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# New quantum numbers in collision theory

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A set of indices $\{\sigma,\tau,\zeta,\eta\}$ is introduced to replace the four orbital quantum numbers that occur in cross-section formulas. Each index belongs to a different species of a $C_{2v}$ point group; only $\sigma$ ranges to infinity whereas all four orbital $l$'s do. Results of an earlier calculation are restructured in terms of $\{\sigma,\tau,\zeta,\eta\}$ affording an improved interpretation of those results. Directions of further investigation are indicated.

## I. INTRODUCTION

Collision theory combines dynamical parameters specific to the colliding partners with geometrical elements which are unspecific and are known in analytical form. Formulas that sort out dynamical from geometrical elements were developed long ago, mainly in nuclear contexts.[1,2] The dynamical parameters take the form of scattering matrix elements, usually in a basis of partial-wave states appropriate to the Hamiltonian's invariance under space rotations. The geometrical elements include instrumental characteristics such as collimators, external fields, polarimeters, etc., which are not invariant under the space rotations that underlie partial-wave analysis. The geometrical connection between the reference frames appropriate to the measuring instruments and to dynamical parameters, respectively, is accordingly complicated.

Further obfuscation of this connection arises from the structure of quantum cross sections which are bilinear in the scattering matrix $S$ and in its Hermitian conjugate $S^{\dagger}$. This structure casts the dynamical parameters in the form of direct products $S^{\dagger}_{a'b'}S_{ba}$, each of whose labels $(a, b, \ldots)$—for initial and final states—stands for a set of indices that includes a quantum number of the orbital motion in the center of mass frame. The sensitivity of observables to the value and kind of dynamical parameters thus requires careful investigation.

Our investigation will depart from the familiar path that centers on deriving theoretical data estimates to be compared with the results of measurements. This procedure is not only liable to amplify inaccuracies of theoretical models but is also often vitiated by redundances of experimental data rooted in inherent, if less than obvious, regularities. Here, as elsewhere, we examine which information is actually contained in experimental and theoretical data and shift their comparison to the parametrization that appears most significant and sensitive.

To this end we shall note that the dynamical parameters $S^{\dagger}_{a'b'}S_{ba}$ form a reducible representation of a $C_{2v}$ point group and shall recast them into linear combinations labeled by quantum numbers $\{\sigma,\tau,\zeta,\eta\}$ that belong to different symmetry species of $C_{2v}$. There will result a hierarchy of such linear combinations which can be only partially unraveled from cross section and orientation data on electron scattering. New classes of experiments may have to be devised to provide further evidence. The present paper merely introduces some new concepts and procedures whose development will require further work.

We refer here specifically to the theoretical analysis[3] of the observed orientation of He$(n\,^1P^o)$ atoms excited by electron collision at 50—80 eV. Even though the transition $^1S\rightarrow{}^1P^o$ is geometrically quite simple, its quantum-mechanical analysis involves infinite sums over a multiplicity of indices including four orbital quantum numbers $(l_a l_b l'_a l'_b)$. We shall probe this connection between observables and dynamical parameters more deeply by carrying the analysis of Ref. 3 through a further symmetry consideration unrelated to space rotations.

## II. SYMMETRICAL PARAMETERS

The differential cross section $d\sigma(\theta)/d\Omega$ for excitation of He$(n\,^1P^o)$ by electron scattering as well as the orientation and alignment of the excited target are expanded according to Ref. 3 into series of Legendre polynomials $P_k(\cos\theta)$ or of their derivatives, i.e., of their associated polynomials. The index $k$ is related to orbital quantum numbers of the partial-wave expansions by the triangular relations of the two triads $(l_a l'_a k)$ and $(l_b l'_b k)$ and by the parity restrictions

$$l_a+l'_a+k=\text{even}\ ,\qquad l_b+l'_b+k=\text{even}\ . \tag{1}$$

Additional triangular and parity relations in Ref. 3 concern the triads $(l_a l_b j_t)$ and $(l'_a l'_b j_t)$ and their parities

$$l_a+l_b+j_t=\text{even}\ ,\qquad l'_a+l'_b+j_t=\text{even}\ . \tag{2a}$$

Here $j_t$ indicates the angular momentum transferred to the target, indicated vectorially by

$$\mathbf{j}_t=\mathbf{l}_a-\mathbf{l}_b=\mathbf{l}'_a-\mathbf{l}'_b\ , \tag{2b}$$

and restricted in magnitude to $j_t=1$ for He$(^1S)\rightarrow$He$(^1P^o)$.

