac{C}{\alpha}\ln\left[\frac{C}{t_* - t}\right]\right]\ . \tag{5.5}$$

The coefficients are $\alpha \approx 1.66$, $\beta \approx 3.3$, $\theta \approx 2.69$, and $C \approx 1.81$ for the G3 initial condition, $\alpha \approx 1.04$, $\beta \approx 2.1$, $\theta \approx 2.95$, and $C \approx 1.140$ for the L3 initial condition.

Note that the $\frac{1}{2}$ power law for the scaling factor $L(t)$ begins to be visible when the amplitude has been amplified by a factor of 10 or even less. It was thus seen in direct numerical integration of Eq. (1.1).[6,11,12] Nevertheless, a correct determination of the asymptotic profile of the solution also requires observation of the logarithmic behavior of the phase at the origin. The present numerical technique permits us to determine accurately this logarithmic behavior anticipated by Zakharov.[6,5]

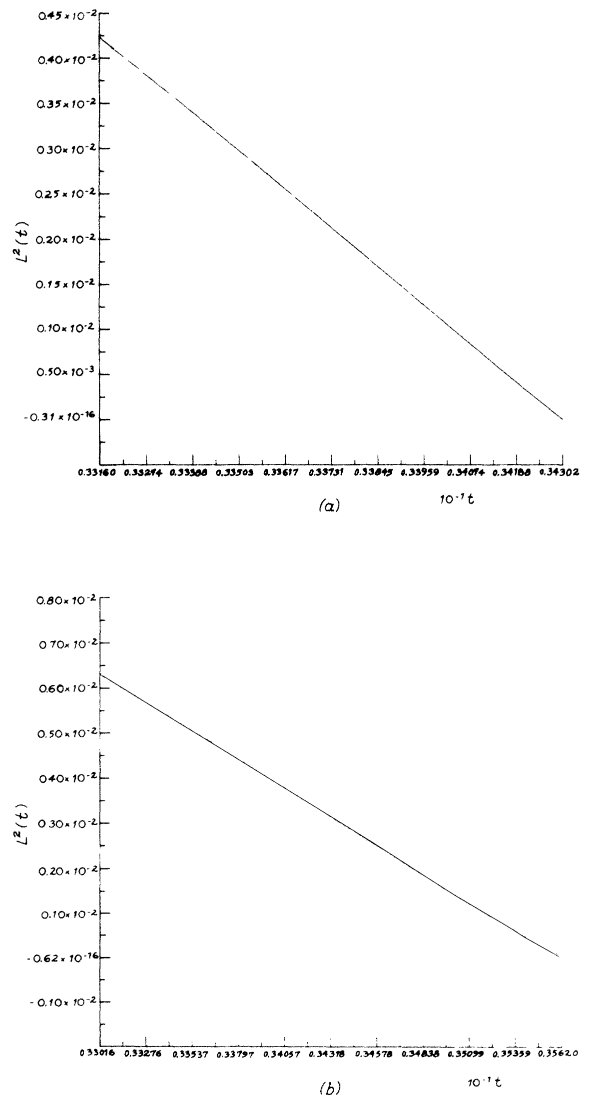



FIG. 5. Scaling factor $L$ vs $t$ in three dimensions. (a) G3 initial conditions. (b) L3 initial conditions.