Notice here how all these relations are invariant under the pair of symmetry operations $P$ and $Q$ defined by

$$P(l_a,l_b)\equiv(l'_a,l'_b),\quad P(l'_a,l'_b)\equiv(l_a,l_b)\ , \tag{3}$$

$$Q(l_a,l'_a)\equiv(l_b,l'_b),\quad Q(l_b,l'_b)\equiv(l_a,l'_a)\ . \tag{4}$$

 

TABLE I. Symmetries of $\{\sigma,\tau,\zeta,\eta\}$.

| $P \backslash Q$ | $+$ | $-$ |
|---|---|---|
| $+$ | $\sigma$ | $\zeta$ |
| $-$ | $\tau$ | $\eta$ |

(These operators are understood here to apply to electron orbital momenta only, rather than to the target's, although both are included in $\{a,b,a',b'\}$.) We replace now the set $\{l_a l_b l_a' l_b'\}$ by the equivalent set

$$\begin{aligned} \sigma &= \tfrac{1}{2}(l_a+l_b+l_a'+l_b')\,, \\ \tau &= \tfrac{1}{2}(l_a+l_b-l_a'-l_b')\,, \\ \zeta &= \tfrac{1}{2}(l_a-l_b+l_a'-l_b')\,, \\ \eta &= \tfrac{1}{2}(l_a-l_b-l_a'+l_b')\,, \end{aligned} \tag{5}$$

whose elements are joint eigenvectors of $P$ and $Q$ with alternative eigenvalues

$$\begin{aligned} &P\sigma=\sigma,\;\; P\tau=-\tau,\;\; P\zeta=\zeta,\;\; P\eta=-\eta\,, \\ &Q\sigma=\sigma,\;\; Q\tau=\tau,\;\; Q\zeta=-\zeta,\;\; Q\eta=-\eta\,. \end{aligned} \tag{6}$$

The triangular conditions on the triads restrict the new quantum numbers—all of them integers—to

$$\sigma\geq k,\;\; -k\leq\tau\leq k,\;\; -j_t\leq\zeta\leq j_t,\;\; -j_t\leq\eta\leq j_t\,. \tag{7a}$$

The parity transfer in the excitation of the He($^1P^o$) combined with Eq. (2a) implies

$$\begin{aligned} &|\,l_a-l_b\,|=|\,l_a'-l_b'\,|=1\,, \\ &\sigma>|\,\tau\,|\,, \end{aligned} \tag{7b}$$

whereby $\sigma$ and $\tau$ cannot simultaneously equal $k$. The Eqs. (2) and (7), with $j_t=1$, also require

$$\zeta+\eta=\pm1\,, \tag{8}$$

meaning that either $\zeta$ or $\eta$ vanishes.

Since the pair of operators $\{P,Q\}$ forms (together with the identity and $PQ$) the group $C_{2v}$, each of the new quantum numbers is seen to belong to one of the four irreducible representations of this group which are called $(a_1,a_2,b_1,b_2)$, as indicated by the parity Table I. Contrast also the ranges of variation of $\{l_a,l_b,l_a',l_b'\}$, each of which runs from 0 to $\infty$, with the ranges (7) of the new quantum numbers only one of which, $\sigma$, spans an infinite range, the others being bounded on both sides.

TABLE II. Compatibility of $(\sigma,\tau)$: $\times$, compatible values of $(\sigma,\tau)$; $\square$, values compatible with $\eta=0$; $\bigcirc$, values compatible with $\zeta=0$.

| $\tau \backslash \sigma$ | $k$ | $k+1$ | $k+2$ | $\infty$ |
|---|---|---|---|---|
| $k$ | | ⊠ | | |
| $k-1$ | ⊗ | | ⊗ | |
| $k-2$ | | ⊠ | | |
| $-k+1$ | ⊗ | | ⊗ | |
| $-k$ | | ⊠ | | |