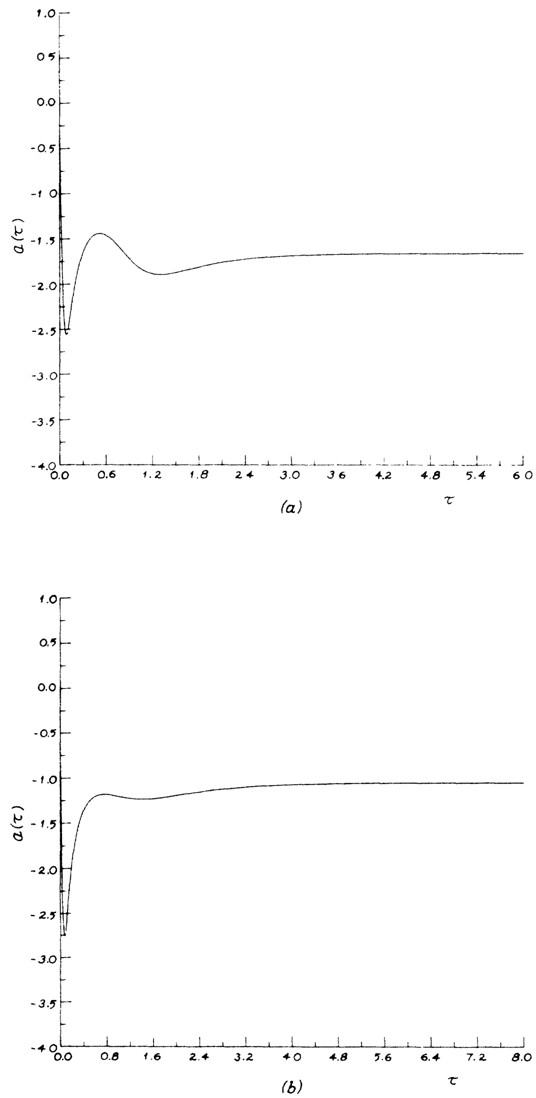



FIG. 6. $a(\tau) = (d/d\tau)\ln L$ vs $\tau$ in three dimensions. (a) G3 initial conditions. (b) L3 initial conditions.

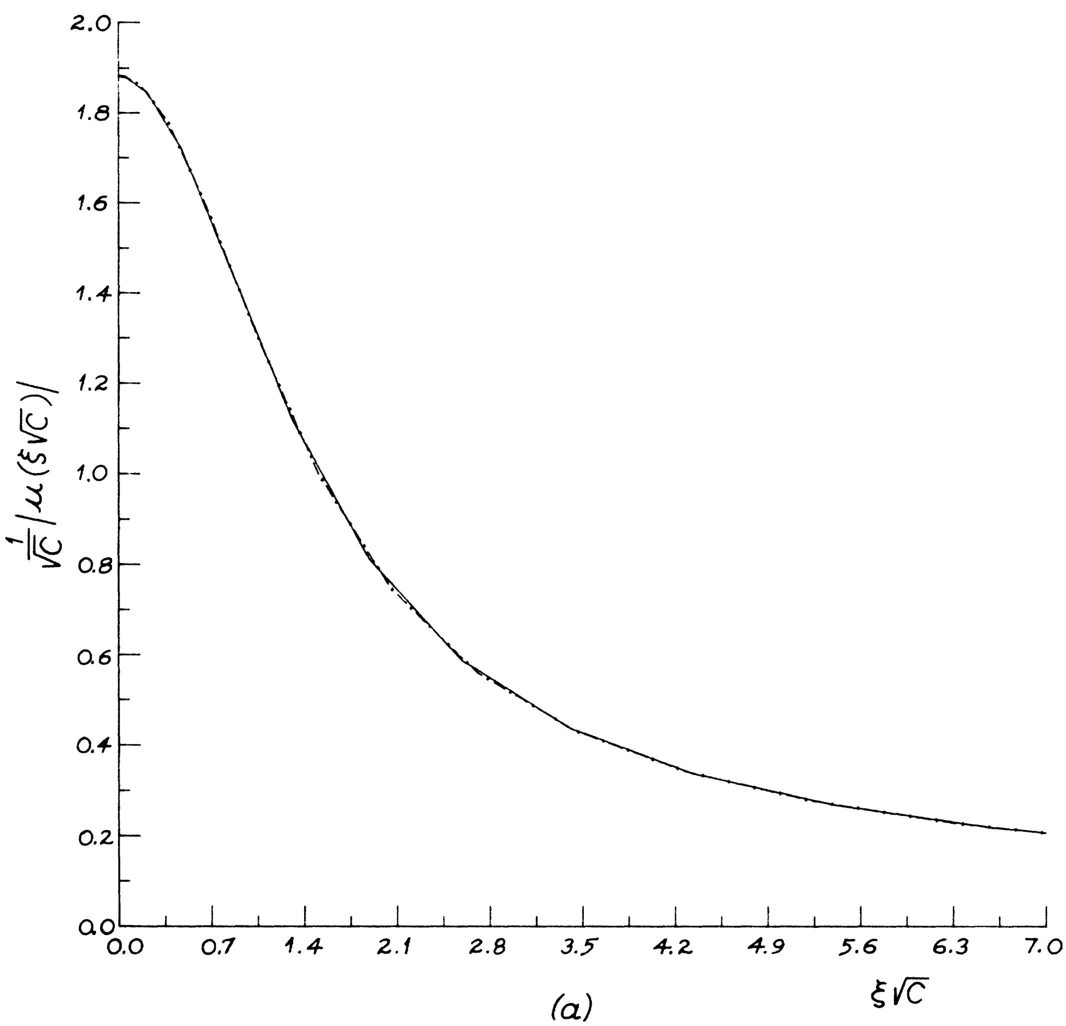



FIG. 7. (a) Asymptotic profiles and (b) phases obtained with initial conditions G3 (solid line) and L3 (dashed line) in three dimensions.

## VI. RESULTS IN TWO DIMENSIONS

The main observation concerning the two-dimensional problem is a drastic slowing down of the dynamics, in the rescaled variables, typical of a critical situation. Computations were performed both with Gaussian and Lorentzian initial conditions: $\psi_0(r)=4e^{-r^2}$ referred to as G2 and $\psi_0(r)=4/(1+r^2)$ referred to as L2. We also used two different nonlinear scalings which preserve either $|\nabla u|_{L^2}$ or $|\nabla u|_{L^4}$. The results in both cases were similar, so we report here only on the $L^4$ scaling. Figure 8 shows a plot of $a(\tau)$. After a short transient, $a(\tau)$ seems to converge to zero but so slowly that we are unable to determine its decay rate. On the other hand, the phase has a relatively fast evolution. Figure 9 shows that the phase at the origin becomes almost linear in $\tau$. However, even at the end of the runs, a slow variation of the slope estimated by a local mean-square fit is still visible. Figure 10 shows that the amplitude at the origin reaches a limiting value rapidly.

Let us rewrite the solution in terms of amplitude and phase

$$u(\xi,\tau)=S(\xi,\tau)e^{i\psi(\xi,\tau)} . \tag{6.1}$$

Equation (3.5) becomes

$$-\psi_\tau S+a(\tau)S\xi\psi_\xi+\Delta S-\psi_\xi^2 S+S^3=0 , \tag{6.2a}$$

$$\tfrac{1}{2}\partial_\tau S^2+\nabla\cdot\left[\left[\nabla\psi-a(\tau)\frac{\xi}{2}\right]S^2\right]=0 . \tag{6.2b}$$