TABLE III. Compatibility of $(\zeta,\eta)$: $\times$, compatible values of $(\zeta,\eta)$; $\square$, $\bigcirc$ as in Table II.

| $\eta \backslash \zeta$ | 1 | 0 | $-1$ |
|---|---|---|---|
| 1 | | ⊗ | |
| 0 | ⊠ | | ⊠ |
| $-1$ | | ⊗ | |

The interpretation and systematic application of the new quantum numbers will be introduced in the following sections and in further papers, but a few of their aspects are immediately apparent. The single new quantum number $\sigma$ with infinite range represents an average orbital number and may accordingly be viewed as an impact parameter at the given energy. The new quantum number $\tau$ relates instead to the interference between the partial-wave expansions of $S_{ba}$ and $S^{\dagger}_{a'b'}$; the limitation (7) to the range of values of $\tau$ implies that each harmonic $P_k(\cos\theta)$ in the expansion of the product $S^{\dagger}_{a'b'}S_{ba}$ reflects only a finite subset of interference terms. The sharp restriction to the ranges of $\zeta$ and $\eta$ in our example implies that their role will be very simple for $j_t=1$ but remains to be explored in broader contexts.

The following properties of the new quantum numbers, implied by (1), (2), (7a), (7b), and (8), will prove essential to the analysis of observables in the next section. (a) The parity transfer in the excitation of He($^1P^o$), Eq. (2), requires that

$$l_a+l_b=\sigma+\tau=\text{odd}\,, \tag{9}$$

thus restricting the pairs $(\sigma,\tau)$ to the values indicated by $\times$ in Table II. (b) Equation (8), also stemming from parities, restricts $(\zeta,\eta)$ to the boxes indicated by $\times$ in Table III analogous to Table II. The range of $\zeta$ and $\eta$ is restricted here to $0,\pm1$ in contrast to the unlimited range of $\sigma$ in (10). (c) Substitution of $l_a+l_a'=\sigma+\zeta$ from (5) into Eq. (1) yields

$$\sigma+\zeta+k=\text{even}\,. \tag{10}$$

This third relation links the entries in Tables II and III into subsets marked by a square and a circle, respectively. The values of the new quantum numbers $\{\sigma,\tau,\zeta,\eta\}$ are thus partitioned into *two separate subsets*. The elements of each subset are mutually compatible. Those marked with a circle include $\eta=\pm1$, which belong to the symmetry $b_2$, those with a square include $\zeta=\pm1$ of the species $a_2$.

### A. Classification of interference effects

Tables II and III display the role of the symmetrical quantum numbers in bringing out implications of diverse symmetries, a role that will be prominent in the analysis of cross-section formulas. The cross sections for elastic collisions, which conserve the orbital momentum, would involve only one pair of indices $(l,l')$ instead of the two pairs appearing in Eq. (4). Only two symmetrical quantum numbers would occur in that case, namely $\sigma$ and $\tau$ indicating the mean and the difference of $l$ and $l'$, respectively. Cross-section terms with $\tau=0$, i.e., with $l'=l$, are regarded as "direct" (or quadratic), whereas the "cross" terms with $\tau\neq 0$ represent interference effects.

Nonconservation of orbital momentum in inelastic collisions has led here to the occurrence of three distinct indices of interference. A zero value of $\tau$ no longer suffices to identify "direct" terms of a cross-section formula, being compatible with $l'_a-l_a=-l'_b+l_b\neq 0$. Each term of such an expansion involves in fact some interference in our example of He excitation where parity transfer excludes $l_b=l_a$. Table I may thus be viewed as providing a classification of interference effects.

## III. CROSS SECTION AND ORIENTATION

Symmetrical quantum numbers have been introduced in Sec. II as a tool to sharpen the analysis of the formulas developed in Ref. 3. As a preliminary we restate those results in a more articulated form.

### A. Summary of earlier results

The differential cross section for the process $e^-+\mathrm{He}(1^1S)\rightarrow e^-+\mathrm{He}(2^1P^o)$ will be considered here as expanded in spherical harmonics

$$\frac{d\sigma}{d\Omega}=\tfrac{1}{4}\lambda\!\!\!^{-2}\sum_k P_k(\cos\theta)\sum_{l_a,l_b,l'_a,l'_b} C_{kj_t}(l_a,l_b,l'_a,l'_b)S^{\dagger}_{a'b'}S_{ba}\,, \tag{11}$$

where $\lambda\!\!\!^{-}$ is the wavelength of the incident electron divided by $2\pi$ and $\theta$ is the scattering angle. The coefficient

$$C_{kj_t}(l_a,l_b,l'_a,l'_b)=(-1)^{1+(l_a-l_b-l'_a+l'_b)/2}(2j_t+1)(2k+1)[(2l_a+1)(2l_b+1)(2l'_a+1)(2l'_b+1)]^{1/2}\times\begin{pmatrix}l_a & l'_a & k\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}l_b & l'_b & k\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix}l_a & l_b & j_t\\ l'_b & l'_a & k\end{Bmatrix}, \tag{12}$$

connects the frame of the observable scattering event to that of its dynamical analysis in partial waves. The dynamical parameters $S_{ba}$, defined as

$$S_{ba}\equiv(2^1P^o_1,l_b\mid S\mid 1^1S_0,l_a)=\sum_m(1,-m,l_b,m\mid S\mid 0,0,l_a,0)(-1)^{l_b-m}(l_a,0,l_b,-m\mid j_t,-m)\,, \tag{13}$$

with $j_t=1$, are independent of any geometrical frame. The triangular and parity relations in Eqs. (1) and (7a) reflect the structure of the $3j$ and $6j$ coefficients in (12).

The orientation of the target state $f(\equiv 2^1P^o_1)$ resulting from scattering with deflection $\theta$ in the $(zx)$ plane was represented in Ref. 3 by the mean value of the orbital momentum component orthogonal to $(zx)$, namely $\langle J_y\rangle_{f\theta}$. Since this orientation vanishes at $\theta\rightarrow 0°$ or 180° in proportion to $\sin\theta$, its dependence on $\theta$ is properly expanded into associate polynomials defined as in Eq. (8.6.6) of Ref. 4,

$$\langle J_y\rangle_{f\theta}=\sum_k P_{k1}(\cos\theta)D_{kf}\,. \tag{14}$$

Reference 3 calculated the product of (11) and (14) which is represented in the present notation by

$$\frac{d\sigma}{d\Omega}\langle J_y\rangle_{f\theta}=-\tfrac{1}{4}\lambda\!\!\!^{-2}\sum_k P_{k1}(\cos\theta)\left[\sum_{k',k''}\left[\frac{k(k+1)}{k'(k'+1)}\right]^{1/2}D_{k'f}(k',0,k'',0\mid k,0)(k,1\mid k',1,k'',0)\times\sum_{l_a,l_b,l'_a,l'_b}C_{k''j_t}(l_a,l_b,l'_a,l'_b)S^{\dagger}_{a'b'}S_{ba}\right]. \tag{15}$$

The calculation of Ref. 3 yields the alternative form of (15)

$$\frac{d\sigma}{d\Omega}\langle J_y\rangle_{f\theta}=\tfrac{1}{4}\lambda\!\!\!^{-2}\sum_k P_{k1}(\cos\theta)i^{-1}\sum_{l_a,l_b,l'_a,l'_b}C_{kj_t}(l_a,l_b,l'_a,l'_b)\frac{\mathbf{j}_t\cdot\mathbf{k}}{k(k+1)}S^{\dagger}_{a'b'}S_{ba}\,. \tag{16}$$

Comparison of (15) and (16) shows that the coefficients $D_{kf}$ of the expansion (14) obey the equations

$$\sum_{k',k''}\left[\frac{k(k+1)}{k'(k'+1)}\right]^{1/2}(k,1\mid k',1,k'',0)D_{k'f}(k',0,k'',0\mid k,0)C_{k''j_t}(l_a,l_b,l'_a,l'_b)=-i^{-1}C_{kj_t}(l_a,l_b,l'_a,l'_b)\frac{\mathbf{j}_t\cdot\mathbf{k}}{k(k+1)}\,. \tag{17}$$