When the amplitude saturates, Eq. (6.2b) reduces to

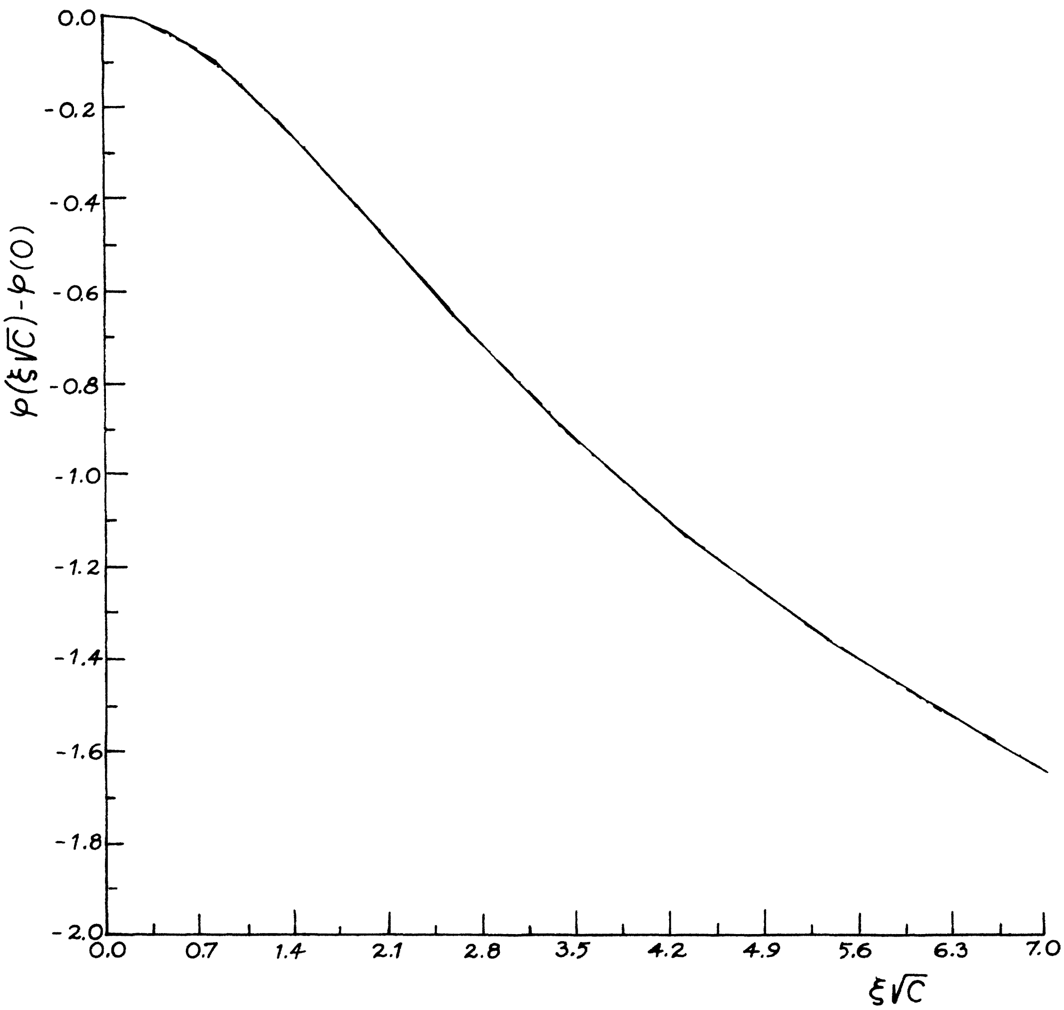
$$\nabla\cdot\left[\nabla\psi-a(\tau)\frac{\xi^{-1}}{2}\right]=0 . \tag{6.3}$$

Because of axial symmetry,

$$\psi_\xi=a(\tau)\frac{\xi}{2} \tag{6.4}$$

and

$$\psi(\xi,\tau)=\psi(0,\tau)+\frac{a(\tau)}{4}\xi^2 . \tag{6.5}$$




(b)

FIG. 7. (*Continued*).

Since the phase at the origin evolves almost linearly, we rewrite (6.5) in the form

$$\phi(\xi,\tau)=C(\tau)\tau+\frac{a(\tau)}{4}\xi^2, \tag{6.6}$$

where $C(\tau)$ has a slow rate of variation. Substituting this in Eq. (6.2a), we obtain

$$\Delta S - C(\tau)S + S^3 = -\left[(\dot{a}-a^2)\frac{\xi^2}{4}-\dot{C}\tau\right]S\ . \tag{6.7}$$

If, as is consistent with our computations, $a$, $\dot{a}$, and $\dot{C}$ vanish and $C(\tau)$ reaches a constant value $C$ when $\tau\to\infty$, a simple rescaling

$$R=\sqrt{C}S,\quad \eta=\frac{1}{\sqrt{C}}\xi, \tag{6.8}$$

indicates that the profile should be asymptotically given by the positive solution (referred to as the ground state) of

TABLE III. Amplitude at the origin in two dimensions for Lorentzian initial conditions as obtained from $\psi(0,t)=L(0)u(0,\tau)$. At longer times, errors in $L(\tau)$ become significant.

| $\tau$ | $t$ | $\lvert\psi(0,t)\rvert$ |
|---|---|---|
| 0 | 0 | 4 |
| 5 | 0.152 059 02 | $1.806\times10^{-1}$ |
| 10 | 0.159 652 34 | $5.145\times10^{-1}$ |
| 15 | 0.160 751 87 | $1.194\times10^{2}$ |
| 20 | 0.160 974 39 | $2.493\times10^{2}$ |
| 25 | 0.161 028 33 | $4.872\times10^{2}$ |
| 30 | 0.161 042 99 | $9.104\times10^{2}$ |
| 35 | 0.161 047 30 | $1.650\times10^{3}$ |
| 40 | 0.161 048 63 | $2.935\times10^{3}$ |
| 45 | 0.161 049 05 | $5.229\times10^{3}$ |
| 50 | 0.161 049 18 | $9.867\times10^{3}$ |
| 55 | 0.161 049 21 | $2.79\times10^{4}$ |
| 55.96 | 0.161 049 22 | $4.961\times10^{4}$ |

$$\Delta R - R + R^3 = 0 , \tag{6.9}$$

as conjectured by Zakharov and Synakh.[13] Note that because of the $\xi^2$ term in Eq. (6.7), this convergence is nonuniform in $\xi$.

The value of $R$ at $\xi=0$ is $R(0) \sim 2.205$. In Fig. 11 the profile obtained with the Gaussian initial conditions is rescaled to fit the value of $R$ at the origin. The dashed line there represents the ground state. A good fit is obtained for $\eta < 1.5$ corresponding to $\xi < 3$. This is also the range where we observe that Eq. (6.5) gives a good representation of the phase. Nevertheless, the scaling factor which we used to fit the values at the origin is 0.2517, while $C(\tau) \approx 0.2654$ for $\tau = 120$. However, $C(\tau)$ is still decreasing slowly with $\tau$ and it may possibly converge to a value close to 0.2517 as $\tau \rightarrow +\infty$. Although the amplitude at the origin has been amplified by a huge factor (Table III), the asymptotic regime has not yet been reached. Extrapolation of the observed behavior suggests that $\tau$ should be increased by several orders of magnitude.

This slowing down in the approach of the asymptotic regime makes the precise nature of the focusing singularity difficult to catch numerically and only partial conclusions can be drawn at present. The scaling factor $L(t)$ appears to differ only slightly from a $\frac{1}{2}$ power law. The deviation is not a modification of the exponent since this would lead to $a(\tau) \sim \tau^{-1}$, which is not supported by the numerical calculations. It is rather a more slowly varying factor. This weak factor has, however, a significant influence on the profile: it makes $a(\tau)$ vanish asymptotically and identifies the limiting profile with the ground state of Eq. (6.9). Note that the simulations done in the primitive variables[13,16–18] ended while the solution was still in an early transient regime. For example, the computations carried out in Ref. 17 with the same Gaussian initial conditions $\psi_0(r) = 4e^{-r^2}$ ended at $t = 0.14$ corresponding to