The vector $\mathbf{k}$, identified in Ref. 3 as $\mathbf{l}'_b-\mathbf{l}_b=\mathbf{l}'_a-\mathbf{l}_a$ and odd under the permutation $P$ of (3), occurs in the expression (16) of $d\sigma/d\Omega\langle J_y\rangle_{f\theta}$ but is absent in the expression (11) of $d\sigma/d\Omega$. It is accordingly characteristic of the orientation $\langle J_y\rangle_{f\theta}$ as stressed in Ref. 3 and again in the following. The vector $\mathbf{j}_t=\mathbf{l}_a-\mathbf{l}_b$ in Eq. (2b) is odd under the permutation $Q$, in contrast to $\mathbf{k}$ which is odd under $P$. The product $\mathbf{j}_t\cdot\mathbf{k}$ in Eq. (17) is accordingly odd under both $P$ and $Q$, thus belonging to the same $a_2$ species as $\eta$.

### B. Transcription to symmetrical quantum numbers

(1) The sums over $\{l_a,l_b,l_a',l_b'\}$ implied by the notation $\sum_{l_a,l_b,l_a',l_b'}$ in (11) and (14) may be replaced by $\sum_{\sigma,\tau,\zeta,\eta}$, but the ranges of these sums are variously restricted, as detailed in Sec. II. From Tables II and III in particular, it follows that

$$\sum_{l_a,l_b,l_b',l_b'} = \left[ \sum_{(\sigma-k)/2=0}^{\infty} \sum_{(\tau+k-1)/2=0}^{k-1} \sum_{\eta=\pm 1} \right]_{\zeta=0} + \left[ \sum_{(\sigma-k-1)/2=0}^{\infty} \sum_{(\tau+k)/2=0}^{k} \sum_{\zeta=\pm 1} \right]_{\eta=0} . \tag{18}$$

The range of $\tau$ depends here on the value of $\sigma$ and on whether $\zeta$ or $\eta$ vanishes; the range ($\pm 1$) of $\zeta$ or $\eta$ is, however, independent of other new quantum numbers.

(2) The coefficient $C_{kj_t}$, Eq. (12), is manifestly invariant under both permutations $P$ and $Q$ in view of the symmetries of $3j$ and $6j$ coefficients. Accordingly only its dependence on the indices $(\tau,\zeta,\eta)$ should be affected by their respective sign reversals. The $3j$ and $6j$ coefficients in (11) are generally polynomials but are monomials in our present case where the lower indices of the $3j$ vanish and $j_t=1$.[5] They are accordingly represented in terms of $(\sigma,\tau,k)$ as ratios of factorials, binomial coefficients, and simpler algebraic expressions, namely

$$C_{kj_t}(\sigma,\tau,0,\eta)=\frac{4(\sigma-k)!\left[\dfrac{\sigma+k+1}{2}!\right]^2}{[(\sigma+1)^2-\tau^2]^{1/2}(\sigma+k)!\left[\dfrac{\sigma-k-1}{2}!\right]^2}\begin{bmatrix} k-\tau \\ \dfrac{k-\tau}{2}\end{bmatrix}\begin{bmatrix} k+\tau \\ \dfrac{k+\tau}{2}\end{bmatrix} . \tag{19a}$$

$$C_{kj_t}(\sigma,\tau,\zeta,0)=\frac{4(\sigma-k)!\left[\dfrac{\sigma+k}{2}!\right]^2(k-\tau)!(k+\tau)!}{[(\sigma+1)^2-\tau^2]^{1/2}(\sigma+k)!\left[\dfrac{\sigma-k}{2}!\right]^2\left[\dfrac{k-1-\tau}{2}!\right]^2\left[\dfrac{k-1+\tau}{2}!\right]^2} . \tag{19b}$$

The new quantum numbers $\zeta$ and $\eta$ fail to appear explicitly on the right of (19) since $C_{kj_t}$ must be independent of their values $\pm 1$.

(3) The factor $\mathbf{j}_t\cdot\mathbf{k}/k(k+1)$ of Eq. (16), odd under both $P$ and $Q$, as noted at the end of Sec. III A transcribes into

$$\mathbf{j}_t\cdot\mathbf{k}=(\sigma+1)\eta+\tau\zeta=\begin{cases}(\sigma+1)\eta \ \text{ when } \zeta=0 & \text{(20a)}\\ \tau\zeta \ \text{ when } \eta=0 . & \text{(20b)}\end{cases}$$

Notice how the oddness of $\mathbf{j}_t\cdot\mathbf{k}$ under $P$ or $Q$ emerges in (20a) because $\eta$ is itself odd under both operations, whereas in (20b) both $\tau$ and $\zeta$ are odd under one permutation and even under the other.