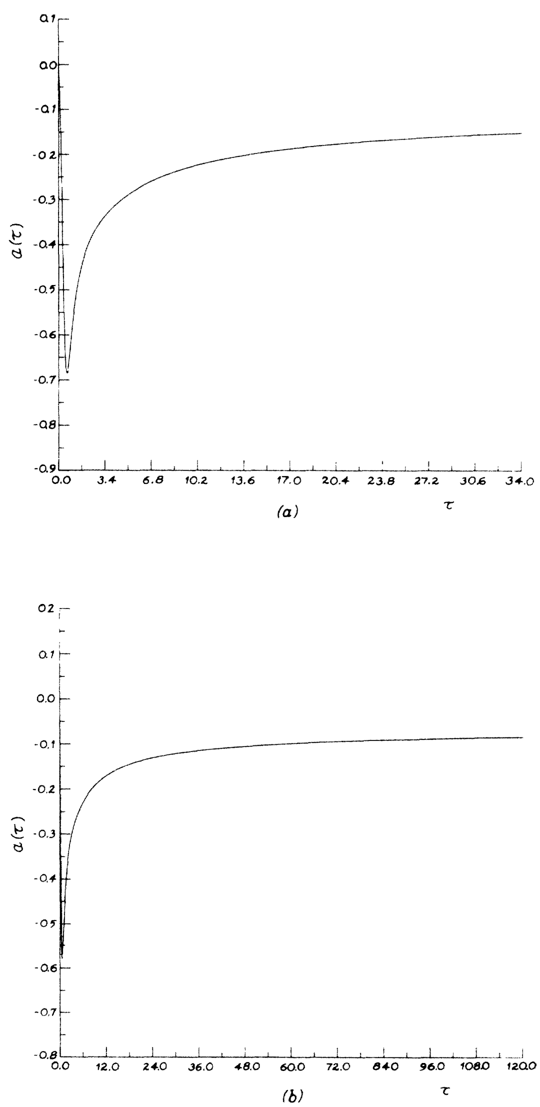



FIG. 8. $a(\tau)$ vs $\tau$ in two dimensions. (a) G2 initial conditions. (b) L2 initial conditions.

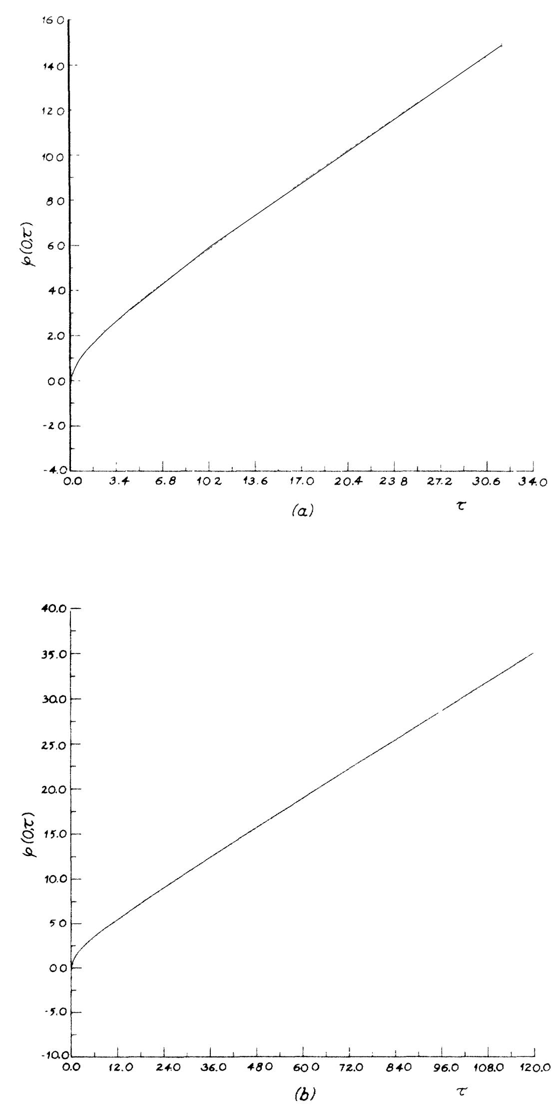



FIG. 9. Phase at the origin vs $\tau$ in two dimensions. (a) G2 initial conditions. (b) L2 initial conditions.

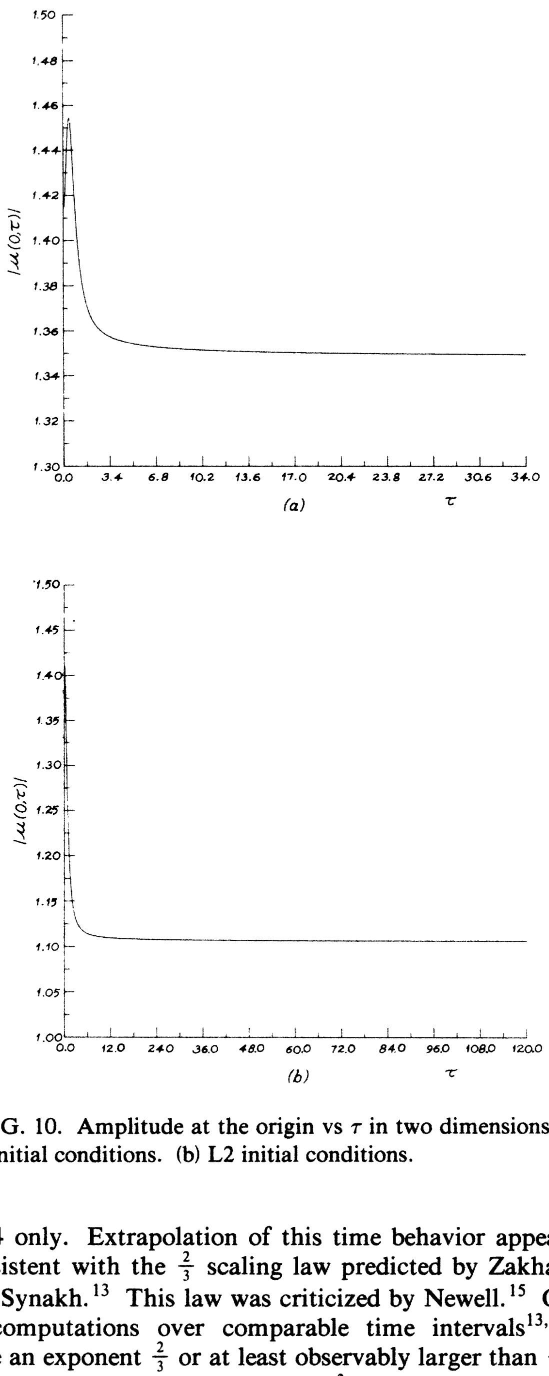



FIG. 10. Amplitude at the origin vs $\tau$ in two dimensions. (a) G2 initial conditions. (b) L2 initial conditions.

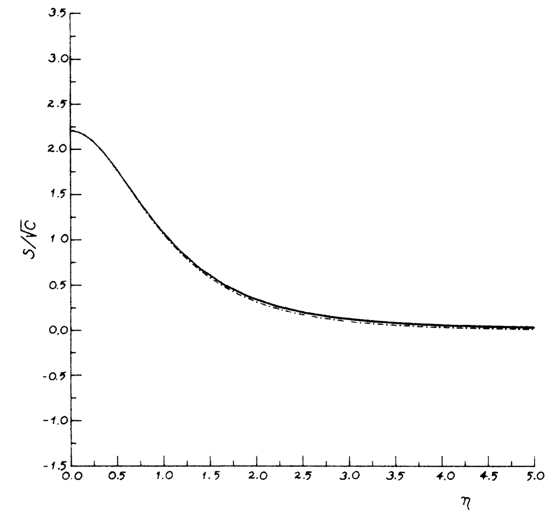



FIG. 11. Comparison of the rescaled profile corresponding to L2 initial condition for $\tau=56-120$ in steps of 4 (solid line) with the ground state of (6.9) (dashed line): the solid lines corresponding to increasing times slowly approach the ground state.

$\tau=4$ only. Extrapolation of this time behavior appeared consistent with the $\frac{2}{3}$ scaling law predicted by Zakharov and Synakh.[13] This law was criticized by Newell.[15] Other computations over comparable time intervals[13,16,17] gave an exponent $\frac{2}{3}$ or at least observably larger than $\frac{1}{2}$.[18] Note that singular solutions with $\frac{2}{3}$ exponent can be constructed analytically.[27] In the analogous critical one-dimensional problem, such singular solutions are given in Ref. 28.

Another computation performed with an adaptive grid by Vlasov, Piskunova and Talanov[14] enables the authors to reach amplifications up to $10^3$. They obtain a scaling factor of the form

$$\left[(t^*-t)/\ln\left[\frac{1}{t_*-t}\right]\right]^{1/2} .$$

This factor was also proposed by Wood[29] on the basis of a systematic but formal analysis. This result seems also unsupported by our computations since it would correspond to an algebraic decay of $a(\tau)$. However, extrapolation of the behavior of $a(\tau)$ from values not large enough might lead to such a conclusion. More detailed numerical results in the critical case are given by Le Mesurier[30] and in Ref. 27.

## ACKNOWLEDGMENTS

We acknowledge very useful discussions with B. Le Mesurier. Two of us (C.S.) and (P.L.S.) thank the Courant Institute of Mathematical Sciences for its hospitality. This work was supported by the U.S. Air Force Office of Scientific Research under Grant No. AFOSR-85-0017.

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