(4) The products of matrix elements $S^{\dagger}_{a'b'}S_{ba}$ will now be indicated as elements of the direct product matrix $S^{\dagger}\times S$ with indices $(\sigma,\tau,\zeta,\eta)$, it being understood that $S$ also implies target transitions ${}^1S\rightarrow{}^1P^o$ and $S^{\dagger}$ the reciprocal transition ${}^1P^o\rightarrow{}^1S$

$$S^{\dagger}_{a'b'}S_{ba}\equiv(S^{\dagger}\times S)_{\sigma,\tau,\zeta,\eta} . \tag{21}$$

These dynamical parameters are *not* eigenvectors of the permutations $P$ and $Q$. The symmetry of $S$ under permuation of its indices has, nevertheless, a consequence that was crucial to Ref. 3, namely,

$$P(S^{\dagger}\times S)_{\sigma,\tau,\zeta,\eta}=(S^{\dagger}\times S)_{\sigma,-\tau,\zeta,-\eta}=(S^{\dagger}\times S)^{*}_{\sigma,\tau,\zeta,\eta} . \tag{22}$$

On the other hand, the symmetry of $S_{ab}$ under permutation of $a$ and $b$ differs from the operation $Q$ which interchanges the orbital numbers $l_a$ and $l_b$ without affecting the target states. The operation $Q$ reverses the "propensity" of the transition $a\rightarrow b$. Propensity means that the value of $S_{ba}$ is much larger when the target excitation ${}^1S\rightarrow{}^1P^o$ is accompanied by a loss of orbital momentum of the electron ($l_b>l_a$) than when accompanied by $l_b<l_a$.[6]

### C. Sums over symmetrical quantum numbers

The coefficients $C_{kj_t}$ of Eqs. (11) and (16) do not depend explicitly on the symmetrical quantum numbers $\zeta$ and $\eta$, according to Eqs. (19), beyond the requirement that the alternative expressions (19a) and (19b) be entered in the separate contributions of $\zeta=0$ or $\eta=0$ terms. Recall also that Eqs. (19) are symmetric in $\tau$ and $-\tau$. Explicit dependence on $\zeta$ and $\eta$ is accordingly confined to the dynamical parameters $(S^{\dagger}\times S)_{\sigma,\tau,\zeta,\eta}$ and, for Eq. (16), to the coefficient $\mathbf{j}_t\cdot\mathbf{k}$, Eqs. (20). The alternative summations over $\zeta=\pm 1$ and $\eta=\pm 1$ can now be worked out explicitly, using the symmetry (22). The resulting expressions will also be symmetric in $\tau$ and $-\tau$ as are the $C_{kj_t}$.

For $\zeta=0$, Eq. (22) yields the expression to be entered in (11)

$$\sum_{\eta=\pm 1}(S^{\dagger}\times S)_{\sigma,\tau,0,\eta} =2[\,|(S^{\dagger}\times S)_{\sigma\,|\tau|\,0\,1}|\cos\phi_{\sigma\,|\tau|\,0\,1} +|(S^{\dagger}\times S)_{\sigma\,|\tau|\,0\,-1}|\cos\phi_{\sigma\,|\tau|\,0\,-1}] , \tag{23a}$$

where $\phi_{\sigma,\tau,\zeta,\eta}$ indicates the complex phase of $(S^{\dagger}\times S)_{\sigma,\tau,\zeta,\eta}$. The corresponding expression to be entered in Eq. (16), including the contribution of the factor $\mathbf{j}_t \cdot \mathbf{k}$ as given by (20b), is

$$(\sigma+1) \sum_{\eta=\pm 1} \eta (S^\dagger \times S)_{\sigma,\tau,0,\eta} = 2i(\sigma+1)[\,|(S^\dagger \times S)_{\sigma\,|\tau|\,01}|\sin\phi_{\sigma\,|\tau|\,01} - |(S^\dagger \times S)_{\sigma\,|\tau|\,0-1}|\sin\phi_{\sigma\,|\tau|\,0-1}] \,. \tag{24a}$$

The analogous expressions for the $\eta=0$ terms are

$$\sum_{\zeta=\pm 1} (S^\dagger \times S)_{\sigma,\tau,\zeta,0} = 2[\,|(S^\dagger \times S)_{\sigma\,|\tau|\,10}|\cos\phi_{\sigma\,|\tau|\,10} + |(S^\dagger \times S)_{\sigma\,|\tau|\,-10}|\cos\phi_{\sigma\,|\tau|\,-10}] \,, \tag{23b}$$

$$\tau \sum_{\zeta=\pm 1} \zeta (S^\dagger \times S)_{\sigma,\tau,\zeta,0} = 2i\,|\tau|\,[\,|(S^\dagger \times S)_{\sigma\,|\tau|\,10}|\sin\phi_{\sigma\,|\tau|\,10} - |(S^\dagger \times S)_{\sigma\,|\tau|\,-10}|\sin\phi_{\sigma\,|\tau|\,-10}] \,. \tag{24b}$$

The dynamical parameters with $\zeta=\pm 1$ are not related by symmetry, as noted above, those with $\zeta=1$ being generally much larger in magnitude owing to a propensity rule.[6] The propensity rule has a major influence on the interpretation of the two groups of terms in (18), those with $\zeta=0$ and those with $\eta=0$, respectively, and on the possibility of sorting out their respective contributions to scattering cross section and orientation. Except for this aspect, to be dealt with in a separate report, the contributions of the two dynamical parameters in Eqs. (23b) and (24b) could not be sorted out from current experiments.

Neither can we foresee at this time how to sort out the contributions to Eqs. (11) and (16) from terms with different values of $\tau$. Terms with different values of $k$ can, of course, be determined separately from measurements of $d\sigma/d\Omega$ and of $\langle J_y \rangle_{f\theta}$ at different scattering angles. From such data it may well be possible to sort out the contributions from the terms with different values of $\sigma$.

## IV. SUMMARY AND PROSPECTS

Use of the symmetry-adapted quantum numbers $\{\sigma,\tau,\zeta,\eta\}$ has enabled us to restructure the results of Ref. 3 into a form that groups the dynamical parameters $S^\dagger_{a'b'}S_{ba}$ into subsets of increasing scope: sums over $\zeta$ and $\eta$, over $\tau$, and finally over $\sigma$. The sums over $\zeta$ and $\eta$ will be interpreted elsewhere in the light of propensity rules. The sums over $\tau$ represent interference effects, whose fuller interpretation seems to require a deeper analysis. The sums over $\sigma$, on the other hand, seem likely to be unraveled by expressing each of their terms as a linear combination of the coefficients of $P_k(\cos\theta)$ or $P_{k1}(\cos\theta)$ in Eqs. (11) and (16), thus relating it to specific features of the angular dependence of observables.

The observables (11) and (16) are themselves invariant under the permutation $P$. Indeed, in Eqs. (23) and (24), the sums over the parameter $\tau$ remain invariant as $\tau \rightarrow -\tau$ even though $\tau$ is itself odd under $P$. On the other hand, the observables (11) and (16) are not invariant under the permutation $Q$. Analysis of the propensity effect on $S_{ab}$—to be reported separately—will, however, relate the terms of Eq. (18) with $\zeta$ or $\eta=0$ to collisions with forward or backward scattering, respectively, as a by-product of symmetry under $Q$.

Beyond these tasks lies the major one of extending the scope of our treatment, which has been restricted here to the He excitations considered in Ref. 3. Target alignment remains to be treated. More important is lifting the restrictions to unit angular momentum transfer ($j_t=1$) and to excitations with parity transfer, both of which hold only for $^1S \rightarrow {}^1P^o$ transitions. Spin-orbit coupling also becomes relevant for higher-$Z$ processes. Most of the treatment of Sec. II should thus be extended, particularly Table II. Sums over alternative values of $j_t$ will occur in the analysis of (11) and (16), including generally cross terms ($j'_t \neq j_t$) whenever target orientation, alignment, or higher multipoles are observed. The appropriate structure of the relevant equations remains thus altogether unexplored.

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